

BOUNDARIES OF ESCAPING FATOU COMPONENTS

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ABSTRACT. Let f be a transcendental entire function and U be a Fatou component of f . We show that if U is an escaping wandering domain of f , then most boundary points of U (in the sense of harmonic measure) are also escaping. In the other direction we show that if enough boundary points of U are escaping, then U is an escaping Fatou component. Some applications of these results are given; for example, if $I(f)$ is the escaping set of f , then $I(f) \cup \{\infty\}$ is connected.

1. INTRODUCTION

Let f be a transcendental entire function and denote by f^n , $n = 0, 1, 2, \dots$, the n th iterate of f . The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n \in \mathbb{N}}$ forms a normal family in some neighborhood of z . The components of $F(f)$ are called *Fatou components*. The complement of $F(f)$ is called the *Julia set* $J(f)$. An introduction to the properties of these sets can be found in [3].

The set $F(f)$ is completely invariant, so for any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \dots$, a component of $F(f)$, which we call U_n , such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of *period* p , assuming p to be minimal. There are then four possible types of periodic components; see [3, Theorem 6]. If $U_n \neq U_m$ for $m \neq n$, then we say that U is a *wandering domain* of f .

The *escaping set*

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

was first studied for a general transcendental entire function f by Eremenko [7]. He proved that

$$(1.1) \quad I(f) \cap J(f) \neq \emptyset \quad \text{and} \quad \partial I(f) = J(f),$$

and also that $\overline{I(f)}$ has no bounded components. Eremenko remarked that it is plausible that all the components of $I(f)$ are unbounded, a statement now known as *Eremenko's conjecture* that remains open in spite of much work on it and many partial results.

Any Fatou component that meets $I(f)$ must lie in $I(f)$ by normality; we call such components *escaping Fatou components*. Escaping wandering domains can be bounded or unbounded, and escaping periodic Fatou components are all unbounded; the latter are called *Baker domains*. This paper gives results about the relationship between an escaping Fatou component and its boundary, and describes some consequences of these results.

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For the function $f(z) = z + 1 + e^{-z}$, studied by Fatou in [8], the set $F(f)$ is a completely invariant Baker domain, whose boundary is $J(f)$. So in this example an escaping Fatou component has many boundary points that are *not* in $I(f)$. It is natural to ask whether every escaping Fatou component of a transcendental entire function must have at least one boundary point in $I(f)$. We have the following partial result.

Theorem 1.1. *Let f be a transcendental entire function and let U be a wandering domain of f such that $U \subset I(f)$. Then*

$$\partial U \cap I(f) \neq \emptyset.$$

Moreover, the set $\partial U \cap I(f)^c$ has zero harmonic measure relative to U .

The proof of Theorem 1.1 can be adapted to show that $\partial U \cap I(f) \neq \emptyset$ for many Baker domains (see Section 2, Remark 2), but it remains open whether this conclusion holds whenever U is a Baker domain.

In the other direction to Theorem 1.1, we can ask whether a Fatou component must be escaping if a large enough subset of its boundary is escaping. We have the following result in this direction.

Theorem 1.2. *Let f be a transcendental entire function and let U be a Fatou component of f .*

- (a) *If $\partial U \cap I(f)$ has positive harmonic measure relative to U , then $U \subset I(f)$.*
- (b) *If $\partial U \cap A(f)$ has positive harmonic measure relative to U , then $U \subset A(f)$.*

In particular, if U is a Fatou component of f such that $\partial U \subset I(f)$, then $U \subset I(f)$, and a similar result holds for $A(f)$.

Here $A(f)$ is the *fast escaping set*, introduced by Bergweiler and Hinkkanen in [5], which can be defined as follows (see [17]):

$$A(f) = \{z : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\}.$$

Here $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ and $R > 0$ is such that $M(r) > r$ for $r \geq R$ or equivalently such that $M^n(R) \rightarrow \infty$ as $n \rightarrow \infty$.

Many of the properties of $A(f)$ are stronger than those of $I(f)$ (see Section 4). For example, $A(f)$ has the following stronger property than that given for $I(f)$ in Theorem 1.1 (see [17, Theorem 1.2]):

$$(1.2) \quad \text{if } U \text{ is a Fatou component of } f \text{ that meets } A(f), \text{ then } \overline{U} \subset A(f).$$

Note that it is possible for a Fatou component to lie in $A(f)$; for example,

$$(1.3) \quad \text{if } U \text{ is a multiply connected Fatou component of } f, \text{ then } \overline{U} \subset A(f),$$

(see [14]) and Bergweiler has constructed a transcendental entire function with both simply connected and multiply connected Fatou components in $A(f)$ (see [4]). All Fatou components in $A(f)$ are wandering domains (see [5]).

We prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. In Section 4 we give two consequences of these theorems, one of which states that if f is a transcendental entire function, then $I(f) \cup \{\infty\}$ is connected, and the other of which is an improved version of a result from [15] which gives a sufficient condition for $I(f)$ to be connected; the proof we give of the latter result was inspired by

an unpublished idea of Professor Noel Baker. This section also includes a short proof that $I(f) \neq \emptyset$, which may be of independent interest.

Section 5 is concerned with the components of $I(f)$; for example, we give a new sufficient condition for the components of $I(f)$ to be unbounded and prove that various sets, such as $I(f)$ and $A(f)$, and their complements, are either connected or have infinitely many components.

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2. PROOF OF THEOREM 1.1

First we recall that for a domain G and a set $E \subset \partial G$, the *harmonic measure* of E at z relative to G , denoted by $\omega(z, E, G)$, is the solution of the Dirichlet problem in G (found by using the Perron method) with boundary values given by the characteristic function χ_E . See [9] or [12], for example, for the solution of the Dirichlet problem and the properties of harmonic measure.

In any simply connected domain G in \mathbb{C} all points $\zeta \in \partial G$ are *regular* for the Dirichlet problem (see [12]); that is, if ϕ is any real-valued function on ∂G which is continuous at $\zeta \in \partial G$ and H_ϕ is the solution of the Dirichlet problem in G with boundary values ϕ , then $H_\phi(z) \rightarrow \phi(\zeta)$ as $z \rightarrow \zeta$.

Proof of Theorem 1.1. Suppose that U is an escaping wandering domain of f . Fix $z_0 \in U$ and, for $n \geq 0$, let $z_n = f^n(z_0)$ and U_n be the Fatou component that contains $f^n(U)$. In view of (1.3), we can assume that each U_n is simply connected.

Let $R > 0$ and put

$$B_n = B_n(R) = \{z \in \partial U : |f^n(z)| \leq R\}, \quad n \in \mathbb{N}.$$

To prove that $\partial U \cap I(f)^c$ has zero harmonic measure relative to U , it is sufficient to show that, for each $R > 0$,

$$(2.1) \quad \bigcap_{m \geq 1} \bigcup_{n \geq m} B_n = \{z \in \partial U : |f^n(z)| \leq R \text{ for infinitely many } n\}$$

has harmonic measure 0 relative to U .

Define $\Delta = \{z : |z| > R\} \cup \{\infty\}$, considered as a disc in the Riemann sphere $\hat{\mathbb{C}}$, and choose $N = N(R)$ such that $|z_n| > 2R$ for $n \geq N$. Then, for $n \geq N$, define

$$E_n = \partial U_n \cap \{z : |z| \leq R\},$$

V_n to be the component of $U_n \cap \Delta$ that contains z_n ,

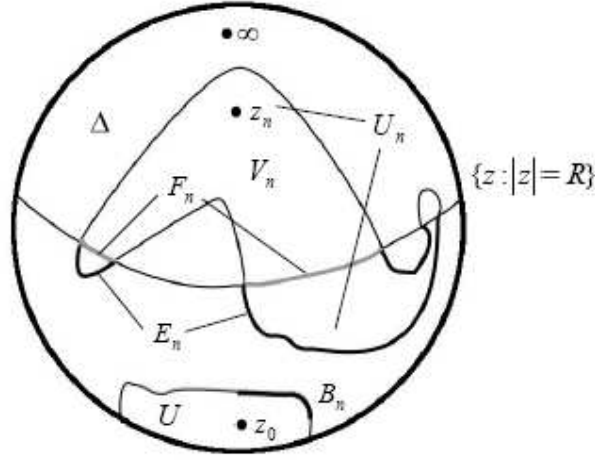
$$F_n = \partial V_n \cap \{z : |z| = R\};$$

see Figure 1. We have

$$(2.2) \quad \omega(z, E_n, U_n) \leq \omega(z, F_n, \Delta), \quad \text{for } z \in V_n, n \geq N.$$

This inequality is a special case of Carleman's domain extension principle, and it is proved as follows. The function

$$u(z) = \begin{cases} 0, & z \in \Delta \setminus V_n, \\ \omega(z, E_n, U_n), & z \in V_n, \end{cases}$$

FIGURE 1. The sets U , B_n , Δ , U_n , V_n , E_n and F_n

is subharmonic in Δ because $\omega(z, E_n, U_n)$ has boundary value 0 at all points of $\partial V_n \cap \Delta$ (since V_n is simply connected and hence regular for the Dirichlet problem). Also,

$$\limsup_{z \rightarrow \zeta} u(z) \leq \begin{cases} 0, & \text{for } \zeta \in \partial\Delta \setminus F_n, \\ 1, & \text{for } \zeta \in F_n, \end{cases}$$

because F_n is a closed subset of $\partial\Delta$. So (2.2) holds by the maximum principle for subharmonic functions (see [12, Theorem 2.3.1]).

By conformally mapping Δ onto $\{z : |z| < 1/R\}$ and applying Harnack's inequality (see [12, Theorem 1.3.1]) in that disc, we obtain

$$(2.3) \quad \omega(z_n, F_n, \Delta) \leq 3\omega(\infty, F_n, \Delta), \quad \text{for } n \geq N,$$

as $|z_n| > 2R$ for $n \geq N$. Since U is a wandering domain, the sets F_n are disjoint, so we deduce from (2.2) and (2.3) that

$$(2.4) \quad \sum_{n \geq N} \omega(z_n, E_n, U_n) \leq 3 \sum_{n \geq N} \omega(\infty, F_n, \Delta) \leq 3\omega(\infty, \partial\Delta, \Delta) = 3.$$

Now $f^n(U) \subset U_n$ and $f^n(B_n) \subset E_n$, so

$$\omega(z, B_n, U) \leq \omega(f^n(z), E_n, U_n), \quad \text{for } z \in U, n \geq N,$$

by [12, Theorem 4.3.8]. In particular,

$$\omega(z_0, B_n, U) \leq \omega(z_n, E_n, U_n), \quad \text{for } n \geq N.$$

Hence, by (2.4),

$$(2.5) \quad \sum_{n \geq N} \omega(z_0, B_n, U) \leq \sum_{n \geq N} \omega(z_n, E_n, U_n) < \infty,$$

so

$$\omega(z_0, \bigcup_{n \geq m} B_n, U) \leq \sum_{n \geq m} \omega(z_0, B_n, U) \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

which gives (2.1). \square

Remarks

1. It is natural to ask if there is an example of an entire function with an escaping wandering domain that has at least one non-escaping boundary point.

2. The proof of Theorem 1.1 can be adapted to show that, for many Baker domains U , the set $\partial U \cap I(f)^c$ has zero harmonic measure relative to U . For example, suppose that U is an invariant Baker domain in which there is an orbit $z_n = f^n(z_0)$, $n \in \mathbb{N}$, such that

$$(2.6) \quad |z_{n+1}| \geq k|z_n|, \quad \text{for } n \in \mathbb{N},$$

where $k > 1$. This is the case for several types of Baker domains; see [13, Section 2] for examples. Then, with the notation from the proof of Theorem 1.1, we have $U_n = U$ and $E_n = \partial U \cap \{z : |z| \leq R\} = E$, say, for $n \in \mathbb{N}$, and

$$(2.7) \quad \omega(z, E, U) \leq \frac{C}{|z|^{1/2}}, \quad \text{for } z \in U \cap \Delta,$$

where $C > 0$ is an absolute constant. Since U is simply connected, this last harmonic measure estimate can be obtained from the Beurling projection theorem (see [9, Theorem 9.2, page 105]), by exchanging the roles of 0 and ∞ . By (2.6) and (2.7), we deduce that (2.5) holds, so the conclusion of Theorem 1.1 also holds in this situation.

3. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 uses the following version of the maximum principle for subharmonic functions (see [9, page 102] or [12, Theorem 3.6.9]).

Lemma 3.1. *Let G be a domain in \mathbb{C} and let u be a subharmonic function in G which is bounded above. If ∂G is not a polar set and*

$$\limsup_{z \rightarrow \zeta} u(z) \leq 0, \quad \text{for } \zeta \in \partial G \setminus E,$$

where E is a polar subset of ∂G , then $u \leq 0$ in G .

Polar sets are defined in [9] and [12] – here we just need the facts that if a set contains a continuum, then it is not polar whereas a finite set is polar.

Proof of Theorem 1.2. In view of (1.3), we can assume throughout the proof that U is simply connected; in particular, ∂U is not a polar set and all points of ∂U are regular for the Dirichlet problem.

Part (a). We consider two cases. First we suppose that $F(f)$ is disconnected. In this case we claim that there is a closed disc which lies outside $\bigcup_{n \geq 0} f^n(U)$. This is clearly true if U is forwards invariant under f . If U is not forwards invariant, then there is a Fatou component V of f different from U such that $f(V) \subset U$. Moreover, we claim that we can choose V to be different from all $f^n(U)$, $n \geq 0$. For otherwise, U is periodic under f , with period $p > 1$, and the p Fatou components of f in the orbit of U are each completely invariant under f^p . This contradicts a theorem of Baker [1] which states that a transcendental entire function can have at most one completely invariant Fatou component.

By making an affine change of variables, we can assume that $\{z : |z| \leq 1\}$ lies outside $\bigcup_{n \geq 0} f^n(U)$. Thus the functions

$$u_n(z) = \log |f^n(z)|, \quad z \in \bar{U}, \quad n \in \mathbb{N},$$

are positive harmonic in U and continuous in \bar{U} . By hypothesis there exists a set $E \subset \partial U$ such that the harmonic measure $\omega(z, E, U) > 0$, for $z \in U$, and

$$u_n(\zeta) \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad \text{for } \zeta \in E.$$

Since $\omega(z, \cdot, U)$ is a positive finite Borel measure on ∂U , we can assume by Egorov's theorem (see [19], for example) that

$$u_n \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad \text{uniformly on } E,$$

and we can assume in addition that E is closed. Thus if $C > 0$ is given, then there exists $N = N(C)$ such that

$$(3.1) \quad u_n(\zeta) \geq C, \quad \text{for } \zeta \in E, \quad n \geq N.$$

Now consider the bounded harmonic functions

$$v_n(z) = C\omega(z, E, U) - u_n(z), \quad z \in U, \quad n \geq N.$$

Then, for $n \geq N$,

$$\limsup_{z \rightarrow \zeta} v_n(z) \leq \begin{cases} C - u_n(\zeta) \leq 0, & \text{for } \zeta \in E, \\ \lim_{z \rightarrow \zeta} C\omega(z, E, U) - u_n(\zeta) \leq 0, & \text{for } \zeta \in \partial U \setminus E. \end{cases}$$

The first inequality follows from (3.1) and the fact that $\omega(z, E, U) \leq 1$ for $z \in U$. The second inequality holds because $u_n(z) > 0$ for $z \in U$ and $\lim_{z \rightarrow \zeta} \omega(z, E, U) = 0$, for $\zeta \in \partial U \setminus E$, since all points of ∂U are regular and E is closed, and $\omega(z, E, U)$ is the solution of the Dirichlet problem in U with boundary values χ_E .

Also, in the case that U is unbounded, the point at infinity acts as a polar set in the boundary of U (see [12, p. 85 and Corollary 3.2.5]).

It follows by Lemma 3.1 that $v_n(z) \leq 0$, for $z \in U$. Hence

$$u_n(z) = \log |f^n(z)| \geq C\omega(z, E, U), \quad \text{for } z \in U, \quad n \geq N.$$

Since $C > 0$ is arbitrary, it follows that

$$u_n(z) = \log |f^n(z)| \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad \text{for } z \in U,$$

so $U \subset I(f)$.

In the second case, we suppose that $U = F(f)$ is connected, from which it follows that $f^n(U) \subset U$, for $n \in \mathbb{N}$. In this case, U is simply connected and unbounded so we can define

$$(3.2) \quad \psi(z) = k\sqrt{z-a} + b, \quad z \in U,$$

where $a \in J(f)$, ψ is conformal on U , and $k > 0$ and $b \in \mathbb{C}$ are chosen so that $\psi(U) \cap \{z : |z| \leq 1\} = \emptyset$. Note that for $z \in U$ we have $\psi(z) \rightarrow \infty$ if and only if $z \rightarrow \infty$. Then put

$$v_n(z) = \log |\psi(f^n(z))|, \quad z \in \bar{U}, \quad n \in \mathbb{N}.$$

Each v_n is positive harmonic in U and continuous in \overline{U} , and we find by using Egorov's theorem again that there exists a closed set $E \subset \partial U$ such that $\omega(z, E, U) > 0$ for $z \in U$ and

$$v_n \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ uniformly on } E.$$

As in the first case, we can use Lemma 3.1 to deduce that

$$v_n(z) = \log |\psi(f^n(z))| \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for } z \in U.$$

Hence $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$ for $z \in U$, as required.

Part (b) This proof is similar to that of part (a). Suppose that $\partial U \cap A(f)$ has positive harmonic measure relative to U . By the definition of $A(f)$, we have

$$A(f) = \bigcup_{L \in \mathbb{N}} A_R^{-L}(f),$$

where $R > 1$ is so large that $R_n = M^n(R) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$A_R^{-L}(f) = \{z : |f^{n+L}(z)| \geq R_n, \text{ for } n \in \mathbb{N}\}, \quad L \in \mathbb{N}.$$

Thus $\partial U \cap A_R^{-L}(f)$ has positive harmonic measure relative to U for some $L \in \mathbb{N}$. Hence there exists a closed set $E \subset \partial U$ of positive harmonic measure relative to U such that

$$(3.3) \quad |f^{n+L}(\zeta)| \geq R_n, \quad \text{for } \zeta \in E, n \in \mathbb{N}.$$

As in the proof of part (a), in the 'F(f) disconnected' case we deduce by Lemma 3.1 that

$$\log |f^{n+L}(z)| \geq (\log R_n) \omega(z, E, U), \quad \text{for } z \in U, n \in \mathbb{N},$$

so

$$|f^{n+L}(z)| \geq R_n^{\omega(z, E, U)}, \quad \text{for } z \in U, n \in \mathbb{N}.$$

For any fixed $z \in U$, we can choose N so large that

$$R_n^{\omega(z, E, U)} \geq R_{n-1}, \quad \text{for } n \geq N,$$

because $\log M(r)/\log r \rightarrow \infty$ as $r \rightarrow \infty$. So for this z we have

$$|f^{n+L}(z)| \geq R_{n-1}, \quad \text{for } n \geq N,$$

and, moreover,

$$|f^{n+L+1}(z)| \geq R_n, \quad \text{for } n \in \mathbb{N},$$

because $|f(z)| \geq M(r)$ implies that $|z| \geq r$ for $r > 0$. Thus $z \in A_R^{-L-1}(f)$.

The proof in the 'F(f) connected' case is similar, using the fact that, for $|z|$ large enough, the function ψ defined in (3.2) satisfies

$$(3.4) \quad \frac{k}{2}|z|^{1/2} \leq |\psi(z)| \leq 2k|z|^{1/2}.$$

It follows from (3.3) and (3.4) that, for some $N \in \mathbb{N}$, we have

$$|\psi(f^{n+L}(\zeta))| \geq (k/2)R_n^{1/2}, \quad \text{for } \zeta \in E, n \geq N,$$

so, using Lemma 3.1 as before, we deduce that

$$\log |\psi(f^{n+L}(z))| \geq (\log((k/2)R_n^{1/2})) \omega(z, E, U), \quad \text{for } z \in U, n \geq N.$$

Hence

$$|\psi(f^{n+L}(z))| \geq ((k/2)R_n^{1/2})^{\omega(z, E, U)}, \quad \text{for } z \in U, n \geq N,$$

so, by (3.4) again, there exists $c > 0$ such that

$$|f^{n+L}(z)| \geq cR_n^{\omega(z,E,U)}, \quad \text{for } z \in U, n \geq N.$$

The rest of the proof is similar to that of the ‘ $F(f)$ connected’ case in part (a). \square

Remarks

1. If the Fatou component U is bounded, then the proofs of both parts of Theorem 1.2 can be simplified considerably since only the ‘ $F(f)$ disconnected’ case can occur, and Lemma 3.1 can be replaced by the ordinary maximum principle.
2. In Theorem 1.2 part (a), the condition that $\partial U \cap I(f)$ has Hausdorff dimension 2 would not imply that $U \subset I(f)$. For example, functions of the form $f(z) = \lambda e^z$, $0 < \lambda < 1/e$, have a completely invariant attracting basin U and the Hausdorff dimension of $\partial U \cap I(f) = J(f) \cap I(f)$ is 2; see [11].

4. SOME CONSEQUENCES OF THEOREMS 1.1 AND 1.2

In this section we prove two results about $I(f)$ which follow from Theorems 1.1 and 1.2, respectively. We thank Lasse Rempe for pointing out that part (b) of Theorem 4.1 can be stated in the more interesting way given in part (c).

Theorem 4.1. *Let f be a transcendental entire function. The following statements hold.*

- (a) *Any bounded component of $I(f)$ meets $J(f)$.*
- (b) *If G is a bounded simply connected domain and $G \cap I(f) \neq \emptyset$, then $\partial G \cap I(f) \neq \emptyset$.*
- (c) *$I(f) \cup \{\infty\}$ is connected.*

To prove Theorem 4.1 we need a lemma whose proof is based on the *blowing up property* of $J(f)$:

if f is an entire function, K is compact, $K \subset \mathbb{C} \setminus E(f)$ and V is an open neighbourhood of $z \in J(f)$, then there exists $N \in \mathbb{N}$ such that $f^n(V) \supset K$, for all $n \geq N$.

Here $E(f)$ is the *exceptional set* of f , that is, the set of points with a finite backwards orbit under f , which has at most one point (see [3]).

Lemma 4.1. *Let f be a transcendental entire function. If G is a bounded simply connected domain such that $G \cap J(f) \neq \emptyset$, then $\partial G \cap I(f) \neq \emptyset$.*

Proof. Since G is a bounded domain, we can define α_n , $n \in \mathbb{N}$, to be the outer boundary component of $f^n(G)$. Then, by the blowing up property of $J(f)$, we have $\text{dist}(\alpha_n, 0) \rightarrow \infty$ as $n \rightarrow \infty$. The compact sets

$$K_n = \{z \in \partial G : f^n(z) \in \alpha_n\}, \quad n \in \mathbb{N},$$

form a nested sequence, since $\alpha_{n+1} \subset f(\alpha_n)$, for $n \in \mathbb{N}$, so $K = \bigcap_{n \geq 1} K_n \neq \emptyset$. All points of K must lie in $I(f)$, so $\partial G \cap I(f) \neq \emptyset$, as required. \square

Proof of Theorem 4.1. Part (a) follows immediately from Theorem 1.1. For if I_0 is a bounded component of $I(f)$ that does not meet $J(f)$, then I_0 must be a bounded Fatou component, and this contradicts the fact that such a component must have a boundary point in $I(f)$, by Theorem 1.1.

To prove part (b) we suppose that G is a bounded simply connected domain that meets $I(f)$. By Lemma 4.1, we can assume that $G \subset F(f)$, so $G \subset U$ where U is an escaping Fatou component of f . There are now two cases:

- either $\partial G \cap U \neq \emptyset$, in which case $\partial G \cap I(f) \neq \emptyset$;
- or $G = U$, in which case U is an escaping wandering domain, so $\partial G \cap I(f) \neq \emptyset$, by Theorem 1.1.

To prove part (c) we note that if $I(f) \cup \{\infty\}$ is disconnected, then there exist disjoint open sets H_1 and H_2 in the Riemann sphere $\hat{\mathbb{C}}$ such that

$$I(f) \cup \{\infty\} \subset H_1 \cup H_2 \quad \text{and} \quad H_i \cap (I(f) \cup \{\infty\}) \neq \emptyset, \text{ for } i = 1, 2.$$

Without loss of generality $\infty \in H_2$ and H_1 is bounded. Since $I(f)$ meets H_1 it also meets ∂H_1 by part (b), and this gives a contradiction.

(Part (c) clearly implies part (b), so these two statements are equivalent.) \square

Remarks

1. The proof above of Lemma 4.1 gives a surprisingly simple argument to show that $I(f)$ is non-empty.
2. An alternative proof of Lemma 4.1 can be given, based on the following properties of the fast escaping set $A(f)$ (see [5] and [14]):

$$(4.1) \quad \partial A(f) = J(f);$$

$$(4.2) \quad \text{all the components of } A(f) \text{ are unbounded.}$$

Since $G \cap J(f) \neq \emptyset$, we have $G \cap A(f) \neq \emptyset$, by (4.1). Hence $\partial G \cap A(f) \neq \emptyset$, by (4.2), which proves Lemma 4.1. In fact, the set K constructed in the proof of Lemma 4.1 is a subset of $A(f)$ by [17, Corollary 2.5].

3. There is also a constructive proof of Theorem 4.1 part (c). First prove Theorem 4.1 part (a) as above. Then consider $A_0 = A(f) \cup \{\infty\}$. By (4.2), A_0 is a union of connected sets in $\hat{\mathbb{C}}$, all containing ∞ , so A_0 is connected. Let I_0 be the component of $I(f) \cup \{\infty\}$ that contains A_0 . Any point of $I(f)$ either lies in an unbounded component of $I(f)$, and so is in I_0 , or it lies in a bounded component of $I(f)$, which must contain a point of $J(f)$, by Theorem 4.1 part (a). By (4.1), such a component of $I(f)$ must also be in I_0 . Hence $I(f) \cup \{\infty\} = I_0$. Next we use Theorem 1.2 to give a sufficient condition for $I(f)$ to be connected.

Theorem 4.2. *Let f be a transcendental entire function and suppose that there exists a bounded domain G such that*

$$(4.3) \quad \partial G \subset I(f) \quad \text{and} \quad G \cap I(f)^c \neq \emptyset.$$

Then

- (a) *for each $n \in \mathbb{N}$, the outer boundary component α_n of $f^n(G)$ is contained in $I(f)$, $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$, and α_n surrounds 0 for sufficiently large n ;*
- (b) *$I(f)$ is connected.*

The statement of Theorem 4.2 is similar to that of [15, Theorem 2], but here we assume that G satisfies (4.3) whereas in [15, Theorem 2] we assumed that

$$(4.4) \quad \partial G \subset I(f) \quad \text{and} \quad G \cap J(f) \neq \emptyset.$$

In [15], we remarked that (4.3) and (4.4) are equivalent, but we later realised that the statement that (4.3) implies (4.4) is not immediate. The following lemma shows that this implication can be deduced from Theorem 1.2.

Lemma 4.2. *Let f be a transcendental entire function. Then the conditions (4.3) and (4.4) are equivalent.*

Proof. It is clear that if $G \cap J(f) \neq \emptyset$, then $G \cap I(f)^c \neq \emptyset$, since $J(f) = \partial I(f)$, by (1.1). Thus (4.4) implies (4.3).

To prove that (4.3) implies (4.4) we argue as follows. If $\partial G \subset I(f)$ and $G \subset F(f)$, then either $\partial G \cap F(f) \neq \emptyset$, in which case $G \subset I(f)$ or G is a component of the Fatou set, so $G \subset I(f)$ by Theorem 1.2 part (a). Hence (4.3) implies (4.4). \square

Theorem 4.2 follows immediately from Lemma 4.2 and [15, Theorem 2]. However, the proof below contains a shorter argument to prove part (b) than that given in [15]. (In [15] a constructive argument was used whereas here we argue by contradiction.)

Proof of Theorem 4.2. By Lemma 4.2, the hypothesis (4.3) implies that $G \cap J(f) \neq \emptyset$.

Since $\partial G \subset I(f)$, the outer boundary component α_n of $f^n(G)$ is contained in $I(f)$, for each $n \in \mathbb{N}$. By the blowing up property of $J(f)$, $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$, and α_n surrounds 0 for sufficiently large n . Thus part (a) holds.

To prove part (b) suppose that $I(f)$ is disconnected. Then there exist disjoint open sets H_1 and H_2 in \mathbb{C} such that

$$I(f) \subset H_1 \cup H_2 \quad \text{and} \quad H_i \cap I(f) \neq \emptyset, \quad \text{for } i = 1, 2.$$

Without loss of generality one of H_1 and H_2 is bounded, say H_1 , for otherwise both H_1 and H_2 have unbounded boundary components that do not meet $I(f)$, which contradicts part (a). We can also assume that H_1 is simply connected.

Fix $z_0 \in H_1$ and take any $m \in \mathbb{N}$. Then there exists $N_m \in \mathbb{N}$ such that

$$(4.5) \quad f^n(z_0) \text{ lies outside } \alpha_m, \quad \text{for } n \geq N_m.$$

Now, for $n \in \mathbb{N}$,

$$(4.6) \quad \partial f^n(H_1) \subset f^n(\partial H_1) \subset I(f)^c,$$

because $\partial H_1 \subset I(f)^c$. Also, $f^n(\partial H_1)$ is connected because ∂H_1 is connected. Thus, since $\alpha_m \subset I(f)$, it follows from (4.5) and (4.6) that

$$\overline{f^n(H_1)} \text{ lies outside } \alpha_m, \quad \text{for } n \geq N_m.$$

Thus $\overline{f^n(H_1)} \rightarrow \infty$ as $n \rightarrow \infty$, so $\partial H_1 \subset I(f)$, a contradiction. Hence $I(f)$ is connected. \square

Remarks

1. The final part of this contradiction argument can be replaced by an argument based on the use of Lemma 4.1 and Theorem 4.1.
2. In Theorem 4.2 the set $I(f)$ has the structure of a ‘spider’s web’, a concept introduced in [17]: we say that a set E is an *(infinite) spider’s web* if E is connected and there exists a sequence (G_n) of bounded simply connected domains with $G_n \subset G_{n+1}$, $\partial G_n \subset E$, for $n \in \mathbb{N}$, and

$$\bigcup_{n=1}^{\infty} G_n = \mathbb{C}.$$

Several examples of functions for which $I(f)$ is a spider’s web, and hence $I(f)$ is connected, were given in [15, Section 6], though the name ‘spider’s web’ was not used there. Many more examples of such functions are given in [17] and [18].

Condition (4.3) can be described by saying that $I(f)$ *has a hole*, so Theorem 4.2 can be stated in the following geometric way:

if $I(f)$ has a hole, then $I(f)$ is a spider’s web.

3. The proof of Theorem 4.2 given above was inspired by unpublished work of Professor Noel Baker. In early 2010 the authors found a note written by Noel Baker some time after 1996 in an old diary that he used for rough work. He stated there that if f is a transcendental entire function with a multiply connected Fatou component, then $I(f)$ is connected. He gave no details of the proof, except that it used the method of showing that $I(f) \neq \emptyset$ given by Domínguez in [6], and there was a sketch which suggested that his argument was by contradiction.

Noel Baker died in 2001 and as far as we know he never mentioned this result to anyone. In [14] we proved the result as a corollary of the fact that all the components of $A(f)$ are unbounded. Thinking about the argument he might have found led us to the proof of Theorem 4.2 above.

5. COMPONENTS OF $I(f)$

In this final section, we give various results about the possible structures of the components of $I(f)$. The first result, which follows from Theorem 1.1, gives a new sufficient condition for all the components of $I(f)$ to be unbounded.

Theorem 5.1. *Let f be a transcendental entire function and E be a set such that $J(f) \subset \overline{E}$. If E is contained in the union of finitely many components of $I(f)$, then*

- (a) $I(f) \cap J(f)$ is contained in one component, I_1 say, of $I(f)$;
- (b) all the components of $I(f)$ are unbounded, and they consist of
 - (i) I_1 , which also contains any escaping wandering domains and any Baker domains of f with at least one boundary point in $I(f)$,
 - (ii) any Baker domains of f with no boundary points in $I(f)$ and the infinitely many preimage components of such Baker domains.

Remark If every Baker domain of f has a point of $I(f)$ on its boundary, then the conclusion of Theorem 5.1 can be strengthened to ‘ $I(f)$ is connected’.

Proof of Theorem 5.1. Suppose that E is contained in the union of finitely many components of $I(f)$, say I_1, I_2, \dots, I_m . Take any $z \in I(f) \cap J(f)$. Since $J(f) \subset \bar{E}$, there exist $z_n \in E$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$. Without loss of generality all terms of this sequence (z_n) lie in a single component, I_j say. Since $z \in I(f)$, we have $z \in I_j$. Hence

$$(5.1) \quad I(f) \cap J(f) \subset I_1 \cup I_2 \cup \dots \cup I_m.$$

We now assume that I_1, I_2, \dots, I_m is the minimal set of components of $I(f)$ such that (5.1) holds. Then $I_j \cap J(f) \neq \emptyset$, for $j = 1, 2, \dots, m$. Note that if the exceptional set $E(f)$ is non-empty, then

$$(5.2) \quad (I_j \setminus E(f)) \cap J(f) \neq \emptyset, \quad \text{for } j = 1, 2, \dots, m.$$

Indeed, if $E(f) = \{\alpha\} \subset I_j \cap J(f)$, then α is a limit point of the backwards orbit of any non-exceptional point in $I(f) \cap J(f)$ (this follows from the blowing up property of $J(f)$) and hence α is the limit of a sequence in $I_i \cap J(f)$, say, by (5.1). Thus $i = j$ and so (5.2) holds.

If $m = 1$, then $I(f) \cap J(f)$ is contained in one component of $I(f)$, as required. If $m > 1$, then we can take $z_1 \in I_1 \cap J(f)$ and an open disc D centred at z_1 so small that

$$(5.3) \quad D \cap (I_2 \cup \dots \cup I_m) = \emptyset.$$

Consider I_j , $j \geq 2$. Then there exists $N \in \mathbb{N}$ such that $f^N(D)$ meets both $I_1 \cap J(f)$ and $I_j \cap J(f)$, by (5.2) and the blowing up property. Hence there exist $w_1, w_j \in D$ such that

$$f^N(w_1) \in I_1 \cap J(f) \quad \text{and} \quad f^N(w_j) \in I_j \cap J(f),$$

so $w_1, w_j \in I_1$ by the backwards invariance of $I(f) \cap J(f)$ and (5.3). Thus $f^N(I_1)$ is a connected subset of $I(f)$ that meets both I_1 and I_j , which is a contradiction. Hence $m = 1$, so part (a) holds.

Clearly the component I_1 is unbounded. If U is a wandering domain in $I(f)$, then $\partial U \cap I(f) \neq \emptyset$ by Theorem 1.1, so $\partial U \cap I_1 \neq \emptyset$ and hence $U \subset I_1$. Similarly, if U is a Baker domain whose boundary meets $I(f)$, then $U \subset I_1$; in particular, this is true if U is completely invariant, since in this case $\partial U = J(f)$.

To complete the proof of part (b) we note that any other component of $I(f)$ must be a Baker domain that has no points of $I(f)$ in its boundary, and so is not completely invariant, or it must be one of its infinitely many preimage Fatou components, all of which are unbounded. \square

Theorem 5.1 includes the following special cases.

Corollary 5.1. *Let f be a transcendental entire function.*

- (a) *Either $I(f)$ is connected or it has infinitely many components.*
- (b) *If $A(f)$ is contained in the union of finitely many components of $I(f)$, then all the components of $I(f)$ are unbounded.*

In part (b) the set $A(f)$ can be replaced by any subset of $I(f)$ whose closure contains $J(f)$. Such subsets of $I(f)$ include the sets $L(f)$, $M(f)$ and $Z(f)$, all of which are defined in [16].

The proof of Theorem 5.1, and hence of Corollary 5.1, used Theorem 1.1 together with the blowing up property of $J(f)$. In fact, we can use just the blowing up property to prove part (a) of Corollary 5.1. More generally, we have the following result which is not directly related to escaping points.

Theorem 5.2. *Let f be a transcendental entire function, and suppose that the set E is completely invariant under f and that $J(f) = \overline{E \cap J(f)}$. Then exactly one of the following holds:*

- (1) E is connected;
- (2) E has exactly two components, one of which is a singleton $\{\alpha\}$, where α is a fixed point of f and $\alpha \in E(f) \cap F(f)$;
- (3) E has infinitely many components.

Proof. We can assume that E is infinite, since any completely invariant finite set must be a singleton.

Suppose that cases (1) and (3) do not hold. Then E has finitely many components E_1, E_2, \dots, E_m , say, with $m > 1$. Since $J(f) = \overline{E \cap J(f)}$, we can assume that $E_1 \neq E(f)$ and that there exists $z_1 \in (E_1 \setminus E(f)) \cap J(f)$. Now suppose that, for some $j \geq 2$, we have $E_j \neq E(f)$. Choose an open disc D centred at z_1 and not meeting $E_2 \cup \dots \cup E_m$. For some $N \in \mathbb{N}$, $f^N(D)$ meets both E_1 and E_j , by the blowing up property, so $f^N(E_1)$ meets both E_1 and E_j by the backwards invariance of E . But $f^N(E_1)$ is a connected subset of E , by the forwards invariance of E , which is a contradiction.

Thus E has two components E_1 and $E(f) = \{\alpha\}$, say. Then $\alpha \in F(f)$, since any point in $E \cap J(f)$ lies in E_1 because it is a limit point of the backwards orbit of z_1 (this backwards orbit lies in E by the backwards invariance of E).

Finally we show that $f(\alpha) = \alpha$. If $f(\alpha) \neq \alpha$, then $f(\alpha) \in E_1$, by the complete invariance of E . Since E_1 is infinite, $f(\alpha)$ is a limit point of E and so therefore is α , by the complete invariance of E and the local behaviour of f near α . Thus we have a contradiction. Hence α is a fixed point of f and case (2) holds. \square

Remarks

1. The function $f(z) = \frac{1}{2}z^2e^{2-z}$ shows that case (2) of Theorem 5.2 can occur, with $E = J(f) \cup \{0\}$. In this case $J(f)$ is connected and 0 is a fixed point of f in $E(f) \cap F(f)$ (see [10, proof of Theorem 4]).

2. Taking E to be the grand orbit of a non-exceptional point in $J(f)$ shows that in case (3) of Theorem 5.2 the set E may have only countably many components.

Note that case (2) of Theorem 5.2 cannot occur if we also know, for example, that $E \subset I(f)$ or that E contains a neighbourhood of every fixed point of f in $F(f)$. Thus if E is $I(f), L(f), M(f), Z(f)$ or $A(f)$, or the complement of one of these sets, then E is either connected or has infinitely many components.

It is known that if f is a transcendental entire function, then $J(f)$ is either connected or it has uncountably many components (see [2, Theorem B]), and it is natural to ask if $I(f)$ and $A(f)$ also have this property.

In Theorem 5.2 the case when E has infinitely many components can be strengthened to state that each neighbourhood of each point $z_0 \in J(f)$ must meet infinitely many components of E . This is implied by the following simple result.

Theorem 5.3. *Let A and B be completely invariant sets for a transcendental entire function f . If, for some $z_0 \in J(f)$, there is a neighbourhood D of z_0 such that $A \cap D$ meets only finitely many components of B , then A meets only finitely many components of B .*

Proof. Suppose that $A \cap D \subset B_1 \cup \dots \cup B_m$, a minimal union of components of B . Let K be any compact set, not meeting the exceptional set $E(f)$. Then, by the blowing up property of $J(f)$, there exists $N \in \mathbb{N}$ such that $f^N(D) \supset K$. Thus $A \cap K$ meets at most m components of B , by the complete invariance of $A \cap B$. Hence A meets exactly m components of B in $\mathbb{C} \setminus E(f)$, namely B_1, \dots, B_m , and so A meets at most $m + 1$ components of B in \mathbb{C} . \square

By applying Theorem 5.3 with $A = \mathbb{C}$ and $B = E$, we deduce that if E is completely invariant and has infinitely many components, then each neighbourhood of each point $z_0 \in J(f)$ must meet infinitely many components of E ; in particular, this is the case in Theorem 5.2 case (3).

Other choices of the sets A and B in Theorem 5.3 give other corollaries. For example, with $A = A(f)$ and $B = I(f)$, Theorem 5.3 and Corollary 5.1 part (b), combine to give the following result.

Corollary 5.2. *Let f be a transcendental entire function. If, for some $z_0 \in J(f)$, there is a neighbourhood D of z_0 such that $A(f) \cap D$ is contained in the union of finitely many components of $I(f)$, then $A(f)$ is contained in the union of finitely many components of $I(f)$, so all the components of $I(f)$ are unbounded.*

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