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Arakelian's Approximation Theorem

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Arakelian's theorem [**1**], [**3**] concerns uniform approximation by entire functions on possibly unbounded closed subsets E of the complex plane $\mathbb{C}$. Our attention was drawn to this theorem because it has recently been used [**2**; p. 164], [**5**; p. 761] to construct interesting holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. The aim of our note is to show that Arakelian's theorem follows very easily from the much better known theorem of Mergelyan [6; Chap. 20] which deals with uniform approximation on *compact* sets.

Moreover—and this is perhaps our main point—in most applications the function that is to be approximated on E is actually holomorphic in a *neighborhood* of E (i.e., in an open set that contains E), and in that case the proof given below relies only on the classical approximation theorem of Runge [7], [**6**; p. 270]. (Incidentally, we recommend Runge's beautiful original paper; it is very readable.) For functions that are holomorphic in a neighborhood of E, Arakelian's theorem thus turns out to be really elementary.

If E is a closed subset of $\mathbb{C}$, we shall use the phrase "hole of E" to denote any *bounded component of the complement of* E. Using this terminology, Runge's theorem states:

If K is a compact subset of $\mathbb{C}$, *without holes, and f is holomorphic in a neighborhood of K, then f can be approximated, uniformly on K, by holomorphic polynomials.*

Mergelyan's theorem derives the same conclusion from a weaker assumption about f, namely: f should be continuous on K and holomorphic in the interior of K.

To motivate the definition that follows, note that if E is a closed set without holes and D is a closed disc in $\mathbb{C}$, then the intersection $E \cap D$ obviously has no holes either, but the union $E \cup D$ may very well have some, even infinitely many.

Definition. A closed set $E \subset \mathbb{C}$, without holes, is an *Arakelian set* if, for every closed disc $D \subset \mathbb{C}$, the *union* of all holes of $E \cup D$ is a bounded set.

Note. In [**1**] and [**3**], Arakelian's theorem is stated for closed sets without holes whose complement is "locally connected at infinity." The preceding definition describes the same class of sets. We chose it because we think that it is more easily understood, and because it explicitly states the property of E that is crucial in our proof.

THEOREM. *If E is an Arakelian set, f is a complex-valued continuous function on E that is holomorphic in the interior of E, and* $\varepsilon > 0$, *then there is an entire function h that satisfies*

$$|h(z) - f(z)| < \varepsilon$$

for every $z \in E$.

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Proof. Since E is an Arakelian set, there are closed discs D_i $(i = 1, 2, 3, \dots)$, centered at the origin, whose union is $\mathbb{C}$, so that the interior of D_{i+1} contains the compact set $D_i \cup \overline{H}_i$, where H_i is the union of the holes of $E \cup D_i$.

Put $E_0 = E$, and $E_i = E \cup D_i \cup \overline{H}_i$ for $i \geqslant 1$. Note that no E_i has holes.

We first deal with the "Runge case" in which f is holomorphic in a neighborhood of E. Put $h_0 = f$, fix $i \geqslant 1$, and assume (as induction hypothesis) that we have a function h_{i-1} that is holomorphic in a neighborhood of E_{i-1}. There is an open disc Δ that contains $D_i \cup \overline{H}_i$ and whose closure $\overline{\Delta}$ lies in the interior of D_{i+1}. Choose a continuously differentiable function ψ on $\mathbb{C}$ so that $0 \leqslant \psi \leqslant 1$, $\psi = 1$ in Δ, $\psi = 0$ outside D_{i+1}.

Since E_{i-1} has no holes, the same is true of $E_{i-1} \cap D_{i+1}$. Runge's theorem therefore furnishes a polynomial P so that

$$|h_{i-1} - P| < 2^{-i-1}\varepsilon \quad \text{on } E_{i-1} \cap D_{i+1} \tag{1}$$

and

$$\frac{1}{\pi}\int_{E_{i-1}} \left|(h_{i-1} - P)(w)(\bar{\partial}\psi)(w)\right| \frac{dm(w)}{|z - w|} < 2^{-i-1}\varepsilon \tag{2}$$

for all $z \in \mathbb{C}$. Here $\bar{\partial}\psi = \partial\psi/\partial\bar{z}$, and m denotes two-dimensional Lebesgue measure. Note that (2) can be achieved because the integrand vanishes outside D_{i+1}.

Now let V be a neighborhood of E_{i-1} in which h_{i-1} is holomorphic, and which is so close to E_{i-1} that (2) holds with V in place of E_{i-1}. Define

$$r(z) = \frac{1}{\pi}\int_V (h_{i-1} - P)(w)(\bar{\partial}\psi)(w)\frac{dm(w)}{z - w} \qquad (z \in \mathbb{C}) \tag{3}$$

and

$$h_i = \psi P + (1 - \psi)h_{i-1} + r \text{ in } \Delta \cup V. \tag{4}$$

(This is well defined because $1 - \psi = 0$ in Δ.)

The fact that $(h_{i-1} - P)\bar{\partial}\psi$ is continuously differentiable in V shows that

$$\bar{\partial}r = (h_{i-1} - P)\bar{\partial}\psi$$

in V; see, for example [**4**; p. 104]. Since $\bar{\partial}h_{i-1} = 0$ in V, we see that

$$\bar{\partial}h_i = (P - h_{i-1})\bar{\partial}\psi + \bar{\partial}r = 0 \text{ in } V. \tag{5}$$

In Δ, $\bar{\partial}\psi = 0$. The integral (3) extends therefore only over $V \setminus \Delta$, so that r is holomorphic in Δ. The same is true of h_i, because $h_i = P + r$ in Δ.

We conclude: h_i is holomorphic in the neighborhood $\Delta \cup V$ of E_i (which is our induction hypothesis, with i in place of $i - 1$) and

$$|h_i - h_{i-1}| = |(P - h_i)\psi + r| < 2^{-i}\varepsilon \text{ on } E_{i-1} \tag{6}$$

by (1); note that $\psi = 0$ outside D_{i+1} and that $|r| < 2^{-i-1}\varepsilon$.

The sets E_{i-1} contain the discs D_{i-1}, and these expand to cover $\mathbb{C}$. Hence (6) shows that $h = \lim h_i$ satisfies the conclusion of the theorem.

In the "Mergelyan case" we get P from Mergelyan's Theorem, define $r(z)$ as above, but with E_{i-1} in place of V in (3), and we conclude that h_i (defined by (4) on $\Delta \cup E_{i-1}$) is continuous on E_i, holomorphic in the interior of E_i, and satisfies (6) on E_{i-1}.

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This concludes the proof.

Remark. On sets without interior, a considerably stronger version of the theorem can be derived from it without any extra effort:

If E is an Arakelian set with empty interior, f and ω are continuous functions on E, f is complex-valued, ω is positive (and $\omega(z) \to 0$ as $z \to \infty$ along E, to make things interesting), then there is an entire function h that satisfies

$$|h(z) - f(z)| < \omega(z)$$

for every $z \in E$.

To prove this, apply the theorem twice: There are entire functions g and h_0 so that

$$\operatorname{Re} g < \log \omega \text{ and } |h_0 - f \cdot \exp(-g)| < 1$$

on E. Put $h = h_0 \cdot \exp(g)$.

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Added in proof: Another reference to Arakelian's theorem is Approximation Uniforme Qualitative sur les Ensembles non Bornés, by P. M. Gauthier and W. Hengartner, Presses de l'Université de Montréal, 1962, p. 37.

The Norm of a Linear Functional

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Bounded linear functionals of the type

$$f(x) = \int_a^b g(t)x(t)\,dt \tag{1}$$

frequently occur in elementary functional analysis and its applications, and one needs to have an expression for $\|f\|$, the norm of f. For example, if $x = x(t)$ is a continuous function of period 2π and X is the Banach space of all such functions, with $\|x\| = \max\{|x(t)|: -\pi \leqslant t \leqslant \pi\}$, then the Fourier coefficients of x are, by definition,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos kt\,dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin kt\,dt, \tag{2}$$

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