six-cusped sextic, whose cusps lie on a conic, and many properties of this projected curve similar to those of the plane quartic can be at once obtained from the above general theorems.

It remains to be added that the nature of the harmonic envelope T_6, and the configuration of its contact-quadrics U_2 in relation to the 36 double-sixes of the cubic surface by which T_6 is defined, require further investigation and elucidation.

ON AN INTEGRAL FUNCTION OF AN INTEGRAL FUNCTION

G. PÓLYA.*

THE following theorem, which I found in replying to a question of Prof. I. Schur, seems to me to exhibit an essential characteristic of the notion of order.

I. *If* $g(z)$ *and* $h(z)$ *are integral functions and* $g\big(h(z)\big)$ *is an integral function of finite order, then there are only two possible cases: either*

(*a*) *the internal function* $h(z)$ *is a polynomial and the external function* $g(z)$ *is of finite order*;† *or else*

(*b*) *the internal function* $h(z)$ *is not a polynomial but a function of finite order, and the external function* $g(z)$ *is of zero order.*

This theorem appears quite simple but, though several proofs are possible, I have not been able to find one that does not involve some rather elaborate result connected with the theorem of Picard. The simplest proof I can present is based on a theorem of H. Bohr,‡ which generalizes one of Landau§ and leads easily to the following theorem, which is interesting in itself.

II. *Suppose that* $f(z)$, $g(z)$, $h(z)$ *are integral functions connected by the relation*

$$f(z) = g\big(h(z)\big). \tag{1}$$

* Received and read 17 February, 1925.

† The case of finite order includes that of zero order, and the latter includes the case of a polynomial.

‡ H. Bohr, *Scripta Univ. atque Biblioth. Hierosolymitanarum*, 1 (1923).

§ E. Landau, *Rend. di Palermo*, 46 (1922), 347–348.

Suppose further that

$$h(0) = 0. \tag{2}$$

Let $F(r)$, $G(r)$, $H(r)$ *denote the maximum moduli of* $f(z)$, $g(z)$, $h(z)$ *respectively in the circle* $|z| \leqslant r$. *Then there is a definite number* c, *greater than* 0 *and less than* 1, *independent of* $g(z)$, $h(z)$, *and* r, *and such that*

$$F(r) \geqslant G\left(cH(\tfrac{1}{2}r)\right). \tag{3}$$

We could substitute any positive fraction for $\frac{1}{2}$ provided c is replaced by some other suitable constant. Observe that the opposite inequality

$$F(r) \leqslant G\left(H(r)\right)$$

is an immediate consequence of the definitions.

Proof of Theorem II.—The theorem of Bohr which we need runs as follows:—

Suppose that ρ *is a given number,* $0 < \rho < 1$, *and* $w = \phi(z)$ *is any function which is regular for* $|z| \leqslant 1$ *and satisfies the conditions*

$$\phi(0) = 0, \qquad \operatorname*{Max}_{|z|=\rho} |\phi(z)| = 1. \tag{4}$$

Let r_ϕ *denote the radius of the largest circle* $|w| = r_\phi$ *whose points all represent values taken by* $\phi(z)$ *in the circular domain* $|z| \leqslant 1$. *Then* r_ϕ *is not less than* C, $C = C(\rho)$ *being a positive number which depends only on* ρ.

To fix our ideas let us take $\rho = \frac{1}{2}$, put $C(\frac{1}{2}) = c$, and apply the theorem to the function

$$\phi(z) = \frac{h(rz)}{H(\frac{1}{2}r)},$$

which satisfies the conditions (4). We see that the function $w = h(z)$ maps the circular domain $|z| \leqslant r$ on a Riemann surface extended over the w-plane whose various sheets cover the whole length of a certain circle of centre $w = 0$ and of radius R, R being not less than $cH(\frac{1}{2}r)$.

Suppose that w_0 is a point on the circle $|w| = R$, such that

$$|g(w_0)| = G(|w_0|) = G(R).$$

Then there is at least one point z_0 inside $|z| \leqslant r$, such that

$$h(z_0) = w_0.$$

It follows that

$$G\left(cH(\tfrac{1}{2}r)\right) \leqslant G(|w_0|) = |g(w_0)| = \left|g\left(h(z_0)\right)\right| \leqslant F(r).$$

Proof of Theorem I.—The case where $g(z)$ or $h(z)$ is a constant is of no interest and will be excluded. Considering, if necessary, $h(z)-h(0)$ instead of $h(z)$, and $g\left(w+h(0)\right)$ instead of $g(w)$, we can and shall assume that (2) is true. Then we have, adopting the notation (1), the inequality (3). Observe that $F(r)$, $G(r)$, $H(r)$ are increasing functions.

We may express the hypothesis that $f(z)$ is of finite order by the inequality

$$F(r) < Ae^{r^a}. \tag{5}$$

Put
$$h(z) = a_1 z + a_2 z^2 + \ldots + a_m z^m + \ldots,$$

and suppose $|a_m| > 0$. We have

$$H(r) \geqslant |a_m| r^m \tag{6}$$

and, in virtue of (3), (5), and (6),

$$G(c|a_m|2^{-m}r^m) \leqslant G\left(cH(\tfrac{1}{2}r)\right) \leqslant F(r) < Ae^{r^a},$$
$$G(c|a_m|2^{-m}r) \leqslant Ae^{r^{a/m}}.$$

That is to say, the order of $g(z)$ does not exceed a/m. If $h(z)$ is not a polynomial, m can be chosen arbitrarily large and in this case the order of $g(z)$ is zero.

In any case there is an inequality for $g(z)$, analogous to (6), let us say

$$G(r) \geqslant |b_n| r^n \quad (|b_n| > 0, \quad n \geqslant 1).$$

Combining this with (3) and (5), we obtain

$$|b_n| c^n \left(H(\tfrac{1}{2}r)\right)^n \leqslant G\left(cH(\tfrac{1}{2}r)\right) \leqslant F(r) < Ae^{r^a}.$$

Thus the order of $h(z)$ is not greater than a. The chief point being settled by Theorem II, there is naturally no difficulty in finding closer relations between the orders of magnitude of $F(r)$, $G(r)$, and $H(r)$.

The case (b) of Theorem I is actually possible. Put

$$g(w) = 1+2^{-1}w+2^{-4}w^2+2^{-9}w^3+\ldots, \qquad h(z) = e^z.$$

The integral function

$$g\left(h(z)\right) = 1+2^{-1}e^z+2^{-4}e^{2z}+\ldots \tag{7}$$

is the "upper half" of a theta-series. The zeros and the order of magnitude of the whole theta-series being perfectly known, we conclude on general principles that the function (7) is of the second order. We can easily obtain more precise information by direct calculation. Let $M(r)$ denote the maximum modulus and $N(r)$ the number of the zeros of the function (7) in the circle $|z| \leqslant r$. Then we have

$$\lim_{r \to \infty} r^{-2} \log M(r) = 2 \lim_{r \to \infty} r^{-2} N(r) = \frac{1}{4 \log 2}.$$

THE ZEROS OF RIEMANN'S ZETA-FUNCTION ON THE CRITICAL LINE

M. FEKETE.*

1. Since Hardy, in his most interesting note "Sur les zéros de la fonction $\zeta(s)$ de Riemann",† proved the existence of an infinity of zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$, several papers have been published on the order of magnitude of the number $N_0(T)$ of zeros of $\zeta(s) = \zeta(\sigma + it)$ for which

$$\sigma = \tfrac{1}{2}, \quad 0 < t < T.$$

1°. In 1915, Landau‡ proved that

$$\liminf_{T \to \infty} \frac{N_0(T)}{\log \log T} > 0.$$

2°. In 1916, de la Vallée Poussin§ proved that

$$\limsup_{T \to \infty} \frac{N_0(T)}{\sqrt{T}} > 0.$$

3°. In 1917, Hardy and Littlewood|| proved that to every $\epsilon > 0$ corresponds a $T_0 = T_0(\epsilon)$, such that

$$N_0(T) > T^{\frac{3}{4} - \epsilon} \quad (T > T_0).$$

4°. In 1921, the same authors ¶ proved that

$$\liminf_{T \to \infty} \frac{N_0(T)}{T} > 0.$$

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* Received 3 June, 1925; read 11 June, 1925.

† *Comptes rendus*, 158 (1914), 1012–1014.

‡ Landau, "Über die Hardysche Entdeckung unendlich vieler Nullstellen der Zetafunktion mit reellem Teil $\frac{1}{2}$", *Math. Annalen*, 76 (1915), 212–243.

§ de la Vallée Poussin, "Sur les zéros de $\zeta(s)$ de Riemann", *Comptes rendus*, 163 (1916), 418–421.

|| Hardy and Littlewood, "Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes", *Acta Math.*, 41 (1918), 119–196.

¶ Hardy and Littlewood, "The zeros of Riemann's zeta-function on the critical line", *Math. Zeitschrift*, 10 (1921), 283–317.