A SHARP GROWTH CONDITION FOR A FAST ESCAPING SPIDER’S WEB

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Abstract. We show that the fast escaping set $A(f)$ of a transcendental entire function $f$ has a structure known as a spider’s web whenever the maximum modulus of $f$ grows below a certain rate. We give examples of entire functions for which the fast escaping set is not a spider’s web which show that this growth rate is best possible. By our earlier results, these are the first examples for which the escaping set has a spider’s web structure but the fast escaping set does not. These results give new insight into a conjecture of Baker and a conjecture of Eremenko.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and denote by $f^n$, $n = 0, 1, 2, \ldots$, the $n$th iterate of $f$. The Fatou set $F(f)$ is the set of points $z \in \mathbb{C}$ such that $(f^n)_{n \in \mathbb{N}}$ forms a normal family in some neighborhood of $z$. The complement of $F(f)$ is called the Julia set $J(f)$ of $f$. An introduction to the properties of these sets can be found in [2].

In recent years, the escaping set defined by $I(f) = \{ z : f^n(z) \to \infty \text{ as } n \to \infty \}$ has come to play an increasingly significant role in the study of the iteration of transcendental entire functions with much of the research being motivated by a conjecture of Eremenko [5] that all the components of the escaping set are unbounded. For partial results on this conjecture see, for example, [9] and [16].

The most general result on Eremenko’s conjecture was obtained in [10] where it was proved that the escaping set always has at least one unbounded component. This result was proved by considering the fast escaping set $A(f) = \bigcup_{n \in \mathbb{N}} f^{-n}(A_R(f))$, where $A_R(f) = \{ z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N} \}$. Here $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$, $M^n(r, f)$ denotes the $n$th iterate of $M$ with respect to $r$, and $R > 0$ is chosen so that $M(r, f) > r$ for $r \geq R$. The set $A(f)$ has many nice properties including the fact that all its components are unbounded – these properties are described in detail in [12].

There are many classes of transcendental entire functions for which the fast escaping set has the structure of a spider’s web – see [12], [8] and [17]. We say
that a set \( E \) has this structure if \( E \) is connected and there exists a sequence of bounded simply connected domains \( G_n \) such that
\[
\partial G_n \subset E, \quad G_n \subset G_{n+1}, \quad \text{for } n \in \mathbb{N}, \quad \text{and } \bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.
\]
As shown in [12], if \( A_R(f) \) has this structure then so do both \( A(f) \) and \( I(f) \), and hence Eremenko’s conjecture is satisfied. Also, the domains \( G_n \) can be chosen so that \( \partial G_n \subset A_R(f) \cap J(f) \) and so \( f \) has no unbounded Fatou components. This gives a surprising link between Eremenko’s conjecture and a conjecture of Baker that all the components of the Fatou set are bounded if \( f \) is a transcendental entire function of order less than \( 1/2 \). Recall that the order of a transcendental entire function \( f \) is defined to be
\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.
\]
For background and recent results on Baker’s conjecture, see [6], [7], [11] and [13]. It was shown in [11] (see also [12]) that all earlier partial results on Baker’s conjecture are in fact sufficient to imply the stronger result that \( A_R(f) \) is a spider’s web. Here we give a sharp condition on the growth of the maximum modulus that is sufficient to imply that \( A_R(f) \) is a spider’s web and hence that Baker’s conjecture and Eremenko’s conjecture are both satisfied. More precisely, we prove the following sufficient condition.

**Theorem 1.1.** Let \( f \) be a transcendental entire function and let \( R > 0 \) be such that \( M(r, f) > r \) for \( r \geq R \). Let
\[
R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log \log M(r)}{\log r}.
\]
If
\[
\sum_{n \in \mathbb{N}} \varepsilon_n < \infty,
\]
then \( A_R(f) \) is a spider’s web.

We obtained a closely related result in [11, Theorem 3] with the stronger hypothesis that \( \sum_{n \in \mathbb{N}} \sqrt{\varepsilon_n} < \infty \) and remarked there that the square root could be removed by introducing a more sophisticated argument. The method of proof given here is quite different, and more enlightening, than that used to prove [11, Theorem 3]. In fact, Theorem 1.1 follows surprisingly easily from a new local version of the classical \( \cos \pi \rho \) theorem; see Theorem 2.1.

**Remark.** Theorem 1.1 can be generalised to apply to the set of points that escape as fast as possible within a direct tract of a transcendental meromorphic function; see [3] for earlier results concerning the fast escaping set in a direct tract.

It turns out that the condition in Theorem 1.1 is, in a strong sense, best possible. In particular, the following result shows that the condition in Theorem 1.1 cannot be replaced by the weaker condition that \( \sum_{n \in \mathbb{N}} (\varepsilon_n)^c < \infty \), for some \( c > 1 \).

**Theorem 1.2.** There exist transcendental entire functions of the form
\[
f(z) = z^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right)^{2p_n},
\]
where...
where \( p_n \in \mathbb{N} \), for \( n \in \mathbb{N} \), and the sequence \((a_n)\) is positive and strictly increasing such that \( A(f) \cap (-\infty, 0) = \emptyset \); in particular, \( A(f) \) is not a spider’s web.

Moreover, if \((\delta_n)\) is a positive sequence such that

\[
\sum_{n \in \mathbb{N}} \delta_n = \infty,
\]

then we can choose the sequence \((a_n)_{n \in \mathbb{N}}\) and a value \( R > 0 \) in such a way that, with

\[
p_n = [\delta_n^{\delta_n/4}], \quad R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log \log M(r)}{\log r},
\]

there exists a subsequence \((n_k)\) such that

\[
(1.2) \quad \varepsilon_{n_k} \leq \delta_k + \frac{1}{2^{n_k}}, \quad \text{for} \ k \in \mathbb{N},
\]

and

\[
(1.3) \quad \varepsilon_{n_k+m} \leq \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \text{for} \ k \in \mathbb{N}, 1 \leq m < n_{k+1} - n_k.
\]

Since it is possible to choose a positive sequence \((\delta_n)\) with

\[
\sum_{n \in \mathbb{N}} \delta_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \delta_n = 0,
\]

Theorem 1.2 implies that there are functions of order zero for which \( A_R(f) \) fails to be a spider’s web. Thus new techniques are needed in order to solve Baker’s conjecture. One such technique is introduced in [13] where we show that all functions of order less than \( 1/2 \) with zeros on the negative real axis satisfy Baker’s conjecture and also satisfy Eremenko’s conjecture with \( I(f) \) being a spider’s web. Since functions of the form (1.1) with \( \limsup_{n \to \infty} \varepsilon_n < 1/2 \) are of this type, this gives the following corollary to Theorem 1.2 which answers a question in [12].

**Corollary 1.3.** There exist transcendental entire functions for which \( I(f) \) is a spider’s web but \( A(f) \) is not a spider’s web.

**Remark.** In fact we show in [13] that functions of order less than \( 1/2 \) with zeros on the negative real axis have the stronger property that \( Q(f) \) contains a spider’s web, where \( Q(f) \) is the quite fast escaping set. Thus Theorem 1.2 provides examples of functions for which \( Q(f) \neq A(f) \); these two sets are equal for many functions, including all functions in the Eremenko-Lyubich class \( \mathcal{B} \) as we show in [15].

The paper is arranged as follows. In Section 2 we prove Theorem 1.1 and then, in Section 3, we prove Theorem 1.2.

2. **Proof of Theorem 1.1**

Let \( f \) be a transcendental entire function and \( R > 0 \) be such that \( M(r) > r \) for \( r \geq R \). Recall that

\[
A_R(f) = \{ z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N} \}.
\]
and that $A_R(f)$ is a spider’s web if $A_R(f)$ is connected and there exists a sequence of bounded simply connected domains $G_n$ such that
\[
\partial G_n \subset A_R(f), \ G_n \subset G_{n+1}, \text{ for } n \in \mathbb{N}, \text{ and } \bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.
\]

In this section we prove Theorem 1.1 which gives a condition that is sufficient to ensure that $A_R(f)$ is a spider’s web. The key ingredient in our proof is the following result which can be viewed as a local version of the classical cosine theorem. For a discussion of results of this type, see [14].

**Theorem 2.1.** Let $f$ be a transcendental entire function. There exists $r(f) > 0$ such that, if
\[
\log M(r) \leq r^\alpha \quad \text{and} \quad r^{1-2\alpha} \geq r(f),
\]
for some $\alpha \in (0, 1/2)$, then there exists $t \in (r^{1-2\alpha}, r)$ such that
\[
\log m(t) > \log M(r^{1-2\alpha}) - 2.
\]

**Proof.** We apply the following result of Beurling [4, page 96]:

Let $f$ be analytic in $\{z : |z| < r_0\}$, let $0 \leq r_1 < r_2 < r_0$, and put
\[
E = \{t \in (r_1, r_2) : m(t) \leq \mu\}, \text{ where } 0 < \mu < M(r_1).
\]
Then
\[
\log \frac{M(r_2)}{\mu} > \frac{1}{2} \exp \left( \frac{1}{2} \int_E \frac{dt}{t} \right) \log \frac{M(r_1)}{\mu}.
\]

Taking $r_2 = r$, $r_1 = r^{1-2\alpha}$, $\mu = M(r^{1-2\alpha})/e^2$, and $r(f) > 0$ such that $M(r(f)) \geq e^2$, we deduce from (2.1) and (2.2) that, if $m(t) \leq \mu$ for $t \in (r^{1-2\alpha}, r)$, then
\[
r^{\alpha} \geq \log M(r) \geq \log \frac{M(r)}{\mu} > \frac{1}{2} \exp \left( \frac{1}{2} \int_{r^{1-2\alpha}}^{r} \frac{dt}{t} \right) \log \frac{M(r^{1-2\alpha})}{\mu} = r^{\alpha}.
\]
This is a contradiction and so there must exist $t \in (r^{1-2\alpha}, r)$ such that $m(t) > \mu$; that is,
\[
\log m(t) > \log \mu = \log M(r^{1-2\alpha}) - 2,
\]
as required. \qed

We also use the following results about spiders’ webs proved in [12].

**Lemma 2.2.** [12, Corollary 8.2] Let $f$ be a transcendental entire function and let $R > 0$ be such that $M(r) > r$ for $r \geq R$. Then $A_R(f)$ is a spider’s web if there exists a sequence $(\rho_n)$ such that, for $n \geq 0$,
\[
\rho_n > M^n(R)
\]
and
\[
m(\rho_n) \geq \rho_{n+1}.
\]

**Lemma 2.3.** [12, Lemma 7.1(d)] Let $f$ be a transcendental entire function, let $R > 0$ be such that $M(r) > r$ for $r \geq R$, and let $R' > R$. Then $A_{R'}(f)$ is a spider’s web if and only if $A_R(f)$ is a spider’s web.

In addition, we need the following property of the maximum modulus function, which was proved in this form in [11].
Lemma 2.4. Let $f$ be a transcendental entire function. Then there exists $R > 0$ such that, for all $r \geq R$ and all $c > 1$, 

$$M(r^c) \geq M(r)^c.$$ 

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $R > 0$ be such that, for $r \geq R$, Lemma 2.4 holds and $M(r) > r$. For $n \in \mathbb{N}$, let 

$$R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log M(r)}{\log r}.$$ 

Suppose that $\sum_{n \in \mathbb{N}} \varepsilon_n < \infty$. Then we can take $N$ sufficiently large to ensure that 

$$(2.5) \quad \sum_{n \geq N} \varepsilon_n < \frac{1}{8},$$ 

and

$$(2.6) \quad M(R_n)^{1/(8n^2)} = R_{n+1}^{1/(8n^2)} \geq e^2, \quad \text{for } n \geq N, \quad \text{and} \quad R_{N+1}^{1/4} \geq R_n \geq r(f),$$ 

where $r(f)$ is as defined in Theorem 2.1. Note that (2.6) is possible since $\log M(r) / \log r \to \infty$ and so, for large $n$, we have $\log R_{n+1} > 4 \log R_n$.

Now let 

$$r_n = M^{n+1} \left( R_{N+1}^{N+n}(1-2\varepsilon_m-1/(8m^2)) \right), \quad \text{for } n \geq 0.$$ 

We note that, for $n \geq 0$, it follows from (2.5) that 

$$\prod_{m=N}^{N+n} \left( 1 - 2\varepsilon_m - \frac{1}{8m^2} \right) > 1 - \sum_{m=N}^{N+n} 2\varepsilon_m - \sum_{m=N}^{N+n} \frac{1}{8m^2} \geq \frac{1}{2},$$ 

and so, by (2.6), 

$$R_{N+n+2} > r_n > M^{n+1}(R_{n+1}^{1/2}) \geq M^{n+1}(R_N^2) = R_{N+n+1}^2.$$ 

We claim that, for $n \geq 0$, there exists $\rho_n \in (R_{N+n+1}, r_n)$ with $m(\rho_n) > r_{n+1}$. Indeed, it follows from Theorem 2.1 (2.5), (2.6) and Lemma 2.4 that, for $n \geq 0$, there exists $\rho_n \in (r_n^{1-2\varepsilon_m} + 1, r_n) \subset (R_{N+n+1}, r_n)$ such that 

$$m(\rho_n) \geq \frac{1}{c^2} \cdot M(r_n^{1-2\varepsilon_m})$$ 

$$\geq M(r_n^{1-2\varepsilon_m} + 1) \cdot (1-1/(8m^2))$$ 

$$\geq M(r_n^{1-2\varepsilon_m} + 2)$$ 

$$\geq M(r_n^{1-2\varepsilon_m} + 1/(8m^2))$$ 

$$= M \left( \left( R_{N+1}^{N+n}(1-2\varepsilon_m-1/(8m^2)) \right)^{(1-2\varepsilon_m-1/(8m^2))} \right)$$ 

$$\geq M^{n+2} \left( R_{N+1}^{N+n}(1-2\varepsilon_m-1/(8m^2)) \right)$$ 

$$= r_{n+1}.$$ 

Thus, for $n \geq 0$, there exists $\rho_n > R_{N+n}$ with $m(\rho_n) \geq \rho_{n+1}$ and so, by Lemma 2.2, $A_{R_{N+1}^n}(f)$ is a spider’s web. It now follows from Lemma 2.3 that $A_R(f)$ is a spider’s web as claimed. □
3. Proof of Theorem 1.2

Let
\[ f(z) = z^3 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{2p_n}, \]
where the sequence \((a_n)\) is positive and strictly increasing. In addition, let \((\delta_n)\) be a positive sequence such that
\[ \sum_{n \in \mathbb{N}} \delta_n = \infty, \]
and let
\[ p_n = \left[ a_n^{\delta_n/4}/4 \right]. \]
Without loss of generality, we assume that
\[ \delta_n < 1/2, \quad \text{for } n \in \mathbb{N}. \]
Note that \(f((-\infty, 0]) \subset (-\infty, 0]\) and that \(m(r) = f(-r)\) and \(M(r) = f(r) > r^3\), for \(r > 0\). Further, \(M(r) > r\) for \(r \geq 1\).

We first show that the sequence \((a_n)\) can be chosen so that \(A(f) \cap (-\infty, 0] = \emptyset\).

We choose the values of \(a_n\) carefully, beginning with \(a_1\), then \(a_2\) and so on. Because of the way in which we choose the values of \(a_n\), it is helpful to introduce the function \(g\) defined by
\[ g(r) = \begin{cases} r^3, & 0 \leq r < a_1, \\ r^3 \prod_{a_n \leq r} \left(1 + \frac{r}{a_n}\right)^{2p_n}, & r \geq a_1. \end{cases} \]

Note that \(g\) is a strictly increasing function and that it is discontinuous at \(a_n\), for \(n \in \mathbb{N}\). A key property of \(g\) which we use repeatedly is that
\[ m(r) = -f(-r) < g(r) < M(r), \quad \text{for } r \geq 0. \]
Since \(g\) is increasing, \((3.4)\) implies that
\[ f([-r, 0]) \subset [-g(r), 0], \quad \text{for } r \geq 0. \]
We now set \(r_0 = 10\) and \(r_{n+1} = g(r_n) = g^{n+1}(10)\), for \(n \in \mathbb{N}\), and note that
\[ r_{n+1} \geq r_n^3, \quad \text{for } n \geq 0. \]
Also, it follows from \((3.5)\) that
\[ f^n((-r_m, 0]) \subset (-r_{m+n}, 0], \quad \text{for } n, m \in \mathbb{N}. \]

We begin by proving the following result.

**Lemma 3.1.** If there exists a sequence \((N_k)\) such that,
\[ f^{N_k}((-r_{N_k}, 0]) \subset (-r_{N_k}, 0] \]
and, for \(k \geq 2\),
\[ f^{N_k}((-r_{N_1 + \ldots + N_k - 1 + 2k}, 0]) \subset (-r_{N_1 + \ldots + N_k}, 0], \]
then \(A(f) \cap (-\infty, 0] = \emptyset\).
Proof. We first note that, if the hypotheses of Lemma 3.1 hold, then it follows from (3.7) and (3.9) that, for \( k \in \mathbb{N} \),
\[
f^{N_1 + \cdots + N_k}((-r_{2k}, 0]) = f^{N_k}(f^{N_1 + \cdots + N_{k-1}}((-r_{2k}, 0])
\subset f^{N_k}((-r_{N_1 + \cdots + 2k+1}, 0])
\subset (-r_{N_1 + \cdots + N_k}, 0].
\]
Thus
\[
f^{N_1 + \cdots + N_k}((-r_{2k}, 0]) \subset (-r_{N_1 + \cdots + N_k}, 0].
\] (3.10)

Now let \( z \in (-\infty, 0] \). There exists \( K \in \mathbb{N} \) such that, for \( k \geq K \), we have \( z \in (-r_k, 0] \) and hence, by (3.7), we have \( f^k(z) \in (-r_{2k}, 0] \). Thus, by (3.10) and (3.4), for \( k \geq K \),
\[
| f^{N_1 + \cdots + N_k + k}(z) | < r_{N_1 + \cdots + N_k} < M^{N_1 + \cdots + N_k}(10)
\]
and hence
\[
z \notin \{ z : |f^{n+k}(z)| \geq M^n(10) \text{ for } n \in \mathbb{N} \}.
\]
Thus \( A(f) \cap (-\infty, 0] = \emptyset \) as required. \( \square \)

We will show that we can choose the values of \( a_n \) in such a way that the hypotheses of Lemma 3.1 hold. In order to do this, it is helpful to set certain restrictions on our choice of values. Firstly, we choose \( a_1 \) and \( a_{n+1}/a_n, \ n \in \mathbb{N} \), sufficiently large to ensure that
\[
a_1^{\delta_1/4} \geq 4, \ a_{n+1} > a_n^2, \ a_{n+1}^{\delta_{n+1}/2} > 16a_n^{\delta_n}
\] (3.11)
and
\[
a_{n+1}^{\delta_{n+1}/16} > a_n^{\delta_n} \log a_{n+1}.
\] (3.12)

We note that (3.11) implies that
\[
p_1 \geq 1 \text{ and } p_{n+1} \geq 2p_n^2, \text{ for } n \in \mathbb{N}.
\] (3.13)

We also place certain restrictions on our choice of the values of \( a_n \) in relation to the values of \( r_n \):
\[
\text{if } a_k \in [r_n, r_{n+1}), \text{ then } a_m \notin [r_n, r_{n+4}) \text{ for } k, m \in \mathbb{N}, \ m \neq k.
\] (3.14)

We now show that, in order to prove that the hypotheses of Lemma 3.1 hold, it is sufficient to prove the following result.

**Lemma 3.2.** Suppose that, for some \( m \in \mathbb{N} \), we have defined the values of \( a_n \) for which \( a_n \leq r_m \) in such a way that they satisfy (3.11), (3.12) and (3.14). Then we can choose \( N \in \mathbb{N} \) and the values of \( a_n \) for which \( r_m < a_n \leq r_{m+N-1} \) in such a way that they satisfy (3.11), (3.12) and (3.14) and, no matter how the later values of \( a_n \) are chosen,
\[
f^N((-r_{m+1}, 0]) \subset (-r_{m+N}, 0].
\]

Proving Lemma 3.2 is the key part of the proof that we can choose the sequence \((a_n)\) so as to ensure that \( A(f) \cap (-\infty, 0] = \emptyset \). Before proving Lemma 3.2 we show that, if this result holds, then the hypotheses of Lemma 3.1 also hold. First, by applying Lemma 3.2 when \( m = 1 \) we see that there exists \( N_{1,1} \in \mathbb{N} \) and a choice of \( a_n \) for \( r_1 < a_n \leq r_{N_{1,1}} \) such that
\[
f^{N_{1,1}}((-r_2, 0]) \subset (-r_{N_{1,1}+1}, 0].
\] (3.15)
We then apply Lemma 3.2 with $m = N_{1,1}$ and deduce that there exists $N_{1,2} \in \mathbb{N}$ and a choice of $a_n$ for $r_{N_{1,1}} < a_n \leq r_{N_{1,1}+N_{1,2}-1}$ such that

$$f^{N_{1,2}}((-r_{N_{1,1}+1}, 0]) \subset (-r_{N_{1,1}+N_{1,2}}, 0].$$

It follows from (3.15) and (3.16) that

$$f^{N_{1,1}+N_{1,2}}(-r_2, 0] \subset f^{N_{1,2}}((-r_{N_{1,1}+1}, 0] \subset (-r_{N_{1,1}+N_{1,2}}, 0].$$

Putting $N_1 = N_{1,1} + N_{1,2}$, we deduce that we can choose the values of $a_n$ for which $r_1 < a_n \leq r_{N_1-1}$ in such a way that

$$f^{N_1}((-r_2, 0]) \subset (-r_{N_1}, 0].$$

Thus (3.8) holds.

Now suppose that, for some $k \geq 2$, we have defined $N_j$, for $1 \leq j \leq k-1$, and defined $a_n$, for $r_1 < a_n \leq r_{N_1+\ldots+N_{k-1}-1}$. We claim that we can use Lemma 3.2 to define $N_k \in \mathbb{N}$ and $a_n$ with $r_{N_1+\ldots+N_{k-1}-1} < a_n \leq r_{N_1+\ldots+N_{k-1}}$ such that (3.9) holds for $k$. The argument is similar to that given above. First, we apply Lemma 3.2 with $m = N_1 + \ldots + N_{k-1} + 2k-1$ to construct $N_{k,1}$ and $a_n$ with

$$r_{N_1+\ldots+N_{k-1}+2k-1} < a_n \leq r_{N_1+\ldots+N_{k-1}+N_{k,1}+2k-2}$$

such that

$$f^{N_{k,1}}((-r_{N_1+\ldots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\ldots+N_{k-1}+N_{k,1}+2k-1}, 0].$$

Then, for $2 \leq j \leq 2k$, we apply Lemma 3.2 repeatedly with

$$m = N_1 + \ldots + N_{k-1} + N_{k,1} + \ldots + N_{k,j-1} + 2k-j$$

to construct $N_{k,j}$ and $a_n$ with

$$r_{N_1+\ldots+N_{k-1}+N_{k,1}+\ldots+N_{k,j-1}+2k-j} < a_n \leq r_{N_1+\ldots+N_{k-1}+N_{k,1}+\ldots+N_{k,j}+2k-j-1}$$

such that

$$f^{N_{k,j}}((-r_{N_1+\ldots+N_{k-1}+N_{k,1}+\ldots+N_{k,j-1}+2k-j+1}, 0]) \subset (-r_{N_1+\ldots+N_{k-1}+N_{k,1}+\ldots+N_{k,j}+2k-j}, 0].$$

Putting $N_k = N_{k,1} + \ldots + N_{k,2k}$, we deduce that $a_n$ can be chosen with

$$r_{N_1+\ldots+N_{k-1}-1} < a_n \leq r_{N_1+\ldots+N_k-1}$$

such that

$$f^{N_k}((-r_{N_1+\ldots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\ldots+N_k}, 0]$$

and hence (3.9) holds for $k$.

So, it remains to prove Lemma 3.2.

We begin by proving four lemmas. The first describes the extent to which $f$ is small close to a zero at $-a_k$, where $k \in \mathbb{N}$.

**Lemma 3.3.** For each $k \in \mathbb{N}$,

$$|f(z)| < 1, \text{ for } z \in (-a_k, -a_k^{1-\delta_k/16}).$$
Proof. This holds since, for such a \( z \), it follows from (3.1), (3.11), (3.12) and (3.13) that

\[
|f(z)| \leq a_k^3 \left( 1 - \frac{a_k^{1-\delta_k/16}}{a_k} \right)^{2pk/k-1} \prod_{m=1}^{k-1} \left( 1 + \frac{a_k}{a_m} \right)^{2p_m} \prod_{m=k+1}^{m} \left( 1 + \frac{a_k}{a_m} \right)^{2p_m}
\]

\[
\leq \left( 1 - \frac{1}{a_k^{\delta_k/16}} \right)^{\frac{a_k^{1/2}}{\delta_k/16}} a_k^{3+2p_1+\ldots+2p_{k-1}} \prod_{m=k+1}^{1/2} \left( 1 + \frac{1}{a_m^{1/2m-k}} \right)
\]

\[
\leq \exp(-\delta_k/16) a_k^{-\delta_k/16} e^{1+1/2+1/4+\ldots}
\]

\[
\leq a_k^{-\delta_k/16} \exp(-\delta_k/16) < 1.
\]

□

The second lemma shows that there is a large increase in the size of \( g(r) \) at \( r = a_k \), where \( k \in \mathbb{N} \).

**Lemma 3.4.** For each \( k \in \mathbb{N} \),

\[
\log g(a_k) \geq p_k^{1/2} \log(a_k^{1-\delta_k/16}).
\]

Proof. For \( k \in \mathbb{N} \), it follows from (3.11) that

\[
g(a_k^{1-\delta_k/16}) < a_k^3 \prod_{m=1}^{k-1} \left( 1 + \frac{a_k}{a_m} \right)^{2p_m}
\]

\[
< a_k^{3+2\sum_{m=1}^{k-1} p_m} \leq a_k^{4p_{k-1}}
\]

and

\[
g(a_k) \geq 2^{2pk}.
\]

Thus, by (3.11), (3.12) and (3.13),

\[
\frac{\log g(a_k)}{\log g(a_k^{1-\delta_k/16})} \geq \frac{2pk \log 2}{4p_{k-1} \log a_k} > \frac{p_k}{3p_{k-1} \log a_k} > p_k^{1/2}.
\]

□

The third lemma shows that \( \log g \) has a convexity property.

**Lemma 3.5.** Let \( r > 0 \) and \( t \geq 2 \). Then

\[
\log g(r^t) \geq t \log g(r).
\]

Proof. Let \( r > 0 \) and \( t \geq 2 \). We have

\[
g(r^t) \geq r^{3t} \prod_{a_m \leq r} \left( 1 + \frac{r^t}{a_m} \right)^{2p_m}
\]

and

\[
g(r)^t = r^{3t} \prod_{a_m \leq r} \left( 1 + \frac{r}{a_m} \right)^{2p_m t}
\]

Thus it is sufficient to show that

\[
\left( 1 + \frac{r}{a_m} \right)^t \leq \left( 1 + \frac{r^t}{a_m} \right)
\]
when \( a_m \leq r \). This is true since it follows from (3.11) that, for \( a_m \leq r \) and \( t \geq 2 \),
\[
\left(1 + \frac{r}{a_m}\right)^t \leq \left(\frac{r \, 1/2}{a_m}\right)^t = \frac{r^t}{a_m^{t/2}} < 1 + \frac{r^t}{a_m}.
\]

The fourth lemma gives an upper bound on the growth of \( g \) on intervals where no point is the modulus of a zero of \( f \).

**Lemma 3.6.** Let \( r > 0 \), \( 0 < s < 1/2 \) and \( t > 1 \) and suppose that there are no values of \( n \in \mathbb{N} \) for which \( a_n \in (r^s, r^t] \). Then
\[
\log g(r^t) \leq t(1 + 2s) \log g(r).
\]

**Proof.** It follows from (3.11) that
\[
g(r^t) = r^{3t} \prod_{a_m \leq r^s} \left(1 + \frac{r^t}{a_m}\right)^{2p_m} < r^{3t} \prod_{a_m \leq r^s} r^{2p_m} = r^{t(3 + \sum_{a_m \leq r^s} 2p_m)}
\]
and
\[
g(r) > r^3 \prod_{a_m \leq r^s} \left(\frac{r}{a_m}\right)^{2p_m} > r^{3 + \sum_{a_m \leq r^s} 2p_m(1-s)}.
\]
Thus
\[
\log g(r^t)/\log g(r) < t/(1-s) \leq t(1 + 2s),
\]
since \( s < 1/2 \).

We are now in a position to prove Lemma \(3.2\).

**Proof of Lemma 3.2.** Suppose that \( m \in \mathbb{N} \) and that we have defined the values of \( a_n \) for which \( a_n \leq r_m \). We now define a sequence \((s_k)\), \(0 \leq k \leq N\), inductively according to certain rules that we give below. Each time we define a value \( s_k \), we also add a zero of \( f \) at \(-s_k\) provided this is allowed by (3.11), (3.12) and (3.14); no other zeros of \( f \) are added. We choose our values \( s_k \) in such a way that
\[
(3.18) \quad r_{m+k} \leq s_k \leq r_{m+k+1}, \text{ for } 0 \leq k < N,
\]
\[
(3.19) \quad s_N \leq r_{m+N}
\]
and
\[
(3.20) \quad f^k((-r_{m+1}, 0]) \subset (-s_k, 0], \text{ for } 0 \leq k \leq N.
\]
The result of Lemma 3.2 follows directly from (3.19) and (3.20). The difficult part of the proof is to show that there exists an \( N \in \mathbb{N} \) for which (3.19) is satisfied.

We define our sequence \((s_k)\) as follows:

- set \( s_0 = r_{m+1} \);
- if \( s_k > r_{m+k} \) and there is a zero of \( f \) at \(-s_k\), then we set
  \[
  (3.21) \quad s_{k+1} = g(s_k^{1 - \delta_{n_k}/16});
  \]
- if \( s_k > r_{m+k} \) and there is no zero of \( f \) at \( s_k \), then we set
  \[
  (3.22) \quad s_{k+1} = g(s_k);
  \]
- if \( s_k \leq r_{m+k} \), then we terminate the sequence \((s_k)\).
It follows from Lemma 3.3 and (3.5) that, with this construction, (3.18), (3.19) and (3.20) are indeed satisfied.

It remains to prove that there exists $K \in \mathbb{N}$ such that the sequence terminates at $s_K$; that is, if

$$T_k = \frac{\log s_k}{\log r_{m+k}},$$

then there exists $K \in \mathbb{N}$ such that $T_K \leq 1$.

We introduce the following terminology. We let $L$ denote the largest integer for which $a_L \leq r_m$ and define a (finite) subsequence $(k_n)$ such that

$$a_{L+n} = s_{k_n}, \text{ for } n = 1, 2, \ldots.$$  \hfill (3.23)

The main idea is to show that, for each $n \geq 2$ we have that $T_{k_n+1}$ is less than $T_{k_n}$, with $k_n$ defined as above. These decreases counteract the small increases that may occur from $T_k$ to $T_{k+1}$ for other values of $k$ and, for $n$ large enough, they will combine together to cause $T_{k_n+1}$ to drop below 1.

We first estimate some quantities that will be useful in our calculations. We begin by noting that it follows from (3.23), (3.18) and Lemma 3.4 that, for $n \geq 1$,

$$\log r_{m+k_n+2} = \log g(r_{m+k_n+1}) \geq \log g(s_{k_n}) \geq p_{L+n}^{1/2} \log g(s_{k_n}^{1-\delta_{L+n}/16}).$$

Thus, by (3.21)

$$\log r_{m+k_n+2} \geq p_{L+n}^{1/2} \log s_{k_n+1}, \text{ for } n \geq 1.$$  \hfill (3.24)

Together with (3.6), (3.24) implies that

$$\log r_{m+k_n+q} \geq 3^{q-2} p_{L+n}^{1/2} \log s_{k_n+1}, \text{ for } q \geq 2, n \geq 1.$$  \hfill (3.25)

Together with Lemma 3.5 (3.24) implies that

$$\frac{\log s_{k_n+q}}{\log r_{m+k_n+q+1}} \leq \frac{\log g^{-1}(s_{k_n+1})}{\log g^{-1}(r_{m+k_n+2})} \leq \frac{\log s_{k_n+1}}{\log r_{m+k_n+2}} \leq \frac{1}{p_{L+n}^{1/2}}, \text{ for } q \geq 2, n \geq 1.$$  \hfill (3.26)

Now fix $n \geq 2$ and write

$$t_{n,q} = T_{k_n+q} = \frac{\log s_{k_n+q}}{\log r_{m+k_n+q}}, \text{ for } q \geq 2.$$  \hfill (3.27)

For $2 \leq q < k_{n+1} - k_n$, there are no zeros of $f$ with modulus in the interval $(s_{k_n}, s_{k_n+q})$ and so it follows from (3.22), Lemma 3.6 and (3.25) that, for such $q$,

$$\log s_{k_n+q+1} = \log g(s_{k_n+q})$$

$$\leq t_{n,q} \left( 1 + 2 \frac{\log s_{k_n}}{\log r_{m+k_n+q}} \right) \log g(r_{m+k_n+q})$$

$$= t_{n,q} \left( 1 + 2 \frac{\log s_{k_n}}{\log r_{m+k_n+q}} \right) \log r_{m+k_n+q+1}$$

$$\leq t_{n,q} \left( 1 + \frac{2}{3^{q-2} p_{L+n}^{1/2}} \right) \log r_{m+k_n+q+1}. $$
Thus, for $2 \leq q < k_{n+1} - k_n$, we have

$$t_{n,q+1} \leq t_{n,q} \left( 1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}} \right).$$  

(3.27)

For $q = k_{n+1} - k_n$, there are no zeros of $f$ with modulus in the interval $(s_{k_n}, s_{k_n+q})$ and so it follows from (3.21), Lemma 3.6 and (3.25) that

$$\log s_{k_n+q+1} = \log g(s_{k_n+q})$$

$$\leq t_{n,q} \left( 1 - \frac{\delta_{L+n+1}}{16} \right) \left( 1 + \frac{2 \log s_{k_n}}{\log r_{m+k_n+q}} \right) \log g(r_{m+k_n+q+1})$$

$$= t_{n,q} \left( 1 - \frac{\delta_{L+n+1}}{16} \right) \left( 1 + \frac{2 \log s_{k_n}}{\log r_{m+k_n+q}} \right) \log r_{m+k_n+q+1}$$

$$\leq t_{n,q} \left( 1 - \frac{\delta_{L+n+1}}{16} \right) \left( 1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}} \right) \log r_{m+k_n+q+1}.$$

Thus, for $q = k_{n+1} - k_n$, we have

$$t_{n,q+1} \leq t_{n,q} \left( 1 - \frac{\delta_{L+n+1}}{16} \right) \left( 1 + \frac{2}{3^{q-2}p_{L+n}^{1/2}} \right).$$

(3.28)

Lastly, it follows from (3.14) that, if $q = k_{n+1} - k_n + 1$, then $q - 1 \geq 2$. Also, there are no zeros of $f$ with modulus in the interval $(s_{k_{n+1}}, s_{k_{n+1}+1}) = (s_{k_{n+1}}, s_{k_n+q})$ and so it follows from Lemma 3.6 and (3.26) that

$$\log s_{k_n+q+1} = \log g(s_{k_n+q})$$

$$\leq t_{n,q} \left( 1 + \frac{2 \log s_{k_{n+1}}}{\log r_{m+k_n+q}} \right) \log g(r_{m+k_n+q})$$

$$= t_{n,q} \left( 1 + \frac{2 \log s_{k_{n+1}}}{\log r_{m+k_n+q}} \right) \log r_{m+k_n+q+1}$$

$$\leq t_{n,q} \left( 1 + \frac{2}{p_{L+n}^{1/2}} \right) \log r_{m+k_n+q+1}.$$

Thus, for $q = k_{n+1} - k_n + 1$, we have

$$t_{n,q+1} \leq t_{n,q} \left( 1 + \frac{2}{p_{L+n}^{1/2}} \right).$$

(3.29)
It follows from (3.27), (3.28), (3.29) and (3.13) that, for $M \geq 2$, we have
\[
T_{kM+1+2} = t_{M,kM+1-kM+2}
\]
\[
= t_{2,2} \prod_{n=2}^{M} \prod_{q=2}^{k_n+1} \frac{t_{n,q+1}}{t_{n,q}}
\]
\[
\leq t_{2,2} \prod_{n=2}^{M} \left(1 + \frac{2}{p_{L+n}}\right)^{\delta_{L+n+1}} \prod_{q=2}^{k_n+1} \left(1 + \frac{2}{3^{q-2}p_{L+n}}\right)
\]
\[
\leq t_{2,2} \prod_{n=2}^{M} \left(1 + \frac{2}{p_{L+n}}\right)^{3^{m+1}} \prod_{q=2}^{k_n+1} \left(1 - \frac{\delta_{L+n+1}}{16}\right).
\]

It follows from (3.13) that $\sum_{n \in \mathbb{N}} \frac{1}{p_{L+n}} < \infty$ and so, since $\sum_{n \in \mathbb{N}} \delta_{L+n+1} = \infty$, we deduce that, for $M$ sufficiently large, $T_{kM+1+2} \leq 1$, as required.

We have now proved Lemma 3.2. As noted earlier, this is sufficient to imply that the hypotheses of Lemma 3.1 hold and hence that $A(f) \cap (-\infty, 0] = \emptyset$ as required.

We complete the proof of Theorem 1.2 by showing that, in addition, conditions (1.2) and (1.3) are satisfied. That is, we prove the following.

**Lemma 3.7.** Let
\[
\varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log \log M(r)}{\log r}.
\]

There exists a subsequence $(n_k)$ such that
\[
\varepsilon_{n_k} \leq \delta_k + \frac{1}{2^{n_k}}, \quad \text{for } k \in \mathbb{N},
\]
and

\[
\varepsilon_{n_k+m} \leq \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \text{for } k \in \mathbb{N}, 1 \leq m < n_{k+1} - n_k.
\]

**Proof.** We begin by setting $R_0 = r_0 = 10$ and defining $R_{n+1} = M(R_n)$, for $n \in \mathbb{N}$. Clearly $R_n \geq r_n$ by (3.1) and
\[
R_{n+1} \geq R_n^3, \quad \text{for } n \in \mathbb{N}.
\]

We claim that
\[
\text{if } a_k \in [R_n, R_{n+1}], \text{ then } a_m \notin [R_n, R_{n+2}) \text{ for } k, m \in \mathbb{N}, \ m \neq k.
\]

In order to deduce this from (3.14), it is sufficient to show that, if $r_p \in [R_n, R_{n+1})$, for some $p, n \in \mathbb{N}$, then $r_{p+2} > R_{n+1}$. We prove this in two steps. Firstly, we note that if $r_p \in [R_n, R_n^3)$, for some $p, n \in \mathbb{N}$, then it follows from (3.6) that $r_{p+1} \geq r_{p}^3 \geq R_n^3$. Secondly, if $r_p \in [R_n^3, R_{n+1})$, for some $p, n \in \mathbb{N}$, then we claim that
\[
r_{p+1} = g(r_p) \geq g(R_n^3) > M(R_n) = R_{n+1}.
\]
This is true since, if \( k \) is the smallest integer such that \( a_k > R_n^3 \), then

\[
g(R_n^3) = R_n^3 \prod_{m=1}^{k-1} \left(1 + \frac{R_n^3}{a_m}\right)^{2p_m}
\]

and so, by (3.1) and (3.11),

\[
M(R_n) = f(R_n) = R_n^3 \prod_{m=1}^{k-1} \left(1 + \frac{R_n^3}{a_m}\right)^{2p_m} < g(R_n^3) R_n^3 \prod_{m=1}^{k-1} \left(1 + \frac{R_n^3}{a_m}\right)^{2p_m} \leq g(R_n^3).
\]

Thus (3.36) does indeed hold and, by the reasoning above, this is sufficient to show that (3.34) holds.

Now, for \( k \in \mathbb{N} \), we choose \( n_k \in \mathbb{N} \) such that \( a_k \in [R_{n_k}, R_{n_k+1}) \). Then, by (3.34), this defines a sequence \((n_k)\) with \( n_j \neq n_k \) for \( j \neq k \). Now suppose that \( r \in [R_{n_k}, R_{n_k+1}] \), for some \( k \in \mathbb{N} \). It follows from (3.11) and (3.34) that

\[
M(r) = f(r) \leq r^3 \left(1 + \frac{r}{a_k}\right)^{2p_k} \prod_{m=1}^{k-1} \left(1 + \frac{r}{a_m}\right)^{2p_m} \prod_{m=k+1}^{\infty} \left(1 + \frac{r}{a_m}\right)^{2p_m} \leq \left(1 + \frac{r}{a_k}\right)^{2p_k} r^{3+2p_1+\cdots+2p_{k-1}} \prod_{m=k+1}^{\infty} \left(1 + \frac{r}{a_m}\right)^{1/2} a_k^{1/2} \leq \left(1 + \frac{r}{a_k}\right)^{a_k^{1/2}} r^{a_k^{1/2} - 1/2} e^{1+1/2+1/4+\cdots}
\]

and so

\[
M(r) < e^{2r^a_k^{1/2} - 1/2} \left(1 + \frac{r}{a_k}\right)^{a_k^{1/2}}.
\]

If \( r < a_k^{1/2} \), then it follows from (3.2) and (3.36) that

\[
M(r) < e^{2r^a_k^{1/2} - 1/2} \left(1 + \frac{r}{a_k}\right)^{a_k^{1/2}} < e^{3r^a_k^{1/2}}
\]

and hence, since \( r \geq R_1 \geq 1000 \),

\[
\frac{\log \log M(r)}{\log r} < \frac{\delta_k \log r + 2 \log \log r}{\log r} = \delta_k + 2 \frac{\log \log r}{\log r} \leq \delta_k + 2 \frac{\log \log R_{n_k}}{\log R_{n_k}}.
\]

It follows from (3.33) that, in this case,

\[
\frac{\log \log M(r)}{\log r} \leq \delta_k + 2 \frac{\log (3^{n_k} \log 10)}{3^{n_k} \log 10} < \delta_k + \frac{1}{2^{n_k}}.
\]
If \( a_k^{1/2} \leq r \leq a_k \), then
\[
\left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}} = \left(1 + \frac{r}{a_k}\right)^{(a_k/r)^{\delta_k} a_k^{\delta_k}} < \left(1 + \frac{r}{a_k}\right)^{(a_k/r)^{\delta_k}} \leq e^{r^{\delta_k}}
\]
and, if \( r > a_k \), then
\[
\left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}} < r^{a_k^{\delta_k}} < r^{r^{\delta_k}}.
\]
So, if \( r \geq a_k^{1/2} \), it follows from (3.36) and (3.11) that
\[
M(r) < e^{2r^{a_k^{\delta_k-1}}} r^{r^{\delta_k}} < e^{2r^{a_k^{\delta_k/2}}} r^{r^{\delta_k}} < e^{2r^{2r^{\delta_k}}}
\]
and hence
\[
\frac{\log \log M(r)}{\log r} < \frac{\delta_k \log r + 2 \log \log r}{\log r} = \delta_k + 2 \frac{\log \log r}{\log r} \leq \delta_k + 2 \frac{\log \log R_{nk}}{\log R_{nk}}.
\]
As before, it follows from (3.33) that
\[
(3.38) \quad \frac{\log \log M(r)}{\log r} \leq \delta_k + \frac{1}{2^{n_k}}.
\]
Together with (3.37), this implies that (3.31) holds.

Now suppose that \( r \in [R_{nk+m}, R_{nk+m+1}) \), for some \( k \in \mathbb{N} \), \( 1 \leq m < n_{k+1} - n_k \).
It follows from (3.11) and (3.33) that
\[
M(r) = f(r) \leq r^3 \prod_{m=1}^{k} \left(1 + \frac{r}{a_m}\right)^{2^{p_m} \prod_{m \geq k+1} \left(1 + \frac{r}{a_m}\right)^{2^{p_m}}} \\
\leq r^{3+2^{p_1+\ldots+2^{p_k}}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}ight) a_m^{1/2} \\
\leq r^{a_k^{\delta_k}} e^{1+1/2+1/4+\ldots} \\
\leq e^{2r^{a_k^{\delta_k}}} \leq e^{2r^{R_{nk+1}^{\delta_k}}} \\
< e^{2r^{r^{\delta_k}/3^{m-1}}}
\]
Thus
\[
\frac{\log \log M(r)}{\log r} < \frac{\delta_k \log r/3^{m-1} + 2 \log \log r}{\log r} < \frac{\delta_k}{3^{m-1}} + 2 \frac{\log \log R_{nk+m}}{\log R_{nk+m}}.
\]
As before, it follows from (3.33) that
\[
(3.39) \quad \frac{\log \log M(r)}{\log r} \leq \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_{m+m}}}
\]
and so (3.32) holds. \(\square\)
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