A SHARP GROWTH CONDITION FOR A FAST ESCAPING SPIDER'S WEB

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Abstract. We show that the fast escaping set $A(f)$ of a transcendental entire function f has a structure known as a spider's web whenever the maximum modulus of f grows below a certain rate. We give examples of entire functions for which the fast escaping set is not a spider's web which show that this growth rate is best possible. By our earlier results, these are the first examples for which the escaping set has a spider's web structure but the fast escaping set does not. These results give new insight into a conjecture of Baker and a conjecture of Eremenko.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [2].

In recent years, the escaping set defined by

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

has come to play an increasingly significant role in the study of the iteration of transcendental entire functions with much of the research being motivated by a conjecture of Eremenko [5] that all the components of the escaping set are unbounded. For partial results on this conjecture see, for example, [9] and [16].

The most general result on Eremenko's conjecture was obtained in [10] where it was proved that the escaping set always has at least one unbounded component. This result was proved by considering the fast escaping set $A(f) = \bigcup_{n\in\mathbb{N}} f^{-n}(A_R(f))$, where

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Here

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|,$$

$M^n(r, f)$ denotes the nth iterate of M with respect to r, and $R > 0$ is chosen so that $M(r, f) > r$ for $r \geq R$. The set $A(f)$ has many nice properties including the fact that all its components are unbounded – these properties are described in detail in [12].

There are many classes of transcendental entire functions for which the fast escaping set has the structure of a spider's web – see [12], [8] and [17]. We say

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that a set E has this structure if E is connected and there exists a sequence of bounded simply connected domains G_n such that

$$\partial G_n \subset E,\ G_n \subset G_{n+1},\ \text{for } n \in \mathbb{N},\ \text{and } \bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}.$$

As shown in [12], if $A_R(f)$ has this structure then so do both $A(f)$ and $I(f)$, and hence Eremenko's conjecture is satisfied. Also, the domains G_n can be chosen so that $\partial G_n \subset A_R(f) \cap J(f)$ and so f has no unbounded Fatou components. This gives a surprising link between Eremenko's conjecture and a conjecture of Baker that all the components of the Fatou set are bounded if f is a transcendental entire function of order less than $1/2$. Recall that the *order* of a transcendental entire function f is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

For background and recent results on Baker's conjecture, see [6], [7], [11] and [13]. It was shown in [11] (see also [12]) that all earlier partial results on Baker's conjecture are in fact sufficient to imply the stronger result that $A_R(f)$ is a spider's web. Here we give a sharp condition on the growth of the maximum modulus that is sufficient to imply that $A_R(f)$ is a spider's web and hence that Baker's conjecture and Eremenko's conjecture are both satisfied. More precisely, we prove the following sufficient condition.

Theorem 1.1. *Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Let*

$$R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n\le r\le R_{n+1}} \frac{\log\log M(r)}{\log r}.$$

If

$$\sum_{n\in\mathbb{N}} \varepsilon_n < \infty,$$

then $A_R(f)$ is a spider's web.

We obtained a closely related result in [11, Theorem 3] with the stronger hypothesis that $\sum_{n\in\mathbb{N}} \sqrt{\varepsilon_n} < \infty$ and remarked there that the square root could be removed by introducing a more sophisticated argument. The method of proof given here is quite different, and more enlightening, than that used to prove [11, Theorem 3]. In fact, Theorem 1.1 follows surprisingly easily from a new local version of the classical $\cos\pi\rho$ theorem; see Theorem 2.1.

Remark. Theorem 1.1, can be generalised to apply to the set of points that escape as fast as possible within a direct tract of a transcendental meromorphic function; see [3] for earlier results concerning the fast escaping set in a direct tract.

It turns out that the condition in Theorem 1.1 is, in a strong sense, best possible. In particular, the following result shows that the condition in Theorem 1.1 cannot be replaced by the weaker condition that $\sum_{n\in\mathbb{N}}(\varepsilon_n)^c < \infty$, for some $c > 1$.

Theorem 1.2. *There exist transcendental entire functions of the form*

$$f(z) = z^3 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{2p_n}, \tag{1.1}$$

where $p_n \in \mathbb{N}$, *for* $n \in \mathbb{N}$, *and the sequence* (a_n) *is positive and strictly increasing such that* $A(f) \cap (-\infty, 0] = \emptyset$; *in particular,* $A(f)$ *is not a spider's web.*

Moreover, if (δ_n) *is a positive sequence such that*

$$\sum_{n\in\mathbb{N}} \delta_n = \infty,$$

then we can choose the sequence $(a_n)_{n\in\mathbb{N}}$ *and a value* $R > 0$ *in such a way that, with*

$$p_n = [a_n^{\delta_n/4}/4], \quad R_n = M^n(R) \quad \textit{and} \quad \varepsilon_n = \max_{R_n \le r \le R_{n+1}} \frac{\log\log M(r)}{\log r},$$

there exists a subsequence (n_k) *such that*

$$\varepsilon_{n_k} \le \delta_k + \frac{1}{2^{n_k}}, \quad \textit{for } k \in \mathbb{N}, \tag{1.2}$$

and

$$\varepsilon_{n_k+m} \le \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \textit{for } k \in \mathbb{N}, 1 \le m < n_{k+1} - n_k. \tag{1.3}$$

Since it is possible to choose a positive sequence (δ_n) with

$$\sum_{n\in\mathbb{N}} \delta_n = \infty \text{ and } \lim_{n\to\infty} \delta_n = 0,$$

Theorem 1.2 implies that there are functions of order zero for which $A_R(f)$ fails to be a spider's web. Thus new techniques are needed in order to solve Baker's conjecture. One such technique is introduced in [13] where we show that all functions of order less than 1/2 with zeros on the negative real axis satisfy Baker's conjecture and also satisfy Eremenko's conjecture with $I(f)$ being a spider's web. Since functions of the form (1.1) with $\limsup_{n\to\infty} \varepsilon_n < 1/2$ are of this type, this gives the following corollary to Theorem 1.2, which answers a question in [12].

Corollary 1.3. *There exist transcendental entire functions for which* $I(f)$ *is a spider's web but* $A(f)$ *is not a spider's web.*

Remark. In fact we show in [13] that functions of order less than 1/2 with zeros on the negative real axis have the stronger property that $Q(f)$ contains a spider's web, where $Q(f)$ is the quite fast escaping set. Thus Theorem 1.2 provides examples of functions for which $Q(f) \neq A(f)$; these two sets are equal for many functions, including all functions in the Eremenko-Lyubich class $\mathcal{B}$ as we show in [15].

The paper is arranged as follows. In Section 2 we prove Theorem 1.1 and then, in Section 3, we prove Theorem 1.2.

2. Proof of Theorem 1.1

Let f be a transcendental entire function and $R > 0$ be such that $M(r) > r$ for $r \ge R$. Recall that

$$A_R(f) = \{z : |f^n(z)| \ge M^n(R), \text{ for } n \in \mathbb{N}\}$$

and that $A_R(f)$ is a spider's web if $A_R(f)$ is connected and there exists a sequence of bounded simply connected domains G_n such that

$$\partial G_n \subset A_R(f), \;\; G_n \subset G_{n+1}, \text{ for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}.$$

In this section we prove Theorem 1.1 which gives a condition that is sufficient to ensure that $A_R(f)$ is a spider's web. The key ingredient in our proof is the following result which can be viewed as a local version of the classical cos $\pi\rho$ theorem. For a discussion of results of this type, see [14].

Theorem 2.1. *Let f be a transcendental entire function. There exists $r(f) > 0$ such that, if*

$$\log M(r) \leq r^\alpha \quad \textit{and} \quad r^{1-2\alpha} \geq r(f), \tag{2.1}$$

for some $\alpha \in (0, 1/2)$, then there exists $t \in (r^{1-2\alpha}, r)$ such that

$$\log m(t) > \log M(r^{1-2\alpha}) - 2.$$

Proof. We apply the following result of Beurling [4, page 96]:

Let f be analytic in $\{z : |z| < r_0\}$, let $0 \leq r_1 < r_2 < r_0$, and put

$$E = \{t \in (r_1, r_2) : m(t) \leq \mu\}, \;\; \text{where } 0 < \mu < M(r_1).$$

Then

$$\log \frac{M(r_2)}{\mu} > \frac{1}{2} \exp\left(\frac{1}{2}\int_E \frac{dt}{t}\right) \log \frac{M(r_1)}{\mu}. \tag{2.2}$$

Taking $r_2 = r$, $r_1 = r^{1-2\alpha}$, $\mu = M(r^{1-2\alpha})/e^2$, and $r(f) > 0$ such that $M(r(f)) \geq e^2$, we deduce from (2.1) and (2.2) that, if $m(t) \leq \mu$ for $t \in (r^{1-2\alpha}, r)$, then

$$r^\alpha \geq \log M(r) \geq \log \frac{M(r)}{\mu} > \frac{1}{2} \exp\left(\frac{1}{2}\int_{r^{1-2\alpha}}^{r} \frac{dt}{t}\right) \log \frac{M(r^{1-2\alpha})}{\mu} = r^\alpha.$$

This is a contradiction and so there must exist $t \in (r^{1-2\alpha}, r)$ such that $m(t) > \mu$; that is,

$$\log m(t) > \log \mu = \log M(r^{1-2\alpha}) - 2,$$

as required. □

We also use the following results about spiders' webs proved in [12].

Lemma 2.2. [12, Corollary 8.2] *Let f be a transcendental entire function and let $R > 0$ be such that $M(r) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if there exists a sequence (ρ_n) such that, for $n \geq 0$,*

$$\rho_n > M^n(R) \tag{2.3}$$

and

$$m(\rho_n) \geq \rho_{n+1}. \tag{2.4}$$

Lemma 2.3. [12, Lemma 7.1(d)] *Let f be a transcendental entire function, let $R > 0$ be such that $M(r) > r$ for $r \geq R$, and let $R' > R$. Then $A_R(f)$ is a spider's web if and only if $A_{R'}(f)$ is a spider's web.*

In addition, we need the following property of the maximum modulus function, which was proved in this form in [11].

Lemma 2.4. [11, Lemma 2.2] *Let f be a transcendental entire function. Then there exists $R > 0$ such that, for all $r \geq R$ and all $c > 1$,*

$$M(r^c) \geq M(r)^c.$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $R > 0$ be such that, for $r \geq R$, Lemma 2.4 holds and $M(r) > r$. For $n \in \mathbb{N}$, let

$$R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log\log M(r)}{\log r}.$$

Suppose that $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$. Then we can take N sufficiently large to ensure that

$$\sum_{n \geq N} \varepsilon_n < \frac{1}{8}, \tag{2.5}$$

and

$$M(R_n)^{1/(8n^2)} = R_{n+1}^{1/(8n^2)} \geq e^2, \text{ for } n \geq N, \text{ and } R_{N+1}^{1/4} \geq R_N \geq r(f), \tag{2.6}$$

where $r(f)$ is as defined in Theorem 2.1. Note that (2.6) is possible since $\log M(r)/\log r \to \infty$ and so, for large n, we have $\log R_{n+1} > 4 \log R_n$.

Now let

$$r_n = M^{n+1}\left(R_{N+1}^{\prod_{m=N}^{N+n}(1-2\varepsilon_m - 1/(8m^2))}\right), \quad \text{for} \quad n \geq 0.$$

We note that, for $n \geq 0$, it follows from (2.5) that

$$\prod_{m=N}^{N+n}\left(1 - 2\varepsilon_m - \frac{1}{8m^2}\right) > 1 - \sum_{m=N}^{N+n} 2\varepsilon_m - \sum_{m=N}^{N+n} \frac{1}{8m^2} \geq \frac{1}{2}$$

and so, by (2.6),

$$R_{N+n+2} > r_n > M^{n+1}(R_{N+1}^{1/2}) \geq M^{n+1}(R_N^2) = R_{N+n+1}^2.$$

We claim that, for $n \geq 0$, there exists $\rho_n \in (R_{N+n+1}, r_n)$ with $m(\rho_n) > r_{n+1}$. Indeed, it follows from Theorem 2.1, (2.5), (2.6) and Lemma 2.4 that, for $n \geq 0$, there exists $\rho_n \in (r_n^{1-2\varepsilon_{n+N+1}}, r_n) \subset (R_{N+n+1}, r_n)$ such that

$$\begin{aligned}
m(\rho_n) &\geq \frac{1}{e^2} M(r_n^{1-2\varepsilon_{n+N+1}}) \\
&\geq M(r_n^{1-2\varepsilon_{n+N+1}})^{1-1/(8(n+N+1)^2)} \\
&\geq M(r_n^{(1-2\varepsilon_{n+N+1})(1-1/(8(n+N+1)^2))}) \\
&\geq M(r_n^{(1-2\varepsilon_{n+N+1}-1/(8(n+N+1)^2))}) \\
&= M\left(\left(M^{n+1}\left(R_{N+1}^{\prod_{m=N}^{N+n}(1-2\varepsilon_m-1/(8m^2))}\right)\right)^{(1-2\varepsilon_{n+N+1}-1/(8(n+N+1)^2))}\right) \\
&\geq M^{n+2}\left(R_{N+1}^{\prod_{m=N}^{N+n+1}(1-2\varepsilon_m-1/(8m^2))}\right) \\
&= r_{n+1}.
\end{aligned}$$

Thus, for $n \geq 0$, there exists $\rho_n > R_{N+n}$ with $m(\rho_n) \geq \rho_{n+1}$ and so, by Lemma 2.2, $A_{R_{N+1}}(f)$ is a spider's web. It now follows from Lemma 2.3 that $A_R(f)$ is a spider's web as claimed. ∎

3. Proof of Theorem 1.2

Let

$$f(z) = z^3 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{2p_n},$$

where the sequence (a_n) is positive and strictly increasing. In addition, let (δ_n) be a positive sequence such that

$$\sum_{n \in \mathbb{N}} \delta_n = \infty,$$

and let

$$p_n = [a_n^{\delta_n/4}/4]. \tag{3.1}$$

Without loss of generality, we assume that

$$\delta_n < 1/2, \text{ for } n \in \mathbb{N}. \tag{3.2}$$

Note that $f((-\infty, 0]) \subset (-\infty, 0]$ and that $m(r) = f(-r)$ and $M(r) = f(r) > r^3$, for $r > 0$. Further, $M(r) > r$ for $r \geq 1$.

We first show that the sequence (a_n) can be chosen so that $A(f) \cap (-\infty, 0] = \emptyset$.

We choose the values of a_n carefully, beginning with a_1, then a_2 and so on. Because of the way in which we choose the values of a_n, it is helpful to introduce the function g defined by

$$g(r) = \begin{cases} r^3, & 0 \leq r < a_1, \\ r^3 \displaystyle\prod_{a_n \leq r} \left(1 + \frac{r}{a_n}\right)^{2p_n}, & r \geq a_1. \end{cases} \tag{3.3}$$

Note that g is a strictly increasing function and that it is discontinuous at a_n, for $n \in \mathbb{N}$. A key property of g which we use repeatedly is that

$$m(r) = -f(-r) < g(r) < M(r), \text{ for } r \geq 0. \tag{3.4}$$

Since g is increasing, (3.4) implies that

$$f([-r, 0]) \subset [-g(r), 0], \text{ for } r \geq 0. \tag{3.5}$$

We now set $r_0 = 10$ and $r_{n+1} = g(r_n) = g^{n+1}(10)$, for $n \in \mathbb{N}$, and note that

$$r_{n+1} \geq r_n^3, \text{ for } n \geq 0. \tag{3.6}$$

Also, it follows from (3.5) that

$$f^n((-r_m, 0]) \subset (-r_{m+n}, 0], \text{ for } n, m \in \mathbb{N}. \tag{3.7}$$

We begin by proving the following result.

Lemma 3.1. *If there exists a sequence (N_k) such that,*

$$f^{N_1}((-r_2, 0]) \subset (-r_{N_1}, 0] \tag{3.8}$$

and, for $k \geq 2$,

$$f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_k}, 0], \tag{3.9}$$

then $A(f) \cap (-\infty, 0] = \emptyset$.

Proof. We first note that, if the hypotheses of Lemma 3.1 hold, then it follows from (3.7) and (3.9) that, for $k \in \mathbb{N}$,

$$\begin{aligned} f^{N_1+\cdots+N_k}((-r_{2k},0]) &= f^{N_k}(f^{N_1+\cdots+N_{k-1}})((-r_{2k},0]) \\ &\subset f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k},0]) \\ &\subset (-r_{N_1+\cdots+N_k},0]. \end{aligned}$$

Thus

$$f^{N_1+\cdots+N_k}((-r_{2k},0]) \subset (-r_{N_1+\cdots+N_k},0]. \tag{3.10}$$

Now let $z \in (-\infty,0]$. There exists $K \in \mathbb{N}$ such that, for $k \geq K$, we have $z \in (-r_k,0]$ and hence, by (3.7), we have $f^k(z) \in (-r_{2k},0]$. Thus, by (3.10) and (3.4), for $k \geq K$,

$$|f^{N_1+\cdots+N_k+k}(z)| < r_{N_1+\cdots+N_k} < M^{N_1+\cdots+N_k}(10)$$

and hence

$$z \notin \{z : |f^{n+k}(z)| \geq M^n(10) \text{ for } n \in \mathbb{N}\}.$$

Thus $A(f) \cap (-\infty,0] = \emptyset$ as required. □

We will show that we can choose the values of a_n in such a way that the hypotheses of Lemma 3.1 hold. In order to do this, it is helpful to set certain restrictions on our choice of values. Firstly, we choose a_1 and a_{n+1}/a_n, $n \in \mathbb{N}$, sufficiently large to ensure that

$$a_1^{\delta_1/4} \geq 4, \quad a_{n+1} > a_n^2, \quad a_{n+1}^{\delta_{n+1}/2} > 16a_n^{\delta_n} \tag{3.11}$$

and

$$a_{n+1}^{\delta_{n+1}/16} > a_n^{\delta_n} \log a_{n+1}. \tag{3.12}$$

We note that (3.11) implies that

$$p_1 \geq 1 \quad \text{and} \quad p_{n+1} \geq 2p_n^2, \text{ for } n \in \mathbb{N}. \tag{3.13}$$

We also place certain restrictions on our choice of the values of a_n in relation to the values of r_n:

$$\text{if } a_k \in [r_n, r_{n+1}), \text{ then } a_m \notin [r_n, r_{n+4}) \text{ for } k,m \in \mathbb{N},\ m \neq k. \tag{3.14}$$

We now show that, in order to prove that the hypotheses of Lemma 3.1 hold, it is sufficient to prove the following result.

Lemma 3.2. *Suppose that, for some $m \in \mathbb{N}$, we have defined the values of a_n for which $a_n \leq r_m$ in such a way that they satisfy* (3.11), (3.12) *and* (3.14)*. Then we can choose $N \in \mathbb{N}$ and the values of a_n for which $r_m < a_n \leq r_{m+N-1}$ in such a way that they satisfy* (3.11), (3.12) *and* (3.14) *and, no matter how the later values of a_n are chosen,*

$$f^N((-r_{m+1},0]) \subset (-r_{m+N},0].$$

Proving Lemma 3.2 is the key part of the proof that we can choose the sequence (a_n) so as to ensure that $A(f) \cap (-\infty,0] = \emptyset$. Before proving Lemma 3.2, we show that, if this result holds, then the hypotheses of Lemma 3.1 also hold. First, by applying Lemma 3.2 when $m = 1$ we see that there exists $N_{1,1} \in \mathbb{N}$ and a choice of a_n for $r_1 < a_n \leq r_{N_{1,1}}$ such that

$$f^{N_{1,1}}((-r_2,0]) \subset (-r_{N_{1,1}+1},0]. \tag{3.15}$$

We then apply Lemma 3.2 with $m = N_{1,1}$ and deduce that there exists $N_{1,2} \in \mathbb{N}$ and a choice of a_n for $r_{N_{1,1}} < a_n \le r_{N_{1,1}+N_{1,2}-1}$ such that

$$f^{N_{1,2}}((-r_{N_{1,1}+1}, 0]) \subset (-r_{N_{1,1}+N_{1,2}}, 0]. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$f^{N_{1,1}+N_{1,2}}(-r_2, 0] \subset f^{N_{1,2}}(-r_{N_{1,1}+1}, 0] \subset (-r_{N_{1,1}+N_{1,2}}, 0].$$

Putting $N_1 = N_{1,1} + N_{1,2}$, we deduce that we can choose the values of a_n for which $r_1 < a_n \le r_{N_1-1}$ in such a way that

$$f^{N_1}((-r_2, 0]) \subset (-r_{N_1}, 0].$$

Thus (3.8) holds.

Now suppose that, for some $k \ge 2$, we have defined N_j, for $1 \le j \le k-1$, and defined a_n, for $r_1 < a_n \le r_{N_1+\cdots+N_{k-1}-1}$. We claim that we can use Lemma 3.2 to define $N_k \in \mathbb{N}$ and a_n with $r_{N_1+\cdots+N_{k-1}-1} < a_n \le r_{N_1+\cdots+N_k-1}$ such that (3.9) holds for k. The argument is similar to that given above. First, we apply Lemma 3.2 with $m = N_1 + \cdots + N_{k-1} + 2k - 1$ to construct $N_{k,1}$ and a_n with

$$r_{N_1+\cdots+N_{k-1}+2k-1} < a_n \le r_{N_1+\cdots+N_{k-1}+N_{k,1}+2k-2}$$

such that

$$f^{N_{k,1}}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_{k-1}+N_{k,1}+2k-1}, 0].$$

Then, for $2 \le j \le 2k$, we apply Lemma 3.2 repeatedly with

$$m = N_1 + \cdots + N_{k-1} + N_{k,1} + \cdots + N_{k,j-1} + 2k - j$$

to construct $N_{k,j}$ and a_n with

$$r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j-1}+2k-j} < a_n \le r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j}+2k-j-1}$$

such that

$$f^{N_{k,j}}((-r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j-1}+2k-j+1}, 0]) \subset (-r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j}+2k-j}, 0].$$

Putting $N_k = N_{k,1} + \cdots + N_{k,2k}$, we deduce that a_n can be chosen with

$$r_{N_1+\cdots+N_{k-1}-1} < a_n \le r_{N_1+\cdots+N_k-1}$$

such that

$$f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_k}, 0]$$

and hence (3.9) holds for k.

So, it remains to prove Lemma 3.2.

We begin by proving four lemmas. The first describes the extent to which f is small close to a zero at $-a_k$, where $k \in \mathbb{N}$.

Lemma 3.3. *For each $k \in \mathbb{N}$,*

$$|f(z)| < 1, \text{ for } z \in (-a_k, -a_k^{1-\delta_k/16}). \tag{3.17}$$

Proof. This holds since, for such a z, it follows from (3.1), (3.11), (3.12) and (3.13) that

$$\begin{aligned}
|f(z)| &\leq a_k^3 \left(1 - \frac{a_k^{1-\delta_k/16}}{a_k}\right)^{2p_k} \prod_{m=1}^{k-1} \left(1 + \frac{a_k}{a_m}\right)^{2p_m} \prod_{m \geq k+1} \left(1 + \frac{a_k}{a_m}\right)^{2p_m} \\
&\leq \left(1 - \frac{1}{a_k^{\delta_k/16}}\right)^{a_k^{\delta_k/4}} a_k^{3+2p_1+\cdots+2p_{k-1}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\
&\leq \exp(-a_k^{\delta_k/16}) a_k^{a_{k-1}^{\delta_{k-1}/2}} e^{1+1/2+1/4+\cdots} \\
&\leq a_k^{a_{k-1}^{\delta_{k-1}}} \exp(-a_k^{\delta_k/16}) < 1.
\end{aligned}$$

□

The second lemma shows that there is a large increase in the size of $g(r)$ at $r = a_k$, where $k \in \mathbb{N}$.

Lemma 3.4. *For each $k \in \mathbb{N}$,*

$$\log g(a_k) \geq p_k^{1/2} \log g(a_k^{1-\delta_k/16}).$$

Proof. For $k \in \mathbb{N}$, it follows from (3.11) that

$$\begin{aligned}
g(a_k^{1-\delta_k/16}) &< a_k^3 \prod_{m=1}^{k-1} \left(1 + \frac{a_k}{a_m}\right)^{2p_m} \\
&< a_k^{3+2\sum_{m=1}^{k-1} p_m} \leq a_k^{4p_{k-1}}
\end{aligned}$$

and

$$g(a_k) \geq 2^{2p_k}.$$

Thus, by (3.11), (3.12) and (3.13),

$$\frac{\log g(a_k)}{\log g(a_k^{1-\delta_k/16})} \geq \frac{2p_k \log 2}{4p_{k-1} \log a_k} > \frac{p_k}{3p_{k-1} \log a_k} > p_k^{1/2}.$$

□

The third lemma shows that $\log g$ has a convexity property.

Lemma 3.5. *Let $r > 0$ and $t \geq 2$. Then*

$$\log g(r^t) \geq t \log g(r).$$

Proof. Let $r > 0$ and $t \geq 2$. We have

$$g(r^t) \geq r^{3t} \prod_{a_m \leq r} \left(1 + \frac{r^t}{a_m}\right)^{2p_m}$$

and

$$g(r)^t = r^{3t} \prod_{a_m \leq r} \left(1 + \frac{r}{a_m}\right)^{2p_m t}.$$

Thus it is sufficient to show that

$$\left(1 + \frac{r}{a_m}\right)^t \leq \left(1 + \frac{r^t}{a_m}\right),$$

when $a_m \le r$. This is true since it follows from (3.11) that, for $a_m \le r$ and $t \ge 2$,

$$\left(1+\frac{r}{a_m}\right)^t \le \left(\frac{r}{a_m^{1/2}}\right)^t = \frac{r^t}{a_m^{t/2}} < 1+\frac{r^t}{a_m}.$$

□

The fourth lemma gives an upper bound on the growth of g on intervals where no point is the modulus of a zero of f.

Lemma 3.6. *Let $r > 0$, $0 < s < 1/2$ and $t > 1$ and suppose that there are no values of $n \in \mathbb{N}$ for which $a_n \in (r^s, r^t]$. Then*

$$\log g(r^t) \le t(1+2s)\log g(r).$$

Proof. It follows from (3.11) that

$$g(r^t) = r^{3t}\prod_{a_m \le r^s}\left(1+\frac{r^t}{a_m}\right)^{2p_m} < r^{3t}\prod_{a_m \le r^s} r^{2p_m t} = r^{t(3+\sum_{a_m\le r^s} 2p_m)}$$

and

$$g(r) > r^3 \prod_{a_m \le r^s}\left(\frac{r}{a_m}\right)^{2p_m} > r^{3+\sum_{a_m \le r^s} 2p_m(1-s)}.$$

Thus

$$\log g(r^t)/\log g(r) < t/(1-s) \le t(1+2s),$$

since $s < 1/2$. □

We are now in a position to prove Lemma 3.2.

Proof of Lemma 3.2. Suppose that $m \in \mathbb{N}$ and that we have defined the values of a_n for which $a_n \le r_m$. We now define a sequence (s_k), $0 \le k \le N$, inductively according to certain rules that we give below. Each time we define a value s_k, we also add a zero of f at $-s_k$ provided this is allowed by (3.11), (3.12) and (3.14); no other zeros of f are added. We choose our values s_k in such a way that

(3.18) $$r_{m+k} \le s_k \le r_{m+k+1}, \text{ for } 0 \le k < N,$$

(3.19) $$s_N \le r_{m+N}$$

and

(3.20) $$f^k((-r_{m+1}, 0]) \subset (-s_k, 0], \text{ for } 0 \le k \le N.$$

The result of Lemma 3.2 follows directly from (3.19) and (3.20). The difficult part of the proof is to show that there exists an $N \in \mathbb{N}$ for which (3.19) is satisfied.

We define our sequence (s_k) as follows:

- set $s_0 = r_{m+1}$;
- if $s_k > r_{m+k}$ and there is a zero of f at $-s_k$, then we set

(3.21) $$s_{k+1} = g(s_k^{1-\delta_{n_k}/16});$$

- if $s_k > r_{m+k}$ and there is no zero of f at s_k, then we set

(3.22) $$s_{k+1} = g(s_k);$$

- if $s_k \le r_{m+k}$, then we terminate the sequence (s_k).

It follows from Lemma 3.3 and (3.5) that, with this construction, (3.18), (3.19) and (3.20) are indeed satisfied.

It remains to prove that there exists $K \in \mathbb{N}$ such that the sequence terminates at s_K; that is, if

$$T_k = \frac{\log s_k}{\log r_{m+k}},$$

then there exists $K \in \mathbb{N}$ such that $T_K \leq 1$.

We introduce the following terminology. We let L denote the largest integer for which $a_L \leq r_m$ and define a (finite) subsequence (k_n) such that

$$a_{L+n} = s_{k_n}, \text{ for } n = 1, 2, \ldots. \tag{3.23}$$

The main idea is to show that, for each $n \geq 2$ we have that T_{k_n+1} is less than T_{k_n}, with k_n defined as above. These decreases counteract the small increases that may occur from T_k to T_{k+1} for other values of k and, for n large enough, they will combine together to cause T_{k_n+1} to drop below 1.

We first estimate some quantities that will be useful in our calculations. We begin by noting that it follows from (3.23), (3.18) and Lemma 3.4 that, for $n \geq 1$,

$$\begin{aligned} \log r_{m+k_n+2} &= \log g(r_{m+k_n+1}) \\ &\geq \log g(s_{k_n}) \geq p_{L+n}^{1/2} \log g(s_{k_n}^{1-\delta_{L+n}/16}). \end{aligned}$$

Thus, by (3.21)

$$\log r_{m+k_n+2} \geq p_{L+n}^{1/2} \log s_{k_n+1}, \text{ for } n \geq 1. \tag{3.24}$$

Together with (3.6), (3.24) implies that

$$\log r_{m+k_n+q} \geq 3^{q-2} p_{L+n}^{1/2} \log s_{k_n+1}, \text{ for } q \geq 2,\ n \geq 1. \tag{3.25}$$

Together with Lemma 3.5, (3.24) implies that

(3.26)

$$\frac{\log s_{k_n+q}}{\log r_{m+k_n+q+1}} \leq \frac{\log g^{q-1}(s_{k_n+1})}{\log g^{q-1}(r_{m+k_n+2})} \leq \frac{\log s_{k_n+1}}{\log r_{m+k_n+2}} \leq \frac{1}{p_{L+n}^{1/2}}, \text{ for } q \geq 2,\ n \geq 1.$$

Now fix $n \geq 2$ and write

$$t_{n,q} = T_{k_n+q} = \frac{\log s_{k_n+q}}{\log r_{m+k_n+q}}, \text{ for } q \geq 2.$$

For $2 \leq q < k_{n+1} - k_n$, there are no zeros of f with modulus in the interval $(s_{k_n}, s_{k_{n+q}})$ and so it follows from (3.22), Lemma 3.6 and (3.25) that, for such q,

$$\begin{aligned} \log s_{k_n+q+1} &= \log g(s_{k_n+q}) \\ &\leq t_{n,q}\left(1 + 2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right) \log g(r_{m+k_n+q}) \\ &= t_{n,q}\left(1 + 2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right) \log r_{m+k_n+q+1} \\ &\leq t_{n,q}\left(1 + \frac{2}{3^{q-2} p_{L+n}^{1/2}}\right) \log r_{m+k_n+q+1}. \end{aligned}$$

Thus, for $2 \le q < k_{n+1} - k_n$, we have

$$t_{n,q+1} \le t_{n,q}\left(1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}}\right). \tag{3.27}$$

For $q = k_{n+1} - k_n$, there are no zeros of f with modulus in the interval $(s_{k_n}, s_{k_{n+q}})$ and so it follows from (3.21), Lemma 3.6 and (3.25) that

$$\begin{aligned}
\log s_{k_n+q+1} &= \log g(s_{k_n+q}^{1-\delta_{L+n+1}/16})\\
&\le t_{n,q}\left(1 - \frac{\delta_{L+n+1}}{16}\right)\left(1 + 2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log g(r_{m+k_n+q})\\
&= t_{n,q}\left(1 - \frac{\delta_{L+n+1}}{16}\right)\left(1 + 2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log r_{m+k_n+q+1}\\
&\le t_{n,q}\left(1 - \frac{\delta_{L+n+1}}{16}\right)\left(1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}}\right)\log r_{m+k_n+q+1}.
\end{aligned}$$

Thus, for $q = k_{n+1} - k_n$, we have

$$t_{n,q+1} \le t_{n,q}\left(1 - \frac{\delta_{L+n+1}}{16}\right)\left(1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}}\right). \tag{3.28}$$

Lastly, it follows from (3.14) that, if $q = k_{n+1} - k_n + 1$, then $q - 1 \ge 2$. Also, there are no zeros of f with modulus in the interval $(s_{k_{n+1}}, s_{k_{n+1}+1}) = (s_{k_{n+1}}, s_{k_n+q})$ and so it follows from Lemma 3.6 and (3.26) that

$$\begin{aligned}
\log s_{k_n+q+1} &= \log g(s_{k_n+q})\\
&\le t_{n,q}\left(1 + 2\frac{\log s_{k_{n+1}}}{\log r_{m+k_n+q}}\right)\log g(r_{m+k_n+q})\\
&= t_{n,q}\left(1 + 2\frac{\log s_{k_n+q-1}}{\log r_{m+k_n+q}}\right)\log r_{m+k_n+q+1}\\
&\le t_{n,q}\left(1 + \frac{2}{p_{L+n}^{1/2}}\right)\log r_{m+k_n+q+1}.
\end{aligned}$$

Thus, for $q = k_{n+1} - k_n + 1$, we have

$$t_{n,q+1} \le t_{n,q}\left(1 + \frac{2}{p_{L+n}^{1/2}}\right). \tag{3.29}$$

It follows from (3.27), (3.28), (3.29) and (3.13) that, for $M \geq 2$, we have

$$\begin{aligned} T_{k_{M+1}+2} &= t_{M,k_{M+1}-k_M+2} \\ &= t_{2,2} \prod_{n=2}^{M} \prod_{q=2}^{k_{n+1}-k_n+1} \frac{t_{n,q+1}}{t_{n,q}} \\ &\leq t_{2,2} \prod_{n=2}^{M} \left(1 + \frac{2}{p_{L+n}^{1/2}}\right)\left(1 - \frac{\delta_{L+n+1}}{16}\right) \prod_{q=2}^{k_{n+1}-k_n} \left(1 + \frac{2}{3^{q-2} p_{L+n}^{1/2}}\right) \\ &\leq t_{2,2} \prod_{n=2}^{M} \left(\left(1 + \frac{2}{p_{L+n}^{1/2}}\right)^3 \left(1 - \frac{\delta_{L+n+1}}{16}\right)\right). \end{aligned}$$

It follows from (3.13) that $\sum_{n\in\mathbb{N}} \frac{1}{p_{L+n}^{1/2}} < \infty$ and so, since $\sum_{n\in\mathbb{N}} \delta_{L+n+1} = \infty$, we deduce that, for M sufficiently large, $T_{k_{M+1}+2} \leq 1$, as required. □

We have now proved Lemma 3.2. As noted earlier, this is sufficient to imply that the hypotheses of Lemma 3.1 hold and hence that $A(f) \cap (-\infty, 0] = \emptyset$ as required.

We complete the proof of Theorem 1.2 by showing that, in addition, conditions (1.2) and (1.3) are satisfied. That is, we prove the following.

Lemma 3.7. *Let*

$$\varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log\log M(r)}{\log r}. \tag{3.30}$$

There exists a subsequence (n_k) *such that*

$$\varepsilon_{n_k} \leq \delta_k + \frac{1}{2^{n_k}}, \text{ for } k \in \mathbb{N}, \tag{3.31}$$

and

$$\varepsilon_{n_k+m} \leq \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \text{for } k \in \mathbb{N}, 1 \leq m < n_{k+1} - n_k. \tag{3.32}$$

Proof. We begin by setting $R_0 = r_0 = 10$ and defining $R_{n+1} = M(R_n)$, for $n \in \mathbb{N}$. Clearly $R_n \geq r_n$ by (3.4) and

$$R_{n+1} \geq R_n^3, \text{ for } n \in \mathbb{N}. \tag{3.33}$$

We claim that

$$\text{if } a_k \in [R_n, R_{n+1}), \text{ then } a_m \notin [R_n, R_{n+2}) \text{ for } k, m \in \mathbb{N},\ m \neq k. \tag{3.34}$$

In order to deduce this from (3.14), it is sufficient to show that, if $r_p \in [R_n, R_{n+1})$, for some $p, n \in \mathbb{N}$, then $r_{p+2} > R_{n+1}$. We prove this in two steps. Firstly, we note that if $r_p \in [R_n, R_n^3)$, for some $p, n \in \mathbb{N}$, then it follows from (3.6) that $r_{p+1} \geq r_p^3 \geq R_n^3$. Secondly, if $r_p \in [R_n^3, R_{n+1})$, for some $p, n \in \mathbb{N}$, then we claim that

$$r_{p+1} = g(r_p) \geq g(R_n^3) > M(R_n) = R_{n+1}. \tag{3.35}$$

This is true since, if k is the smallest integer such that $a_k > R_n^3$, then

$$g(R_n^3) = R_n^9 \prod_{m=1}^{k-1} \left(1 + \frac{R_n^3}{a_m}\right)^{2p_m}$$

and so, by (3.1) and (3.11),

$$\begin{aligned} M(R_n) = f(R_n) &= R_n^3 \prod_{m=1}^{\infty} \left(1 + \frac{R_n}{a_m}\right)^{2p_m} \\ &< \frac{g(R_n^3)}{R_n^6} \left(1 + \frac{R_n}{a_k}\right)^{2p_k} \prod_{m \geq k+1} \left(1 + \frac{a_k}{a_m}\right)^{2p_m} \\ &< \frac{g(R_n^3)}{R_n^6} \left(1 + \frac{1}{a_k^{1/2}}\right)^{a_k^{1/2}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &\leq \frac{g(R_n^3)}{R_n^6} e^{1+1+1/2+1/4+\cdots} < g(R_n^3). \end{aligned}$$

Thus (3.35) does indeed hold and, by the reasoning above, this is sufficient to show that (3.34) holds.

Now, for $k \in \mathbb{N}$, we choose $n_k \in \mathbb{N}$ such that $a_k \in [R_{n_k}, R_{n_k+1})$. Then, by (3.34), this defines a sequence (n_k) with $n_j \neq n_k$ for $j \neq k$. Now suppose that $r \in [R_{n_k}, R_{n_k+1}]$, for some $k \in \mathbb{N}$. It follows from (3.11) and (3.34) that

$$\begin{aligned} M(r) = f(r) &\leq r^3 \left(1 + \frac{r}{a_k}\right)^{2p_k} \prod_{m=1}^{k-1} \left(1 + \frac{r}{a_m}\right)^{2p_m} \prod_{m \geq k+1} \left(1 + \frac{r}{a_m}\right)^{2p_m} \\ &\leq \left(1 + \frac{r}{a_k}\right)^{2p_k} r^{3+2p_1+\cdots+2p_{k-1}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &< \left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}} r^{a_{k-1}^{\delta_{k-1}}} e^{1+1/2+1/4+\cdots} \end{aligned}$$

and so

$$M(r) < e^2 r^{a_{k-1}^{\delta_{k-1}}} \left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}}. \tag{3.36}$$

If $r < a_k^{1/2}$, then it follows from (3.2) and (3.36) that

$$M(r) < e^3 r^{a_{k-1}^{\delta_{k-1}}} < e^3 r^{r^{\delta_k}}$$

and hence, since $r \geq R_1 \geq 1000$,

$$\frac{\log\log M(r)}{\log r} < \frac{\delta_k \log r + 2\log\log r}{\log r} = \delta_k + 2\frac{\log\log r}{\log r} \leq \delta_k + 2\frac{\log\log R_{n_k}}{\log R_{n_k}}.$$

It follows from (3.33) that, in this case,

$$\frac{\log\log M(r)}{\log r} \leq \delta_k + 2\frac{\log(3^{n_k}\log 10)}{3^{n_k}\log 10} < \delta_k + \frac{1}{2^{n_k}}. \tag{3.37}$$

If $a_k^{1/2} \le r \le a_k$, then

$$\left(1+\frac{r}{a_k}\right)^{a_k^{\delta_k}} = \left(1+\frac{r}{a_k}\right)^{(a_k/r)^{\delta_k} r^{\delta_k}} < \left(1+\frac{r}{a_k}\right)^{(a_k/r) r^{\delta_k}} \le e^{r^{\delta_k}}$$

and, if $r > a_k$, then

$$\left(1+\frac{r}{a_k}\right)^{a_k^{\delta_k}} < r^{a_k^{\delta_k}} < r^{r^{\delta_k}}.$$

So, if $r \ge a_k^{1/2}$, it follows from (3.36) and (3.11) that

$$M(r) < e^2 r^{a_{k-1}^{\delta_{k-1}}} r^{r^{\delta_k}} < e^2 r^{a_k^{\delta_k/2}} r^{r^{\delta_k}} < e^2 r^{2r^{\delta_k}}$$

and hence

$$\frac{\log\log M(r)}{\log r} < \frac{\delta_k \log r + 2\log\log r}{\log r} = \delta_k + 2\frac{\log\log r}{\log r} \le \delta_k + 2\frac{\log\log R_{n_k}}{\log R_{n_k}}.$$

As before, it follows from (3.33) that

$$\frac{\log\log M(r)}{\log r} \le \delta_k + \frac{1}{2^{n_k}}. \tag{3.38}$$

Together with (3.37), this implies that (3.31) holds.

Now suppose that $r \in [R_{n_k+m}, R_{n_k+m+1})$, for some $k \in \mathbb{N}$, $1 \le m < n_{k+1} - n_k$. It follows from (3.11) and (3.33) that

$$\begin{aligned} M(r) = f(r) &\le r^3 \prod_{m=1}^{k} \left(1+\frac{r}{a_m}\right)^{2p_m} \prod_{m\ge k+1} \left(1+\frac{r}{a_m}\right)^{2p_m} \\ &\le r^{3+2p_1+\cdots+2p_k} \prod_{m\ge k+1} \left(1+\frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &\le r^{a_k^{\delta_k}} e^{1+1/2+1/4+\cdots} \\ &\le e^2 r^{a_k^{\delta_k}} \le e^2 r^{R_{n_k+1}^{\delta_k}} \\ &< e^2 r^{r^{\delta_k/3^{m-1}}} \end{aligned}$$

Thus

$$\frac{\log\log M(r)}{\log r} < \frac{\delta_k \log r/3^{m-1} + 2\log\log r}{\log r} < \frac{\delta_k}{3^{m-1}} + 2\frac{\log\log r}{\log r} \le \frac{\delta_k}{3^{m-1}} + 2\frac{\log\log R_{n_k+m}}{\log R_{n_k+m}}.$$

As before, it follows from (3.33) that

$$\frac{\log\log M(r)}{\log r} \le \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_m+m}} \tag{3.39}$$

and so (3.32) holds. □

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