ON MULTIPLY CONNECTED WANDERING DOMAINS OF MEROMORPHIC FUNCTIONS

P.J. RIPPON AND G.M. STALLARD

Abstract. We describe conditions under which a multiply connected wandering domain of a transcendental meromorphic function with a finite number of poles must be a Baker wandering domain, and we discuss the possible eventual connectivity of Fatou components of transcendental meromorphic functions. We also show that if f is meromorphic, U is a bounded component of $F(f)$ and V is the component of $F(f)$ such that $f(U) \subset V$, then f maps each component of ∂U onto a component of the boundary of V in $\hat{\mathbb{C}}$. We give examples which show that our results are sharp; for example, we prove that a multiply connected wandering domain can map to a simply connected wandering domain, and vice versa.

1. Introduction

Throughout this paper $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function and we denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighborhood of z. The complement of $F(f)$ in $\hat{\mathbb{C}}$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [9]. In this paper we study the components of $F(f)$, known as *Fatou components*, and their boundaries. Note that the notions of closure and complements are always taken with respect to $\hat{\mathbb{C}}$. However, we need to consider both the boundary of a set U in $\mathbb{C}$, for which we use the notation ∂U, and the boundary of U in $\hat{\mathbb{C}}$, for which we use $\hat{\partial} U$.

The set $F(f)$ is completely invariant under f, as is $J(f)$ in the sense that $z \in J(f)$ if and only if $f(z) \in J(f)$ whenever $f(z)$ is defined. Therefore, any component of $F(f)$ must map into a component of $F(f)$, though this mapping may not be onto because of the possible presence of finite asymptotic values; see Lemma 5 for more detail on this phenomenon. Similar remarks apply to components of $J(f) \cap \mathbb{C}$ and components of ∂U, where U is a Fatou component; see Example 5.

For any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of *period* p, assuming p to be minimal. There are then five possible types of periodic components; see [9, Theorem 6]. If U_n is not eventually periodic, then we say that U is a *wandering component* of $F(f)$, or a *wandering domain*.

We use the name *Baker wandering domain* to denote a wandering component U of $F(f)$ such that, for n large enough, U_n is a bounded multiply connected component of $F(f)$ which surrounds 0, and $U_n \to \infty$ as $n \to \infty$. An example of this phenomenon with f an entire function was first given by Baker in [2] and examples with either a finite or an infinite number of poles can be obtained by minor modifications of this construction; see [29].

If f is a transcendental entire function and U is a multiply connected component of $F(f)$, then U is a Baker wandering domain; see [1]. This need not be the case for meromorphic functions, even those with finitely many poles; see [13] for examples of

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meromorphic functions with one pole which have invariant multiply connected components of $F(f)$. There are also examples of meromorphic functions with multiply connected wandering domains that are not Baker wandering domains. For example, in [6] Baker, Kotus and Lü used techniques from approximation theory to construct several meromorphic functions, each with infinitely many poles, having multiply connected wandering domains of various types. In particular, for $k \in \{2, 3, \ldots\}$, they constructed a meromorphic function with a k-connected bounded wandering domain which is not a Baker wandering domain; recall that a domain is *k-connected* or, equivalently, it has *connectivity* k if $\hat{\mathbb{C}} \setminus U$ has k components.

Baker, Kotus and Lü also showed, in [8], that any invariant Fatou component of a meromorphic function is simply connected, doubly connected (in which case the component is a Herman ring) or infinitely connected. This result (apart from the Herman ring statement) was generalised by Bolsch [11] to periodic Fatou components of functions that are meromorphic outside a small set of essential singularities.

In this paper, we first study the set M_F of transcendental meromorphic functions with only finitely many poles and we give conditions under which a multiply connected wandering domain of a function in M_F must be a Baker wandering domain. We also construct examples to show that if $f \in M_F$, then a multiply connected wandering domain of f need not be a Baker wandering domain. For any meromorphic function f we let $\operatorname{sing}(f^{-1})$ denote the set of inverse function singularities of f, which consists of the critical values and finite asymptotic values of f.

In Section 2, we prove the following result. Recall that for a component U of $F(f)$ and for $n = 0, 1, 2, \ldots$, we denote by U_n the component of $F(f)$ such that $f^n(U) \subset U_n$.

Theorem 1. *Let $f \in M_F$ and let U be a multiply connected wandering domain of f.*

(a) *The component U is a Baker wandering domain if and only if infinitely many of the components U_n, $n = 0, 1, 2, \ldots$, are multiply connected.*

(b) *If*

$$sing(f^{-1}) \cap \bigcup_{n \geq 1} U_n = \emptyset, \tag{1.1}$$

then U_n is multiply connected for $n = 0, 1, 2, \ldots$, so U is a Baker wandering domain.

Remark After submitting this paper, we learnt of the paper [25] by Qiu and Wu, which contains a result closely related to our Theorem 1(a). Their hypothesis is that U is wandering and all U_n are multiply connected, and they conclude that $U_n \to \infty$ as $n \to \infty$ and U_n surrounds 0 for large n. From this they deduce that f has infinitely many weakly repelling fixed points. By Theorem 1(a), this conclusion follows also from the hypothesis that U is wandering and infinitely many U_n are multiply connected.

Note that Theorem 1(a) is false without the hypothesis that $f \in M_F$. This is shown by the finitely connected example of Baker, Kotus and Lü [6] mentioned earlier. In Section 4, we construct an infinitely connected example to show this, as follows.

Example 1. *There exists a meromorphic function f with infinitely many poles and a wandering domain U such that each component U_n, $n = 0, 1, 2, \ldots$, is bounded and infinitely connected, but U is not a Baker wandering domain.*

Our second example shows that there does exist a meromorphic function f with a multiply connected wandering domain U such that, for $n \geq 1$, the components U_n are simply connected. As far as we know, this is the first such example.

Example 2. *There exists a function $f \in M_F$ with a bounded doubly connected wandering domain U such that each component U_n, $n = 1, 2, \ldots$, is bounded and simply connected.*

Next we discuss some general connectivity properties of Fatou components of transcendental meromorphic functions. Following Kisaka and Shishikura [19], we define the *eventual connectivity* of a component U of $F(f)$ to be c provided that U_n has connectivity c for all large values of n. Kisaka and Shishikura [19, Theorem A] showed that if f is entire and U is a multiply connected component of $F(f)$, and hence a Baker wandering domain, then the eventual connectivity of U exists and is either 2 or ∞. Moreover, they constructed the first example of an entire function f with a Baker wandering domain with eventual connectivity 2, thus answering an old question; see [6] and [9, page 167]. Earlier, Baker [3] constructed an example with infinite eventual connectivity.

For meromorphic functions the situation is less straightforward since a wandering domain can be multiply connected without being a Baker wandering domain. The following theorem on connectivity properties of bounded components of $F(f)$ is a collection of known results by other authors, stated together for convenience; see Section 3 for references. Here we denote the connectivity of a domain U by $c(U)$.

Theorem 2. *Let f be meromorphic, let U be a bounded component of $F(f)$ and let V be the component of $F(f)$ such that $f(U) \subset V$.*

(a) *We have*

$$f(U) = V \quad \text{and} \quad f(\partial U) = \hat{\partial} V.$$

(b) *If U is finitely connected, then $c(U) \geq c(V)$.*
(c) *If U is infinitely connected, then V is infinitely connected.*

We remark that if a pole of f lies in ∂U, then ∂V is unbounded and $\hat{\partial} V = \partial V \cup \{\infty\}$. The following corollary of Theorem 2 is immediate.

Corollary 1. *Let f be meromorphic, let U be a component of $F(f)$ and suppose that the components U_n, $n = 0, 1, 2, \ldots$, are all bounded.*

(a) *If U is finitely connected, then*

$$c(U_n) \geq c(U_{n+1}), \quad \text{for } n = 0, 1, 2, \ldots,$$

so the eventual connectivity of U exists and is finite.
(b) *If U is infinitely connected, then each U_n, $n = 0, 1, 2, \ldots$, is infinitely connected, so the eventual connectivity of U is ∞.*

Note that in Corollary 1 we have $f^n(U) = U_n$, for $n \in \mathbb{N}$, by Theorem 2(a).

Using Theorem 1(a) and Corollary 1, we obtain the following result. Part (b) generalises to M_F a result of Kisaka and Shishikura [19, Theorem A] for entire functions, mentioned above.

Theorem 3. *Let $f \in M_F$ and let U be a wandering domain of f.*

(a) *If U is not a Baker wandering domain, then the eventual connectivity of U is 1.*
(b) *If U is a Baker wandering domain, then the eventual connectivity of U is either 2 or ∞.*

In the example of Baker, Kotus and Lü mentioned after Theorem 1, it can be shown that the wandering domains have eventual connectivity k, where $k \in \{2, 3, \ldots\}$. Thus part (a) of Theorem 3 is false without the assumption that $f \in M_F$. By modifying their example, we can obtain a meromorphic function f with a Baker wandering domain whose eventual connectivity is k, where $k \in \{2, 3, \ldots\}$, so Theorem 3(b) is also false without the assumption that $f \in M_F$. The idea of the modification is to replace the sequence of k-connected domains used in the original construction, which are almost invariant under the mapping $z \mapsto z + 10$, by a sequence of similarly shaped domains which are almost invariant under $z \mapsto 10z$; we omit the details which are routine but lengthy.

We now discuss several examples related to Theorem 2. First, it is well known that Theorem 2(a) is false if U is unbounded. For example, the function $f(z) = e^z - 1$ has an unbounded immediate parabolic basin U, which contains the singularity -1, such that $f(U) = U \setminus \{-1\}$. On the other hand, for almost all λ with $|\lambda| = 1$, the function $f(z) = \lambda(e^z - 1)$ has an unbounded invariant Siegel disc U, whose boundary contains the singularity $-\lambda$, such that $f(\partial U) \subset \partial U \setminus \{-\lambda\}$; see [26] and [27].

Next we show that the requirement that U is bounded is essential in Theorem 2(b), as is the requirement that all U_n are bounded in the statement that $c(U_n) \geq c(U_{n+1})$, for $n = 0, 1, 2, \dots$, in Corollary 1(a).

Example 3. *There exists a function $f \in M_F$ with a bounded simply connected wandering domain U such that*

(a) *$f(U)$ is an unbounded simply connected component of $F(f)$ and $\partial f(U)$ consists of two unbounded components;*
(b) *$f^2(U)$ is a bounded doubly connected component of $F(f)$;*
(c) *$f^n(U)$, $n \geq 3$, are bounded simply connected components of $F(f)$.*

Thus $U_1 = f(U)$ is unbounded and $c(U_1) = 1 < 2 = c(U_2)$.

The requirement that U is bounded is also essential in Theorem 2(c), as is the requirement that all U_n are bounded in Corollary 1(b).

Example 4. *There exists a function $f \in M_F$ with a bounded infinitely connected wandering domain U such that*

(a) *$f(U)$ is an unbounded infinitely connected component of $F(f)$;*
(b) *$f^2(U)$ is contained in a bounded doubly connected component of $F(f)$;*
(c) *$f^n(U)$, $n \geq 3$, are contained in bounded simply connected components of $F(f)$.*

Thus $U_1 = f(U)$ is unbounded and infinitely connected, and the eventual connectivity of U_1 is 1.

The following result is closely related to Theorem 2. This result may also be known, but we have not been able to find a reference to it in this generality. Note that Theorem 4 gives an alternative proof of Theorem 2(c).

Theorem 4. *Let f be meromorphic, let U be a bounded component of $F(f)$ and let V be the component of $F(f)$ such that $f(U) \subset V$. Then f maps each component of ∂U onto a component of $\hat{\partial} V$.*

We remark that if a pole of f lies in a component of ∂U, then the image of that component may be the union of more than one component of ∂V together with $\{\infty\}$.

Our final example shows that Theorem 4 is false if U is unbounded.

Example 5. *The function $f(z) = ze^z$ has an unbounded immediate parabolic basin U whose boundary ∂U has components α and α' such that $f(\alpha) = \alpha' \setminus \{0\}$.*

Finally, for an unbounded component U of $F(f)$, we can obtain the following result relating the boundary connectedness properties of U to those of the component of $F(f)$ which contains $f(U)$.

Theorem 5. *Let f be a transcendental meromorphic function, let U be an unbounded component of $F(f)$ and let V be the component of $F(f)$ such that $f(U) \subset V$.*

(a) *We have*
$$\hat{\partial} V = \overline{f(\partial U)}.$$
(b) *If ∂U has only a finite number N of components, then $\hat{\partial} V$ has at most N components.*

(c) *If $c(V) > c(U)$, then there exists at least one unbounded component of ∂U which has a bounded image.*

Example 3 shows that the situation in Theorem 5(c) can occur, since in this example we have $c(U_2) > c(U_1)$.

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2. Proof of Theorem 1

First, we give several results needed in the proof of Theorem 1.

Lemma 1. *Let $f \in M_F$. There exists $r_0 > 0$ such that if U is a component of $F(f)$ which contains a Jordan curve surrounding $\{z : |z| \le r_0\}$, then U is a Baker wandering domain.*

Proof. In [29, Theorem 3] we proved that if $f \in M_F$, then there exists $r_0 > 0$ such that if U is a component of $F(f)$ and $\{z : |z| \le r_0\}$ lies in a bounded complementary component of U, then U is a Baker wandering domain. The proof given there depends only on the fact that U contains a Jordan curve which winds round $\{z : |z| \le r_0\}$ and so it yields the above more general result. □

Now we denote by M the set of transcendental meromorphic functions f with at least one pole which is not an omitted value of f; in the language of [7], f satisfies Assumption A or is a 'general meromorphic function'. We also introduce the notation $\widetilde{E}$ to denote the union of a set E and its bounded complementary components.

Lemma 2. *Let $f \in M$ and let U be a component of $F(f)$. If there is a Jordan curve γ in U such that $\widetilde{\gamma}$ meets $J(f)$, then for some $n \ge 0$, $\widetilde{f^n(\gamma)}$ contains a pole of f.*

Proof. This follows from the fact that for $f \in M$ we have $J(f) = \overline{O^-(\infty)}$, by [7, Lemma 1], together with the fact that if f^n is analytic on $\widetilde{\gamma}$, then $\partial f^n(\widetilde{\gamma}) \subset f^n(\gamma)$. □

In the next lemma we use the classification of periodic components of $F(f)$ into five types: attracting basins, parabolic basins, Siegel discs, Herman rings and Baker domains; see [9, Theorem 6]. Here, and in the proof of Theorem 1(b), we use ideas from [30, Lemma 3.3].

Lemma 3. *Let $f \in M \cap M_F$ and let U be a component of $F(f)$. If there is a Jordan curve γ in U such that $\widetilde{f^n(\gamma)}$ contains a point of $J(f)$ for infinitely many n, then U is either a Herman ring (or its pre-image) or a Baker wandering domain.*

Proof. Suppose that U is not a Herman ring (nor its pre-image). Clearly U is not a Siegel disc (nor its pre-image). Therefore U is a wandering domain or an immediate attracting or parabolic basin of $F(f)$, or a Baker domain of f (or a pre-image of one of these). Hence all locally uniformly convergent subsequences of f^n have constant limit functions in U; see [8, Lemma 2.1] and [9, page 163]. Thus the spherical diameter of $\gamma_n = f^n(\gamma)$ tends to 0 along any such subsequence. Since $f \in M$ and $f \in M_F$, we deduce by Lemma 2 that, for infinitely many n, $\widetilde{\gamma_n}$ contains the same pole of f, say p. Thus there is a sequence n_k such that $p \in \widetilde{\gamma_{n_k}}$ for all k and f^{n_k} tends to either ∞ or p, locally uniformly in U.

In the first case, $\operatorname{dist}(\gamma_{n_k}, 0) \to \infty$. Also, $p \in \widetilde{\gamma_{n_k}}$ and hence $0 \in \widetilde{\gamma_{n_k}}$, for all large enough k. Thus U is a Baker wandering domain by Lemma 1. In the second case, $\operatorname{dist}(\gamma_{n_k}, p) \to 0$, so $\operatorname{dist}(f(\gamma_{n_k}), 0) \to \infty$ and $0 \in \widetilde{f(\gamma_{n_k})}$, for all large enough k. Thus U is again a Baker wandering domain by Lemma 1. □

Proof of Theorem 1(a). First, if f is a transcendental entire function, then Theorem 1(a) is well-known; see [1]. Next, suppose that f is a transcendental meromorphic function with exactly one pole, which is an omitted value of f. Then f cannot have a multiply connected wandering domain [4, Theorem 1], so there is nothing to prove. Hence we can assume without loss of generality that $f \in M \cap M_F$.

It is obvious that if U is a Baker wandering domain, then infinitely many U_n are multiply connected. We now prove the opposite implication by contradiction. Let U be a wandering domain such that infinitely many of the components U_n are multiply connected and suppose that U is not a Baker wandering domain. Since U is a wandering domain, we deduce, by Lemma 3, that

> if γ is a Jordan curve in U_N, where $N \geq 0$, then $\widetilde{f^n(\gamma)}$ contains a pole of f for at most finitely many n.

Choose n_0 such that U_{n_0} is multiply connected, and then take any Jordan curve γ_0 in U_{n_0} such that $\widetilde{\gamma_0}$ meets $J(f)$. By Lemma 2, we can choose $m_0 \geq 0$ such that $\widetilde{f^{m_0}(\gamma_0)}$ contains a pole of f. If $\widetilde{f^{m_0+1}(\gamma_0)}$ meets $J(f)$, then we can apply Lemma 2 again to find $m_0' > m_0$ such that $\widetilde{f^{m_0'}(\gamma_0)}$ contains a pole of f. Repeating this argument as often as necessary we deduce, by the above displayed statement, that we can redefine m_0 to be a non-negative integer such that $\widetilde{f^{m_0}(\gamma_0)}$ contains a pole of f and $\widetilde{f^{m_0+1}(\gamma_0)}$ does not meet $J(f)$.

Since infinitely many of the components U_n are multiply connected, we can now choose $n_1 \geq n_0 + m_0 + 1$ and take a Jordan curve γ_1 in U_{n_1} such that $\widetilde{\gamma_1}$ meets $J(f)$. By the above reasoning, there exists $m_1 \geq 0$ such that $\widetilde{f^{m_1}(\gamma_1)}$ contains a pole of f and $\widetilde{f^{m_1+1}(\gamma_1)}$ does not meet $J(f)$. Repeating this argument, we obtain sequences of non-negative integers n_k, m_k, and Jordan curves γ_k, such that, for $k \geq 0$,

$$n_{k+1} \geq n_k + m_k + 1, \tag{2.1}$$

$$\gamma_k \subset U_{n_k} \text{ and } \widetilde{\gamma_k} \text{ meets } J(f), \tag{2.2}$$

$$f^{m_k}(\gamma_k) \subset U_{n_k+m_k} \text{ and } \widetilde{f^{m_k}(\gamma_k)} \text{ contains a pole of } f, \tag{2.3}$$

$$f^{m_k+1}(\gamma_k) \subset U_{n_k+m_k+1} \text{ and } \widetilde{f^{m_k+1}(\gamma_k)} \text{ does not meet } J(f). \tag{2.4}$$

Since $f \in M_F$, we can assume by (2.3) and (2.4) that n_k and m_k have been chosen such that, for some pole p of f,

$$U_{n_k+m_k} \text{ contains a Jordan curve } \Gamma_k \text{ such that } p \in \widetilde{\Gamma_k}, \tag{2.5}$$

$$\widetilde{f(\Gamma_k)} \text{ does not meet } J(f). \tag{2.6}$$

Since U is a wandering domain, the components U_n are disjoint. Thus, for $k \geq 0$, the Jordan curves Γ_k are disjoint by (2.1) and (2.5), as are the image curves $f(\Gamma_k)$. Hence, for $0 \leq k < l < \infty$, we must have Γ_k inside Γ_l, or vice versa. Since $f \in M_F$, there must exist integers k_1 and k_2, $0 \leq k_1 < k_2 < \infty$, such that f has no poles in the closure of the ring domain A lying between Γ_{k_1} and Γ_{k_2}. Thus $f(A)$ is bounded and

$$\partial f(A) \subset f(\partial A) = f(\Gamma_{k_1}) \cup f(\Gamma_{k_2}),$$

so $f(A)$ is a subset of at least one of $\widetilde{f(\Gamma_{k_1})}$, $\widetilde{f(\Gamma_{k_2})}$. This contradicts (2.6), however, because $A \cap J(f) \neq \emptyset$ (since Γ_{k_1} and Γ_{k_2} lie in different components of $F(f)$) so $f(A) \cap J(f) \neq \emptyset$. This completes the proof of Theorem 1(a).

Proof of Theorem 1(b) Part (b) now follows from part (a) by a standard argument which we give for completeness. Suppose that

$$\operatorname{sing}(f^{-1}) \cap \bigcup_{n\geq 1} U_n = \emptyset. \tag{2.7}$$

By part (a), it is sufficient to prove that if γ is any Jordan curve in U which is not null-homotopic, then the image $\gamma_n = f^n(\gamma)$ is not null-homotopic in U_n, for $n \in \mathbb{N}$. But if $z_0 \in \gamma$ and $\gamma_n \sim f^n(z_0)$ in U_n, for some $n \geq 1$, then the branch, g say, of f^{-n} such that $g(f^n(z_0)) = z_0$ can be continued analytically (and univalently) to a simply-connected neighbourhood of γ_n in U_n, by (2.7). Then g lifts the homotopy $\gamma_n \sim f^n(z_0)$ in U_n to a homotopy $\gamma \sim z_0$ in U, which is a contradiction. This completes the proof of Theorem 1(b). □

3. Proofs of Theorems 2, 3, 4 and 5

Theorem 2 is a combination of the following two known results which together show that a meromorphic function f maps bounded components of $F(f)$ in a nice way. An analytic function defined on a domain U is called a *proper* map if f has a topological degree; see [31, pages 4–9] for a discussion of proper maps.

Lemma 4. *Let f be meromorphic and let U be a bounded domain in which f is analytic.*

(a) *Then $f : U \to f(U)$ is proper if and only if $\hat{\partial} f(U) = f(\partial U)$ or, equivalently, if and only if pre-images of relatively compact subsets of $f(U)$ are relatively compact subsets of U.*

(b) *If $f : U \to f(U)$ is proper with degree k and there are N critical points of f in U (counted according to multiplicity), then*
$$c(U) - 2 = k(c(f(U)) - 2) + N;$$
in particular, $c(U) \geq c(f(U))$.

Lemma 4(a) is proved in [31, page 5, Theorem 1] and Lemma 4(b) is the Riemann–Hurwitz formula; see [31, page 7] for the case of finite connectivity and [11, Lemma 4] for the case of infinite connectivity, in an even more general context.

Lemma 5. *Let f be meromorphic and let $f : U \to V$, where U and V are components of $F(f)$.*

(a) *Then $|V \setminus f(U)| \leq 2$ and for any $w_0 \in V \setminus f(U)$ there exists a path $\Gamma \subset U$ such that $f(z) \to w_0$ as $z \to \infty$, $z \in \Gamma$.*

(b) *If U is also bounded, then $f(U) = V$ and $f(\partial U) = \hat{\partial} V$.*

Lemma 5(a) and the first assertion of Lemma 5(b) are results of Herring [17, Theorems 1 and 2]; see also [11]. Also, if U is a bounded Fatou component, then it is well-known that $f : U \to V$ is proper; that is, $f(\partial U) = \hat{\partial} f(U) = \hat{\partial} V$.

All parts of Theorem 2 follow immediately from Lemmas 4 and 5.

Proof of Theorem 3. The proof follows that of [19, Theorem A]. Let $f \in M_F$ and suppose that U is a wandering domain. If U is not a Baker wandering domain, then by Theorem 1(a) all but a finite number of the components U_n are simply connected, so the eventual connectivity of U is 1. If U is a Baker wandering domain which is infinitely connected, then its eventual connectivity is ∞ by Corollary 1(b). If U is a Baker wandering domain which is finitely connected, then the eventual connectivity, c say, of U exists, by Corollary 1(a), and $2 \leq c < \infty$. If $c > 2$, then $f : U_n \to U_{n+1}$ is univalent, for large n, by Lemma 4(b). Moreover, for n large enough f maps the outer boundary of U_n to the outer boundary of U_{n+1}; see [13, proof of Theorem F] or [29, Lemma 4]. Thus, since $f \in M_F$, we can use the argument principle to show that f takes each value in $\mathbb{C}$ at most finitely often, and this is impossible by Picard's theorem. Hence $c = 2$, as required. □

Proof of Theorem 4. For the case when U is of finite connectivity, see [31, page 6], and also [20] for the case when in addition $U = V$.

Let α be any component of ∂U which is mapped into but not onto a component β of $\hat{\partial} V$. Choose a point $w_0 \in \beta \setminus f(\alpha)$, possibly $w_0 = \infty$. Since U is bounded and f is meromorphic, there exist only finitely many pre-images of w_0 in ∂U, say z_k, $k = 1, \ldots, p$, none of which lies in α.

Let V_n, $n = 1, 2, \ldots$, be a smooth exhaustion of V; that is, the sets V_n are smooth bounded domains such that $\overline{V_n} \subset V_{n+1}$, for $n = 1, 2, \ldots$ and $\bigcup V_n = V$. Then β lies in a unique component of the complement of V_n, for each n, so there exists a unique component, H_n say, of $V \setminus \overline{V_n}$ such that $\beta \subset \overline{H_n}$. Note that $\beta \subset \overline{H_{n+1}} \subset \overline{H_n}$, for $n = 1, 2, \ldots$, so $\bigcap \overline{H_n}$ is a connected subset of $\hat{\partial} V$ and hence $\bigcap \overline{H_n} = \beta$.

We now wish to choose, for each n, a component G_n of $U \cap f^{-1}(H_n)$ such that $\alpha \subset \overline{G_n}$. In order to do this, we construct a path $\Gamma : \gamma(t)$, $t \in [0, \infty)$, in U which approaches α in the sense that $\mathrm{dist}_\chi(\gamma(t), \alpha) \to 0$ as $t \to \infty$ and $\alpha \subset \overline{\Gamma}$, where χ denotes the spherical metric on $\hat{\mathbb{C}}$. Such a path Γ can be constructed by using a smooth exhaustion U_m of U and choosing Γ to lie eventually outside each U_m and to accumulate at each point of a dense subset of α. Then $\mathrm{dist}_\chi(f(\gamma(t)), \beta) \to 0$ as $t \to \infty$. Thus, for each $n = 1, 2, \ldots,$ we have $f(\gamma(t)) \in H_n$ for t large enough, so we can define G_n to be the component of $U \cap f^{-1}(H_n)$ such that $\gamma(t) \in G_n$ for t large enough. By the properties of H_n and the fact that $\alpha \subset \overline{\Gamma}$, we have $\alpha \subset \overline{G_{n+1}} \subset \overline{G_n}$, for $n = 1, 2, \ldots$. Thus $\bigcap \overline{G_n}$ is connected, contains α, and is a subset of ∂U (because any point in $\bigcap \overline{G_n}$ must be mapped by f to a point in β). Hence $\bigcap \overline{G_n} = \alpha$, so we can choose n such that $\overline{G_n} \cap \bigcup_{k=1}^{p} \{z_k\} = \emptyset$.

For such a choice of n, let w_m be a sequence in H_n which converges to w_0. Since $f : G_n \to H_n$ is proper, there exists a sequence z_m in G_n such that $f(z_m) = w_m$, for $m = 1, 2, \ldots$, and we may assume that $z_m \to z_0$, where $f(z_0) = w_0$. Then $z_0 \in \overline{G_n}$, a contradiction to the above choice of n. □

To prove Theorem 5, we need some ideas from the theory of cluster sets. First, for an unbounded domain U, with $z_0 \in \hat{\partial} U$, we define the cluster sets

$$C_U(f, z_0) = \{w_0 \in \hat{\mathbb{C}} : \exists\, z_n \in U \text{ with } z_n \to z_0,\ f(z_n) \to w_0\}$$

and

$$C_{\partial U}(f, \infty) = \{w_0 \in \hat{\mathbb{C}} : \exists\, z_n \in \partial U \text{ with } z_n \to \infty,\ f(z_n) \to w_0\},$$

where we assume that ∂U is unbounded.

We shall use the following result, which is a special case of the Beurling–Kunugui theorem; see [23, page 23, Theorem 7].

Lemma 6. *Let f be meromorphic and let U be an unbounded domain such that ∂U is unbounded. Suppose that the set*

$$\Omega = C_U(f, \infty) \setminus C_{\partial U}(f, \infty)$$

is non-empty and Ω' is any component of Ω. Then every value from Ω', with at most two exceptions, is assumed by f infinitely often in $U \cap \{z : |z| > R\}$, for all $R > 0$.

The set Ω defined in Lemma 6 is open (see [23, page 17, Theorem 4]) and hence Ω has at most countably many such components Ω'. In particular, in Lemma 6 the set $\Omega \setminus f(U)$ is at most countable.

In the general Beurling–Kunugui theorem, the function f is assumed to be meromorphic only in U, so f need not have a continuous extension to ∂U (as is the case here), and the cluster set $C_{\partial U}(f, \infty)$ is defined in terms of the values of $C_U(f, z)$, for $z \in \partial U$.

Proof of Theorem 5. Let f be a transcendental meromorphic function and let U be an unbounded component of $F(f)$. Then ∂U is unbounded, since $J(f)$ is unbounded, so Lemma 6 can be applied. It is a straightforward matter to check that

$$\hat{\partial} f(U) = f(\partial U) \cup (C_U(f,\infty) \setminus f(U)). \tag{3.1}$$

Thus, by Lemma 5(a),

$$\hat{\partial} V = f(\partial U) \cup (C_U(f,\infty) \setminus (f(U) \cup E)), \tag{3.2}$$

where V is the component of $F(f)$ such that $f(U) \subset V$ and $E = V \setminus f(U)$, $|E| \leq 2$. Note that $f(\partial U) \cap E = \emptyset$, since there are no isolated points of ∂U. Since $f(\partial U) \subset \hat{\partial} V$, we deduce that

$$\overline{f(\partial U)} \subset \hat{\partial} V.$$

To prove the desired statement that $\overline{f(\partial U)} = \hat{\partial} V$, we suppose that there exists $w_0 \in \hat{\partial} V \setminus \overline{f(\partial U)}$. Then there is an open disc Δ in $\hat{\mathbb{C}}$ with centre w_0 such that $\Delta \cap \overline{f(\partial U)} = \emptyset$. Since $\hat{\partial} V$ is perfect, as can easily be checked by using the fact that $J(f)$ is perfect, the disc Δ contains uncountably many points w such that $w \in \hat{\partial} V \setminus \overline{f(\partial U)}$. Therefore, by (3.2), the set

$$\hat{\partial} V \setminus \overline{f(\partial U)} = C_U(f,\infty) \setminus \left(f(U) \cup E \cup \overline{f(\partial U)}\right)$$

is uncountable. Since $|E| \leq 2$ and $C_{\partial U}(f,\infty) \subset \overline{f(\partial U)}$, the set

$$C_U(f,\infty) \setminus (f(U) \cup C_{\partial U}(f,\infty)) = \Omega \setminus f(U)$$

is also uncountable, which contradicts the statement following Lemma 6. This completes the proof of Theorem 5(a).

The proof of part (b) is clear since $\hat{\partial} V = \overline{f(\partial U)}$, by part (a), and $f(\partial U)$ can have at most N components.

To prove part (c), we suppose that $c(V) > c(U)$. Then U must have a finite number of bounded boundary components, $\alpha_1, \ldots, \alpha_m$ say, and there must exist at least one bounded boundary component, β_0 say, of V which does not contain any of $f(\alpha_1), \ldots, f(\alpha_m)$. Let $\beta_1, \ldots, \beta_n$ denote those bounded boundary components of V which contain at least one of the sets $f(\alpha_1), \ldots, f(\alpha_m)$; clearly $n \leq m$.

Now suppose that β_0 is not the outer boundary of V. Let Γ be a Jordan curve in V which separates β_0 from $\beta_1 \cup \cdots \cup \beta_n$, such that β_0 lies in the bounded complementary component, G say, of Γ. This is possible by repeated applications of the result [22, page 143, Theorem 3.3] to the closed set $\hat{\mathbb{C}} \setminus V$. By part (a), we have $f(\partial U) \cap G \neq \emptyset$. However, $f(\partial U) \cap \Gamma = \emptyset$, since $f(\partial U) \subset J(f)$. Thus if we choose $z_0 \in \partial U$ such that $f(z_0) \in G$, then the component E_0 of ∂U which contains z_0 is unbounded but its image lies entirely inside Γ and so is bounded, as required.

In the case when β_0 is the outer boundary of V (which can only occur when V is bounded), a similar argument applies, except that in this case β_0 lies in the unbounded complementary component of Γ and the image of E_0 is bounded because it lies in $\overline{V}$. This completes the proof of Theorem 5. □

4. Examples

Our first example shows that Theorem 1(a) is false without the hypothesis that $f \in M_F$.

Example 1. *There exists a meromorphic function f with infinitely many poles and a wandering domain U such that each component U_n, $n = 0, 1, 2, \ldots,$ is bounded and infinitely connected, but U is not a Baker wandering domain.*

Proof. The construction of Example 1 is based on the entire function

$$h(z) = 2 + 2z - 2e^z,$$

which is derived from Bergweiler's example $z \mapsto 2 - \ln 2 + 2z - e^z$ in [10] by shifting the super-attracting fixed point from $\ln 2$ to 0. Here we consider the closely related meromorphic function

$$f(z) = 2 + 2z - 2e^z + \frac{\varepsilon}{e^z - e^a},$$

where a and ε are positive constants to be chosen suitably small. Note that

$$\phi(z) = f(z) - 2z$$

is $2\pi i$-periodic.

First we claim that if $0 < a < 1/32$ and $0 < \varepsilon \le a^2/16$, then the set

$$\Delta_a = \{z : |z| \le 2a, |z - a| \ge a/2\}$$

is mapped by f into $\{z : |z| < a/2\} \subset \Delta_a$. For $|z| \le 1$ we have

$$|2 + 2z - 2e^z| = |z^2 + z^3/3 + \cdots| \le |z|^2(1 + |z|/3 + |z|^2/3^2 + \cdots) < 2|z|^2. \tag{4.1}$$

Similarly, $|e^z - 1| \ge \frac{1}{2}|z|$, for $|z| \le \frac{1}{2}$, so

$$\left|\frac{\varepsilon}{e^z - e^a}\right| = \frac{\varepsilon}{e^a|e^{z-a} - 1|} \le \frac{4\varepsilon}{a} \le \frac{a}{4}, \quad \text{for } a/2 \le |z - a| \le 1/2. \tag{4.2}$$

The estimates (4.1) and (4.2) give

$$|f(z)| < 8a^2 + \frac{a}{4} < \frac{a}{2}, \quad \text{for } z \in \Delta_a,$$

since $0 < a < 1/32$. Therefore $f(\Delta_a) \subset \{z : |z| < a/2\} \subset \Delta_a$, as required.

Thus f has a fixed point, z_0 say, in the interior of Δ_a, which must be attracting. The corresponding immediate attracting basin U_0 of f contains Δ_a but not the point a, where f has a pole, so U_0 is multiply connected. Hence U_0 must be infinitely connected by [8, Theorem 3.1].

It is shown in [18, proof of Theorem 4] that the immediate super-attracting basin of h which contains the super-attracting fixed point 0 is bounded. This is done by specifying a Jordan curve Γ which winds round 0 (and is contained in $\{z : |\Im(z)| < \pi\}$), such that $h(\Gamma)$ lies in the unbounded component of the complement of Γ. This property remains true for $f(\Gamma)$ as long as we choose ε small enough and hence U_0 is bounded.

Since $f(z) = 2z + \phi(z)$, where ϕ is $2\pi i$-periodic, the set $J(f)$ is $2\pi i$-periodic; see [28, Corollary 1], for example. Thus, for each $n \in \mathbb{Z}$, the set $U_n = U_0 + 2n\pi i$ is a bounded infinitely connected component of $F(f)$. Now, for $n \in \mathbb{Z}$, we have

$$2n\pi i \in \Delta_a + 2n\pi i \subset U_n, \quad f(2n\pi i) = 4n\pi i + \frac{\varepsilon}{1 - e^a} \quad \text{and} \quad \left|\frac{\varepsilon}{1 - e^a}\right| \le \frac{a^2/16}{a} < \frac{a}{2},$$

so $f(U_n) \subset U_{2n}$, for $n \in \mathbb{Z}$. Thus U_1 is a bounded infinitely connected wandering domain which is not a Baker wandering domain, as required. □

Note that in this example the Fatou components which contain $f^n(U_1)$ are all infinitely connected, as expected by Corollary 1(b).

A similar construction to Example 1 can be carried out starting with

$$h(z) = z - 1 + e^{-z} + 2\pi i.$$

The function $z \mapsto z - 1 + e^{-z}$ has congruent super-attracting basins containing the super-attracting fixed points $2n\pi i$, $n \in \mathbb{Z}$, and it was shown by Herman that these components form an orbit of wandering domains of h; see [16]. In this case, the construction in Example 1 gives a meromorphic function with an orbit of unbounded infinitely connected wandering domains. We omit the details.

Our next example shows that there does exist a meromorphic function with a multiply connected wandering domain U such that U_n is simply connected for $n \geq 1$.

Example 2. *There exists a function $f \in M_F$ with a bounded doubly connected wandering domain U such that each component U_n, $n = 1, 2, \ldots,$ is bounded and simply connected.*

Proof. The construction of Example 2 is based on the entire function

$$g(z) = z + \lambda \sin(z + a),$$

where $\lambda > 0$ and $a \in \mathbb{R}$ are chosen so that $g(2n\pi) = (2n + 2)\pi$, $n \in \mathbb{Z}$, and g has critical points at each $2n\pi$, $n \in \mathbb{Z}$. Thus

$$\lambda \sin a = 2\pi, \quad 1 + \lambda \cos a = 0, \tag{4.3}$$

so $a = \pi - \tan^{-1}(2\pi) = 1.728\ldots$ and $\lambda = \sqrt{1 + 4\pi^2} = 6.362\ldots$. Devaney showed in [12] that g has a wandering domain containing 0. Here we consider the closely related function

$$f(z) = g(z) + \frac{\varepsilon}{z} = z + \frac{\varepsilon}{z} + \lambda \sin(z + a),$$

where ε is a positive constant to be chosen suitably small. In particular, we require that $0 < \varepsilon < 1/2$, which implies by a calculation that

$$f(\pi/2 - a) = \pi/2 - a + \frac{\varepsilon}{\pi/2 - a} + \lambda > 0,$$

so f has a zero in the interval $(\pi/2 - a, 0)$. Thus $f \in M$, since 0 is a pole of f.

We write $B(z, r) = \{w : |w - z| < r\}$, $r > 0$. Since g has critical points at $2n\pi$, $n \in \mathbb{Z}$, and $g(z + 2\pi) = g(z) + 2\pi$, we can choose a constant r_1 such that $0 < r_1 < 1/2$ and

$$|g'(z)| \leq \frac{1}{4}, \quad \text{for } |z - 2n\pi| \leq r_1,\ n \in \mathbb{Z}. \tag{4.4}$$

Hence

$$g(B(2n\pi, r)) \subset B((2n + 2)\pi, r/4), \quad \text{for } 0 < r \leq r_1,\ n \in \mathbb{Z}.$$

(See (4.8) for a more precise estimate of the behaviour of g near 0.) Therefore, we can choose $\varepsilon > 0$ and r_2, $0 < r_2 < r_1$, such that $6\sqrt{\varepsilon} < r_1$ and

$$\overline{f(B(2n\pi, r_1))} \subset B((2n + 2)\pi, r_2), \quad \text{for } n \geq 1. \tag{4.5}$$

In particular, note that $0 < \varepsilon < (r_1/6)^2 < 1/144$.

Now let

$$\Delta_0 = \{z : \sqrt{\varepsilon}/2 < |z| < 2\sqrt{\varepsilon}\} \quad \text{and} \quad \Delta_n = B(2n\pi, r_1),\ n \geq 1.$$

The function $z \mapsto z + \varepsilon/z$ is a Joukowski function which maps Δ_0 in a 2-to-1 manner onto an ellipse contained in $B(0, 3\sqrt{\varepsilon}\,)$. Also, by (4.4) with $n = 0$, we have

$$\begin{aligned}|\lambda \sin(z + a) - 2\pi| &= |g(z) - 2\pi - z| \\ &\leq |g(z) - 2\pi| + |z| \\ &\leq \tfrac{1}{2}\sqrt{\varepsilon} + 2\sqrt{\varepsilon} < 3\sqrt{\varepsilon}, \quad \text{for } z \in \Delta_0.\end{aligned}$$

Hence

$$f(\Delta_0) \subset B(2\pi, 3\sqrt{\varepsilon} + 3\sqrt{\varepsilon}\,) \subset B(2\pi, r_1). \tag{4.6}$$

Therefore, by (4.5) and (4.6),

$$f^n(\Delta_m) \subset \Delta_{m+n}, \quad \text{for } m, n \geq 0, \tag{4.7}$$

so

$$\Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \cdots \subset F(f),$$

by Montel's theorem. For $n \geq 0$, let U_n be the component of $F(f)$ which contains Δ_n. Clearly U_0 is multiply connected, since $0 \in J(f)$, and $f^n \to \infty$ locally uniformly in each U_n, $n \geq 0$, by (4.7). Hence U_0 is not a Herman ring (nor its pre-image). Also note that $J(f)$ is symmetric with respect to the real axis and each interval of the form $[(2n+1)\pi, (2n+2)\pi]$, $n \geq 0$, contains a repelling fixed point of f, since $0 < \varepsilon < 1/144$.

We now show that the components U_n, $n \geq 0$, are all different. Suppose, for a contradiction, that $U_p = U_q$, where $0 \leq p < q$. Then there is a Jordan curve γ in U_p, which is symmetric with respect to the real axis and passes through Δ_p and Δ_q. Hence $f^n(\gamma)$, $n \geq 0$, is a closed curve in $F(f)$, symmetric with respect to the real axis, which passes through Δ_{p+n} and Δ_{q+n}. It follows that, for $n \geq 0$, the set $\widetilde{f^n(\gamma)}$ contains the repelling fixed point of f located in the interval $[(2(p+n)+1)\pi, (2(p+n)+2)\pi]$. Thus U_0 is a Baker wandering domain, by Lemma 3. Therefore

$$\frac{\ln \ln |f^n(z)|}{n} \to \infty, \quad \text{for } z \in U_0,$$

by [29, Theorem 1(d)], and this contradicts the fact that $f^n(\Delta_0) \subset \Delta_n$, for $n \geq 0$. Hence the components U_n are indeed different and so U_0 is a wandering domain but not a Baker wandering domain.

We now show that the components U_n are all bounded. For $n \geq 0$, put

$$C_n = \{z : |z - 2n\pi| = 0.5\} \quad \text{and} \quad C'_n = \{z : |z - 2n\pi| = 0.6\}.$$

Lemma 7. *We can choose $\varepsilon > 0$ so small that, for $n \geq 0$, we have*

(a) *$f(C_n)$ winds twice positively round C'_{n+1};*
(b) *$f'(C_n)$ winds once positively round $\{z : |z| = 1\}$;*
(c) *U_n lies inside C_n.*

Proof. Recall that $g(z) = z + \lambda \sin(z + a)$ and $f(z) = g(z) + \varepsilon/z$. In view of (4.3), we have

$$g(z) = z - \sin z + 2\pi \cos z = 2\pi - \pi z^2 \left(1 - \frac{z}{3!\pi} - \frac{2z^2}{4!} + \cdots\right). \tag{4.8}$$

Part (a) now follows immediately from the estimate

$$\left| -\frac{z}{3!\pi} - \frac{2z^2}{4!} + \cdots \right| < 0.1, \quad \text{for } |z| \leq 0.5, \tag{4.9}$$

and the facts that $g(z + 2\pi) = g(z) + 2\pi$ and $0 < \varepsilon < 1/144$. Part (b) follows by a similar argument with

$$g'(z) = -2\pi z \left(1 - \frac{z}{2!2\pi} - \frac{z^2}{3!} + \cdots\right).$$

To prove part (c), we first show that, for each $N \geq 0$, the family

$$\phi_n(z) = f^n(z) - 2(n + N)\pi, \quad n \geq 0,$$

is normal in U_N. This holds because the components U_n, $n \geq 0$, are disjoint, so $f^n(z) \neq 2m\pi$, for $m > n + N$, $z \in U_N$, and hence each function ϕ_n omits in U_N the three values

$$\infty, \quad 2(n + 1 + N)\pi - 2(n + N)\pi = 2\pi \quad \text{and} \quad 2(n + 2 + N)\pi - 2(n + N)\pi = 4\pi.$$

Using (4.4) and making a smaller choice of ε if necessary, we deduce that

$$|f'(z)| \leq c, \quad \text{for } |z - 2n\pi| \leq r_1,\ n \geq 1,$$

for some c, $0 < c < 1$. Thus f is contracting on each disc Δ_n, $n \geq 1$. By (4.7), for each $N \geq 0$, we have $\operatorname{diam} f^n(\Delta_N) \to 0$ as $n \to \infty$, so there exists a_N with $|a_N| \leq r_1 < 1/2$ and a subsequence n_k such that

$$\phi_{n_k}(z) \to a_N \ \text{ as } k \to \infty, \quad \text{locally uniformly in } U_N. \tag{4.10}$$

Now suppose for a contradiction that $U_N \cap C_N \neq \emptyset$, for some $N \geq 0$. Then we can join a point z_N of Δ_N to a point $w_N \in C_N$ by a compact curve Γ lying in U_N. Since $f^n(z_N) \in \Delta_{n+N}$ for all $n > 0$, we deduce that $f^n(\Gamma)$ meets C_{n+N} and C'_{n+N} for all $n > 0$. This contradicts (4.10) and completes the proof of Lemma 7. □

We now continue the proof of Example 2. Since the components U_n are all bounded, we deduce that $U_n = f^n(U_0)$, $n \geq 0$, by Lemma 5(b).

We can now deduce that the components U_n, $n \geq 1$, are all simply connected. Indeed, if $N \geq 1$ and γ_N is a Jordan curve in U_N which is not null-homotopic in U_N, then for some $n \geq 0$ the set $\widetilde{f^n(\gamma_N)}$ must contain a pole of f, by Lemma 2, and this is impossible by Lemma 7(c).

Finally, we show that U_0 is doubly connected. To do this we use the Riemann–Hurwitz formula

$$c(U_0) - 2 = k_0(c(U_1) - 2) + N_0, \tag{4.11}$$

where k_0 is the degree of the (proper) mapping $f : U_0 \to U_1$ and N_0 is the number of critical points of f in U_0; see Lemma 4(b).

By Lemma 7(a), with $n = 0$, and the argument principle, the set $\{z \in \operatorname{int} C_0 : f(z) = 2\pi\}$ contains three points, counted according to multiplicity. By (4.8) and (4.9), and the fact that $f(z) = g(z) + \varepsilon/z$, these three points are close to $re^{2\pi ik/3}$, $k = 0, 1, 2$, where $r = \sqrt[3]{\varepsilon/\pi}$. Each of these three pre-images of 2π must lie in U_0, since

$$f(\Delta_0 \cup \Delta_0') \subset B(2\pi, 6\sqrt{\varepsilon}) \subset \Delta_1 \subset U_1, \quad \text{where } \Delta_0' = \{z : 2\sqrt{\varepsilon} \leq |z| \leq \sqrt[3]{\varepsilon}\},$$

as can easily be checked using (4.6), (4.8) and (4.9). Note that $\sqrt[3]{\varepsilon} > 2\sqrt{\varepsilon}$, since $0 < \varepsilon < 1/144$. Hence $k_0 = 3$, by Lemma 7(c). By Lemma 7(b), with $n = 0$, and the argument principle, the set $\{z \in \operatorname{int} C_0 : f'(z) = 0\}$ contains three points, counted according to multiplicity, so $N_0 \leq 3$. Also, $c(U_1) = 1$, so

$$c(U_0) = 2 + 3(-1) + N_0 \leq 2,$$

by (4.11). Since U_0 is multiply connected, we deduce that $c(U_0) = 2$, as required. □

Our next example shows that Theorem 2(b) is false for an unbounded Fatou component, even for $f \in M_F$. Here we use the approximation technique introduced by Eremenko and Lyubich [14].

Example 3. *There exists a function $f \in M_F$ with a bounded simply connected wandering domain U such that*

(a) *$f(U)$ is an unbounded simply connected component of $F(f)$ and $\partial f(U)$ consists of two unbounded components;*
(b) *$f^2(U)$ is a bounded doubly connected component of $F(f)$;*
(c) *$f^n(U)$, $n \geq 3$, are bounded simply connected components of $F(f)$.*

Thus $U_1 = f(U)$ is unbounded and $c(U_1) = 1 < 2 = c(U_2)$.

Proof. Throughout this construction the parameters λ, a and ε are the same as in Example 2, as are the sets Δ_n, $n \geq 0$. In particular, $0 < \varepsilon < 1/144$. We then define

$$g_1(z) = z + \lambda \sin(z + a), \quad g_2(z) = 4e^z - \varepsilon/z \quad \text{and} \quad g_3(z) = 0.$$

Note that g_1 is the function called g in Example 2. Also, let

$$E_1 = \{z : \Re(z) \geq -0.6\}, \quad E_2 = \{z : \Re(z) \leq -1.4\} \quad \text{and} \quad E_3 = \{z : |z + 1| \leq 0.2\}.$$

It follows from Arakelyan's theorem [15] that, for any $\delta > 0$, there exists a transcendental entire function g such that

$$|g(z) - g_k(z)| < \delta/2, \quad \text{for } z \in E_k,\ k = 1, 2, 3, \tag{4.12}$$

and g is symmetric with respect to the real axis. The following lemma then completes the proof of Example 4. □

Lemma 8. *We can choose $\delta > 0$ such that if g is constructed as above, then the transcendental meromorphic function*

$$f(z) = g(z) + \frac{\varepsilon}{z} + \frac{\delta/5}{z+1} \tag{4.13}$$

has the following properties.

(a) *$F(f)$ has a sequence of components V_n, $n \geq 0$, with similar properties to the components U_n in Example 2 (and Lemma 7); in particular, V_0 is doubly connected, V_n, $n \geq 1$, are simply connected, and*
$$\Delta_n \subset V_n \subset \{z : |z - 2n\pi| < 0.5\}, \quad \text{for } n \geq 0.$$

(b) *$F(f)$ has an unbounded simply connected component U' whose boundary $\partial U'$ consists of two unbounded components, such that $f(U') = V_0$.*

(c) *$F(f)$ has a bounded simply connected component U such that $f(U) = U'$.*

Proof. Let $f_1(z) = g_1(z) + \varepsilon/z$, so f_1 is the function called f in Example 2. The proof of Example 2 depended on several properties of f_1. Part (a) of Lemma 8 will follow if we show that these properties are also true for the function f in this example.

First, f_1 is symmetric in the real axis and belongs to $M_F \cap M$, properties which are also true for the function f defined by (4.13).

Next, the proof of Example 2 depended on a finite number of statements, such as (4.5) and Lemma 7, all involving values of z in E_1 and various small positive constants such as r_1, which are true for the function f_1 and which remain true for the function f if we choose $\delta > 0$ small enough; for example, we have

$$|f(z) - f_1(z)| = \left| g(z) - g_1(z) + \frac{\delta/5}{z+1} \right| < \delta, \quad \text{for } z \in E_1,$$

so (4.5) is true for f if $\delta > 0$ is small enough, and

$$|f'(z) - f_1'(z)| \leq 10\delta, \quad \text{for } \Re(z) \geq -0.5,$$

by Cauchy's estimate. Thus the statement (4.10) in the proof of Lemma 7 is also true for f if $\delta > 0$ is small enough.

To prove part (b), we show that a certain component U' of the pre-image of V_0 under f is an unbounded simply connected component of $F(f)$. First, recall that

$$\Delta_0 = \{z : \sqrt{\varepsilon}/2 < |z| < 2\sqrt{\varepsilon}\}.$$

It follows from (4.12) and (4.13) that if $\delta > 0$ is small enough, then there exists $\rho > 0$, depending on ε but not on δ, such that V_0 surrounds $\{z : |z| \leq \rho\}$. In particular, $\rho \leq \sqrt{\varepsilon}/2$. Then we take C such that $8e^{-C} < \rho$, put

$$S = \{z : -C < \Re(z) < -2\},$$

and further require that $0 < \delta < 2e^{-C}$.

Let $\phi(z) = f(z) - 4e^z$. Then, by (4.12) and (4.13), we have

$$|\phi(z)| = \left| g(z) - g_2(z) + \frac{\delta/5}{z+1} \right| < \delta, \quad \text{for } z \in E_2,$$

and hence

$$|\phi'(z)| < \frac{\delta}{0.6} < 2\delta, \quad \text{for } z \in S,$$

by Cauchy's estimate. Now,

$$|f(z)| \geq |4e^z| - |\phi(z)| > 4e^{-C} - \delta > 2e^{-C}, \quad \text{for } z \in S,$$

so any path in S which tends to ∞ is mapped by f to a path which winds infinitely often round $\{z : |z| \le 2e^{-C}\}$. Hence f has no finite asymptotic values in S. Also, since $0 < \delta < 2e^{-C} < \rho/4 \le \sqrt{\varepsilon}/8 < 1/96$, we have

$$|f(z)| > 4e^{-2} - \delta > 0.5, \quad \text{for } \Re(z) = -2,$$
$$0 < 4e^{-C} - \delta < |f(z)| < 4e^{-C} + \delta < \rho, \quad \text{for } \Re(z) = -C,$$

and

$$|f'(z)| = |4e^z + \phi'(z)| \ge 4e^{-C} - 2\delta > 0, \quad \text{for } z \in S.$$

It follows that $f : S \to f(S)$ is a covering map and $\partial f(S)$ lies outside V_0, by part (a). Also, since $0 < \delta < \sqrt{\varepsilon}/8$, the vertical line $\{z : \Re(z) = \ln(\sqrt{\varepsilon}/4)\}$ in S is mapped by f to a path in $\Delta_0 \subset V_0$, which winds infinitely often round 0. Thus $f^{-1}(V_0)$ has a component U' which is an unbounded simply connected domain contained in S, bounded by two unbounded continua in S which are components of the pre-images under f of the inner and outer components of ∂V_0. Thus U' is a Fatou component of f and $f(U') = V_0$, by Lemma 5(a).

Now we show that f is univalent on the punctured disc $D = \{z : 0 < |z+1| < \sqrt{\delta}/2\}$, which is contained in $E_3 = \{z : |z+1| \le 0.2\}$. Put $h(z) = g(z) + \varepsilon/z$. Then, by (4.12) and (4.13),

$$|h(z)| \le \frac{\delta}{2} + \frac{\varepsilon}{0.8} < \frac{1}{50}, \quad \text{for } z \in E_3,$$

since $0 < \varepsilon < 1/144$ and $0 < \delta < 1/96$. Thus, by Cauchy's estimate,

$$|h'(z)| \le \frac{1}{50\,(0.2 - \sqrt{\delta}/2)} < 1/5, \quad \text{for } z \in \overline{D}.$$

Now suppose that $f(z_1) = f(z_2)$, where $z_1, z_2 \in D$. Then

$$\left| \frac{\delta/5}{z_1+1} - \frac{\delta/5}{z_2+1} \right| = |h(z_1) - h(z_2)| \le \frac{1}{5}|z_1 - z_2|,$$

so $\delta \le |z_1+1|\,|z_2+1| \le (\sqrt{\delta}/2)^2$, which is false. Hence f is one-one on D.

Also, for $z \in \partial D \setminus \{-1\}$, we have

$$|f(z)| = \left| h(z) + \frac{\delta/5}{z+1} \right| \le |h(z)| + \frac{\delta/5}{|z+1|} \le \frac{\delta}{2} + \frac{\varepsilon}{0.8} + \frac{2\sqrt{\delta}}{5} \le \sqrt{\varepsilon},$$

provided that we also have $0 < \delta < \varepsilon$. For such δ, the function f maps D univalently onto a domain which contains $\{z : |z| > \sqrt{\varepsilon}\}$ and hence contains the component U', since $\{z : |z| = \sqrt{\varepsilon}\} \subset V_0$. Therefore $f^{-1}(U')$ has a bounded simply connected component U in D, which is a component of $F(f)$ such that $f(U) = U'$ and $-1 \in \overline{U}$. This completes the proof of Lemma 8. □

Our next example shows that Theorem 2(c) is also false for an unbounded Fatou component, even for $f \in M_F$.

Example 4. *There exists a function $f \in M_F$ with a bounded infinitely connected wandering domain U such that*

(a) *$f(U)$ is an unbounded infinitely connected component of $F(f)$;*
(b) *$f^2(U)$ is contained in a bounded doubly connected component of $F(f)$;*
(c) *$f^n(U)$, $n \ge 3$, are contained in bounded simply connected components of $F(f)$.*

Thus $U_1 = f(U)$ is unbounded and infinitely connected, and the eventual connectivity of U_1 is 1.

Proof. The proof is similar to that of Example 3, but we replace the function g_2 used in that proof by

$$g_2(z) = e^z - \sqrt{\varepsilon} - \frac{\varepsilon}{z},$$

and then define g and f, as before, to be symmetric in the real axis and satisfy (4.12) and (4.13). Recall that $\Delta_0 = \{z : \sqrt{\varepsilon}/2 < |z| < 2\sqrt{\varepsilon}\}$, so $-\sqrt{\varepsilon} \in \Delta_0$, and also that $0 < \varepsilon < 1/144$.

As in Lemma 8(a), we can take $\delta > 0$ so small in (4.12) and (4.13) that $F(f)$ has a sequence of components V_n, $n \geq 0$, with similar properties to the components U_n in Example 2 (and Lemma 7); in particular, V_0 is doubly connected, V_n, $n \geq 1$, are simply connected, and

$$\Delta_n \subset V_n \subset \{z : |z - 2n\pi| < 0.5\}, \quad \text{for } n \geq 0. \tag{4.14}$$

Now, we introduce the connected compact set

$$K = \{z : |z| = 3\sqrt{\varepsilon}/2\} \cup [-3\sqrt{\varepsilon}/2, -5\sqrt{\varepsilon}/4] \cup \{z : |z + \sqrt{\varepsilon}| = \sqrt{\varepsilon}/4\},$$

which is a subset of Δ_0, and put

$$L = \exp^{-1}(K + \sqrt{\varepsilon}).$$

Then L is an unbounded 'vertical ladder' (the left edge straight and the right edge wavy), which has infinitely many horizontal rungs and is invariant under translation by $2\pi i$. We have $L \subset E_2$, since $\ln(5\sqrt{\varepsilon}/2) < -1.4$. By (4.12) and (4.13), we have

$$|f(z) - e^z + \sqrt{\varepsilon}| = \left| g(z) - g_2(z) + \frac{\delta/5}{z+1} \right| < \delta, \quad \text{for } z \in E_2, \tag{4.15}$$

so

$$f(z) \in \Delta_0 \subset V_0, \quad \text{for } z \in L, \tag{4.16}$$

provided that $0 < \delta < \sqrt{\varepsilon}/4$. Thus the set L must lie in an unbounded component U' of $F(f)$ such that $f(U') \subset V_0$. Now, the inner boundary component, α_0 say, of the doubly connected component V_0 is surrounded by Δ_0. Thus (4.15) and (4.16) imply that the image under f of the boundary of each hole of the ladder L must wind once round α_0. Hence, by the argument principle, each of the holes of L must contain a pre-image of α_0 under f, so the component U' is infinitely connected.

To complete the proof, we again use the fact that, for small enough $\delta > 0$, the function f maps the punctured disc $D = \{z : 0 < |z+1| < \sqrt{\delta}/2\}$ univalently onto a domain which contains $\{z : |z| > \sqrt{\varepsilon}\}$. □

Our final example shows that Theorem 4 is false if U is unbounded. See [24, Theorem 1] and [5, Theorem 6.1] for related properties of the Julia set of this function.

Example 5. *The function $f(z) = ze^z$ has an unbounded immediate parabolic basin U whose boundary ∂U has components α and α' such that $f(\alpha) = \alpha' \setminus \{0\}$.*

Proof. The function f has a parabolic fixed point at 0, with an associated immediate parabolic basin U that contains $(-\infty, 0)$. The only singular values of f are the finite asymptotic value 0 and the critical value $f(-1) = -1/e$.

Let $\Omega = \{z : \Re(z) \leq 0, |\Im(z)| \leq \pi/2\}$ and let $\Gamma^{\pm}$ be the parts of $\partial\Omega$ in the upper and lower open half-planes. Simple estimates show that

$$f(\Omega \setminus \{0\}) \subset \operatorname{int} \Omega,$$

so $\Omega \setminus \{0\} \subset U$. Then take $G = \mathbb{C} \setminus \Omega$. Let g be the branch of f^{-1} such that $g(0) = 0$, defined on a neighbourhood of 0, and analytically continue g to $\mathbb{C} \setminus (-\infty, 0]$ by using the monodromy theorem. Then $g(G) \supset (0, \infty)$, but

$$g(G) \cap \partial G = \emptyset, \quad \text{since} \quad f(\partial G \setminus \{0\}) \subset \Omega.$$

Thus $g(G) \subset G$, so $\overline{g^n(G)}$, $n = 1, 2, \ldots$, forms a decreasing sequence of continua in $\hat{\mathbb{C}}$ with intersection Δ, say, containing $[0, \infty)$. Then $\Delta \setminus \{\infty\}$ is completely invariant under g.

Now let $S = \{z : \Re(z) \geq 0, |\Im(z)| \leq \pi\}$ and $H = \{z : \Im(z) > 0\}$. By considering the effect of f on each of the half-lines

$$\{x + iy : x \geq 0\}, \quad 0 \leq y \leq \pi,$$

we see that f maps the interior of $S \cap H$ univalently onto a simply connected domain which contains $G \cap H$. Thus $g(G) \subset S$ and hence $\Delta \setminus \{\infty\} \subset S$. We can then deduce that $\Delta \setminus \{\infty\} = [0, \infty)$ by considering a point of Δ with maximal argument, and using the fact that $\arg f(z) = \arg z + y$, for $z \in S$.

We have $(0, \frac{1}{2}\pi i) \subset U \cap \partial(S \cap H)$ and $f((0, \frac{1}{2}\pi i)) \subset \operatorname{int}\Omega \cap H \subset U \cap H$. Thus $g(\operatorname{int}\Omega) \cap \operatorname{int}\Omega \neq \emptyset$, so both $g(\operatorname{int}\Omega)$ and $g(\Gamma^+)$ are subsets of U, and the same therefore holds for $g^n(\Gamma^+)$, for all $n \geq 0$. Since $[0, \infty)$ does not meet U and the curves $g^n(\Gamma^+)$ tend to $[0, \infty)$, we deduce that $\alpha' = [0, \infty)$ is contained in ∂U and moreover forms a component of ∂U.

Next let h denote the branch of f^{-1} which maps the interval $[-1/e, 0)$ to $(-\infty, -1]$. We can analytically continue h to H, and from H across the three intervals of $\mathbb{R} \setminus \{0, -1/e\}$. Therefore the image of H under h is a domain bounded by three curves

$$h((-\infty, -1/e)), \quad h([-1/e, 0)) = (-\infty, -1], \quad h((0, \infty)),$$

each of which is a solution curve of the equation $\Im(ze^z) = 0$. In particular, the curve $\alpha = h((0, \infty))$ is a complete branch of the graph $x = -y \cot y$.

Now $\alpha \subset J(f)$, since $(0, \infty) \subset J(f)$. Also, $h(U \cap H) \subset U$, so $\alpha \subset \partial U$, since $(0, \infty) \subset \partial U$. Moreover α is a component of ∂U since it is a maximal connected subset of $f^{-1}([0, \infty))$. However, $f(\alpha) = (0, \infty) = \alpha' \setminus \{0\}$ is not a component of ∂U, so the proof is complete. □

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The Open University, Department of Mathematics, Walton Hall, Milton Keynes MK7 6AA, UK
E-mail address: p.j.rippon@open.ac.uk

The Open University, Department of Mathematics, Walton Hall, Milton Keynes MK7 6AA, UK
E-mail address: g.m.stallard@open.ac.uk