In memory of Noel Baker

ENTIRE FUNCTIONS WITH BOUNDED FATOU COMPONENTS

AIMO HINKKANEN

Abstract. Starting with the work of I.N. Baker that appeared in 1981, many authors have studied the question of under what circumstances every component of the Fatou set of a transcendental entire function must be bounded. In particular, such functions have no domains now known as Baker domains, and no completely invariant domains. There may be wandering domains but not the familiar and more easily constructed unbounded ones that often appear for functions defined by simple explicit formulas.

Two types of criteria are involved in the partial answers obtained for this question: the order of growth, and the regularity of the growth of the function.

Baker himself showed that a function of sufficiently slow growth has only bounded Fatou components and noted that order 1/2 minimal type is the best condition one could hope for. Subsequently his results have been extended to order less than 1/2 except for wandering domains, and also to wandering domains if the growth satisfies, in addition, a mild regularity condition. Similar results have been obtained also for certain functions of faster growth provided that the growth is sufficiently regular.

In this paper we review the results achieved and the methods involved in this area.

1. Introduction

Let f be a transcendental entire function. We consider the question, initiated by I.N. Baker in 1981, of under what circumstances all the components of the Fatou set of f are bounded. The conjecture that this is the case whenever the growth of f does not exceed order 1/2, minimal type, is still open even though a lot of progress has been made.

We first review some definitions and notations. We write $f^1 = f$, and $f^n = f \circ f^{n-1}$ for $n \geq 2$ for the iterates of f. The Fatou set or the set of normality $\mathcal{F}(f)$ of f consists of all z in the complex plane $\mathbb{C}$ that

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have a neighbourhood U such that the family $\{f^n|U : n \geq 1\}$ of the restrictions of the iterates of f to U is a normal family. The Julia set $\mathcal{J}(f)$ of f is $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$. The set $\mathcal{F}(f)$ is trivially open, while by the results of Fatou and of Julia, $\mathcal{J}(f)$ is a non-empty perfect set which coincides with $\mathbb{C}$, or is nowhere dense in $\mathbb{C}$. For the fundamental results in the iteration theory of rational and entire functions we refer to the original papers of Fatou [12, 13, 14] and of Julia [19] and to the books of Beardon [7], Carleson and Gamelin [10], Milnor [20], and Steinmetz [23]. Fatou's paper [14] was the first to address the dynamics of transcendental entire functions.

We say that a set E is (forward) *invariant* under f if $f(E) \subset E$, and *backward invariant* under f if $f^{-1}(E) \subset E$. We say that E is *completely invariant* under f if E is both invariant and backward invariant under f. Each of the sets $\mathcal{F}(f)$ and $\mathcal{J}(f)$ is completely invariant under f.

We next consider the components of $\mathcal{F}(f)$. Let D be a component of $\mathcal{F}(f)$. Then $f^n(D) \subset D_n$ where D_n is a component of $\mathcal{F}(f)$. Incidentally, equality holds if f is rational, and Herring [16] has shown that if f is entire, then $D_n \setminus f^n(D)$ contains at most one point. If all the domains D_n are disjoint, then D is called a *wandering domain*. If there is a smallest positive integer p such that $D_p = D$ then D is *periodic* of period p. In particular, if $p = 1$, then D is called *invariant*. Otherwise, D is called *preperiodic*, and then $D_n = D_{n+p}$ for some $n, p \geq 1$, while for all $p \geq 1$ we have $D_p \neq D$.

We have $\mathcal{F}(f^p) = \mathcal{F}(f)$ for all $p \geq 1$. Hence a periodic domain of period p is an invariant domain for f^p. There is a standard *classification of invariant components* D of $\mathcal{F}(f)$ into the following five types.

There may be a fixed point α of f in D such that $\lim_{n\to\infty} f^n(z) = \alpha$, locally uniformly for $z \in D$. Then the *multiplier* $\lambda = f'(\alpha)$ satisfies $|\lambda| < 1$. If $\lambda = 0$, we call α a superattracting fixed point of f and call D a *superattracting* domain. If $0 < |\lambda| < 1$, we call α an attracting fixed point of f and call D an *attracting* domain.

There may be a fixed point α of f on ∂D such that $\lim_{n\to\infty} f^n(z) = \alpha$, locally uniformly for $z \in D$. Then $f'(\alpha) = 1$. We call α a parabolic fixed point of f and call D a *parabolic* domain.

It may be that D is simply connected, containing a fixed point α of f, such that if φ is a conformal mapping of D onto the unit disk with $\varphi(\alpha) = 0$, then $(\varphi \circ f \circ \varphi^{-1})(z) \equiv e^{2\pi i\beta} z$ for some real irrational number β. In this case D is called a *Siegel disk* with centrum α.

Finally, it may be that $\lim_{n\to\infty} f^n(z) = \infty$, locally uniformly for $z \in D$. Then D is called a *Baker domain*.

We note that this classification differs from that for rational functions in two ways. Rational functions have no Baker domains, whose possible existence stems from the impossibility to define f at the essential singularity at infinity. A component of $\mathcal{F}(f)$ for a rational f that behaves like a Baker domain would have to be a parabolic domain with the parabolic fixed point at infinity. Also, as Baker has shown, it follows from the maximum principle that entire functions (including polynomials) have no Herman rings. A *Herman ring* is a doubly connected invariant component D of $\mathcal{F}(f)$ such that there is a conformal mapping φ of D onto the annulus $\{z : 1 < |z| < R < \infty\}$ for which the conjugated map $(\varphi \circ f \circ \varphi^{-1})(z) \equiv e^{2\pi i\beta} z$ for some real irrational number β. If an entire function f had a Herman ring D, then all points on the inner component of ∂D would have to lie in $\mathcal{J}(f)$. On the other hand, all the iterates of f remain uniformly bounded on an invariant Jordan curve in D and hence, by the maximum principle, in the bounded component of the complement of D, which implies that the iterates of f form a uniformly bounded family and hence a normal family in the union of D and the bounded component of the complement of D. Hence this larger set must lie in $\mathcal{F}(f)$ and so cannot intersect $\mathcal{J}(f)$, which is a contradiction.

Clearly a Baker domain must be unbounded. It is easily seen that for a transcendental entire f, any completely invariant component of $\mathcal{F}(f)$ is unbounded.

For a polynomial f, the set $\mathcal{F}(f)$ contains a neighbourhood of infinity, corresponding to the superattracting fixed point at infinity, and hence has a unique unbounded component, which, furthermore, is completely invariant. One can ask what can be said about the boundedness or unboundedness properties of the components of the Fatou set of a transcendental entire function. Many examples of transcendental entire functions have unbounded components in their Fatou sets, for example, the function $z + 1 + e^{-z}$ considered by Fatou in 1926 [14]. This particular function, however, has order 1.

We now review some concepts related to the growth of an entire function. We use the following notations for the *maximum* and *minimum modulus* of f:

$$M(r, f) = \max\{|f(z)| : |z| = r\}, \qquad m(r, f) = \min\{|f(z)| : |z| = r\}.$$

(The notation $m(r, f)$ is also the standard notation for the proximity function of f in the Nevanlinna theory but we will not need that concept.) Recall that the *order* $\rho(f)$ and *lower order* $\lambda(f)$ of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \qquad \lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

If $0 < \rho(f) = \rho < +\infty$, we define the *type* of f by

$$\tau(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}}.$$

If $\tau(f) = 0$, we say that f is of *minimal type*. If $0 < \tau(f) < +\infty$, we say that f is of *mean type*. If $\tau(f) = +\infty$, we say that f is of *maximal type*.

I.N. Baker [2] asked in 1981 whether every component of $\mathcal{F}(f)$ is bounded if the growth of f is sufficiently small. The appropriate growth condition would appear to be of order 1/2, minimal type at most. Baker ([2], p. 489) observed that this condition would be best possible in the following sense. For any sufficiently large positive a, the function $f(z) = z^{-1/2}\sin\sqrt{z} + z + a$ is of order 1/2, mean type, and has an unbounded component D of $\mathcal{F}(f)$ containing a segment $[x_0, \infty)$ of the positive real axis, such that $f^n(z) \to \infty$ as $n \to \infty$, locally uniformly in D.

As Baker noted ([2], p. 484), it is possible to have a function of order 1/2 and of arbitrarily small type with the same properties. He considered $f(z) = \cos\left\{(\varepsilon^2 z + (9/4)\pi^2)^{1/2}\right\}$ where $0 < \varepsilon < \sqrt{3\pi}$. This function is of order 1/2 and type ε. Now $f(0) = \cos(3\pi/2) = 0$ and $0 < f'(0) = \varepsilon^2/(3\pi) < 1$ so that f has an attracting fixed point at the origin. We have $f([0,\infty)) = [-1,1]$. Furthermore,

$$0 < f'(x) = -\varepsilon^2 \sin\sqrt{(9/4)\pi^2 + \varepsilon^2 x}/(2\sqrt{(9/4)\pi^2 + \varepsilon^2 x}) < 1$$

for $-1 \le x \le 1$, which implies first together with $f(0) = 0$ that $|f(x)| < |x|$ for $x \in [-1,1] \setminus \{0\}$, and then that $\lim_{n\to\infty} f^n(x) = 0$ for $-1 \le x \le 1$. It follows that the component of $\mathcal{F}(f)$ containing the attracting fixed point at the origin contains the positive real axis and is therefore unbounded.

The conjecture that every component of $\mathcal{F}(f)$ is bounded if the growth of f is of order 1/2, minimal type at most, remains open in general. However, numerous cases have been settled and there is an open problem only for wandering domains, and even then only for functions whose growth is quite irregular, in a sense to be made more precise later on. Several techniques have been developed to attack various instances of this problem. In this paper we survey the results obtained and the techniques developed. We also reproduce a number of proofs related to this subject.

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2. Notation

We denote the set of complex numbers by $\mathbb{C}$, and write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the extended complex plane, that is, the Riemann sphere. If $z \in \mathbb{C}$ and $r > 0$, we write $B(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$, $S(z, r) = \{w \in \mathbb{C} : |w - z| = r\}$, and $\overline{B}(z, r) = \{w \in \mathbb{C} : |w - z| \le r\}$. We denote the unit disk $B(0, 1)$ by $\mathbb{D}$.

If γ is a Jordan curve in the complex plane $\mathbb{C}$, and hence bounded, we denote the bounded component of the complement of γ by $\operatorname{int} \gamma$ and call it the *interior* of γ. We denote the unbounded component of $\overline{\mathbb{C}} \setminus \gamma$ by $\operatorname{ext} \gamma$ and call it the *exterior* of γ.

3. Hyperbolic geometry and Schottky's theorem

Hyperbolic geometry is an indispensable tool in complex analysis, and complex dynamics is no exception. We review a few concepts and results that we shall need.

In the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ we define the density of the hyperbolic or Poincaré metric $\lambda(z) = \lambda_{\mathbb{D}}(z)$ by

$$\lambda(z) = \frac{1}{1 - |z|^2}.$$

If U is a simply connected domain in $\overline{\mathbb{C}}$ whose complement contains at least 2 points, there is a conformal mapping φ of $\mathbb{D}$ onto U, and we define the density of the hyperbolic metric $\lambda_U(z)$ in U by

$$\lambda_U(\varphi(z))|\varphi'(z)| = \frac{1}{1 - |z|^2}. \tag{1}$$

Then $\lambda_U(z)$ does not depend on the choice of φ.

If U is any domain in $\overline{\mathbb{C}}$ for which there exists an analytic universal covering map φ of $\mathbb{D}$ onto U, we define the density of the hyperbolic metric $\lambda_U(z)$ in U by (1). Then $\lambda_U(z)$ does not depend on the choice of φ, nor, for a fixed φ and for a given $w \in U$, on the choice of $z \in \mathbb{D}$ such that $\varphi(z) = w$.

We write $h_U(z, w)$ for the hyperbolic distance between $z, w \in U$. Thus $h_U(z, w) = \inf_\gamma \int_\gamma \lambda_U(\zeta)\,|d\zeta|$ where γ runs over all rectifiable paths joining z to w in U.

If U is the punctured unit disk $U = \{z : 0 < |z| < 1\}$, then a calculation shows that

$$\lambda_U(z) = -\frac{1}{2|z|\log|z|} \tag{2}$$

since the map $z \mapsto \exp\{(z-1)/(z+1)\}$ is the universal covering map of the unit disk onto U.

Consider the thrice punctured sphere $V = \overline{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$. Suppose that f is analytic in $\mathbb{D}$ and that f omits the values 0 and 1 there. Then f maps $\mathbb{D}$ into V, and by the distance decreasing property of analytic functions, we have

$$h_V(f(z), f(w)) \le h_{\mathbb{D}}(z, w)$$

for all $z, w \in \mathbb{D}$. If $f(0) = a$ and $|z| \le r < 1$, then

$$h_V(f(z), a) \le \frac{1}{2}\log\frac{1+r}{1-r},$$

which imposes an upper bound of $|f(z)|$ as well as a lower bound on each of $|f(z)|$ and $|f(z)-1|$. This result is known as Schottky's theorem. Explicit bounds have been obtained by several authors; for our purposes, any explicit bounds will do. Therefore we only mention the result ([15], Theorem 5.6, p. 60) that

$$\log^+|f(z)| \le (\pi/2 + 2\log 3 + \log^+|f(0)|)\frac{1+|z|}{1-|z|}. \tag{3}$$

Applying (3) to $1/f$ and to $1/(1-f)$ instead of f, one obtains lower bounds for $|f(z)|$ and $|f(z)-1|$.

In complex dynamics, one often normalizes a situation by conjugating an entire function by a linear polynomial. Since such a conjugation does not change dynamics, it is a natural thing to do. In that way, one can, for example, move certain two points to two simpler points such as 0 and 1. However, we will often deal with conditions involving circles centred at the origin, and since such a conjugation may change those circles, it might not be clear whether the conjugated function satisfies similar conditions any more. Therefore, it will often be convenient for us not to conjugate the function but to consider the original points. For this purpose, we need a version of Schottky's theorem involving general finite distinct points α and β instead of 0 and 1. This can be achieved as follows. If f is analytic in $\mathbb{D}$ and omits finite distinct points α and β there, then define $g(z) = (f(z)-\alpha)/(\beta-\alpha)$. Now g omits 0 and 1 and satisfies the assumptions of Schottky's theorem. We conclude after a computation that

$$\log^+ |f(z)| \leq A\frac{1+|z|}{1-|z|} + B, \tag{4}$$

where

$$A = \pi/2 + 2\log 3 + \log^+ |(f(0)-\alpha)/(\beta-\alpha)|, \tag{5}$$

$$B = \log^+ |\alpha| + \log^+ |\beta-\alpha| + \log 2, \tag{6}$$

whenever $|z| < 1$.

Using these estimates, and the estimates of the hyperbolic metric of the thrice punctured sphere that they are based on, one can prove the following consequence of Schottky's theorem, which is frequently used in complex dynamics.

Theorem 3.1. *Let D be a plane domain whose complement in the sphere contains at least three points. Let f be an analytic function omitting the finite distinct points α and β in D. Let K be a compact subset of D. Then there are constants $B > 1$ and $C > 1$ depending only on K, α, and β such that for all $z, w \in K$ we have*

$$|f(w) - \alpha| \leq B \max\{1, |f(z)-\alpha|\}^C. \tag{7}$$

4. Growth of entire functions and the $\cos \pi\rho$–theorem

Let f be an entire function. To discuss and properly understand even those results that are not directly in terms of the growth of the function, it is necessary to review a few notions concerning the growth of an entire function. We recall the $\cos \pi\rho$-theorem in the form given by P. Barry in [5], p. 294 (we note in passing that Barry has also obtained an analogous result involving lower order [6]).

If $E \subset [1, \infty)$, the *lower logarithmic density* of the set E is defined by

$$\underline{\text{log dens}}\ E = \liminf_{R\to\infty} \frac{1}{\log R} \int_{E\cap(1,R)} \frac{dt}{t}.$$

Theorem 4.1. *Let f be a transcendental entire function of order $\rho < 1/2$, and suppose that $\rho < \alpha < 1/2$. Then if*

$$E = \{\, r \geq 1 \,:\, \log m(r,f) > (\cos \pi\alpha)\, \log M(r,f) \,\}$$

we have

$$\underline{\text{log dens}}\ E \geq 1 - \frac{\rho}{\alpha}.$$

In particular, $\limsup_{r\to\infty} m(r,f) = \infty$.

If f is of order $1/2$, minimal type, then $m(r,f)$ is unbounded, and in fact, for each positive integer n, the quantity $m(r,f)/r^n$ is unbounded.

To see that for each positive integer n, the quantity $m(r,f)/r^n$ is unbounded, write the Taylor series of f at the origin as $f(z)=\sum_{k=0}^{\infty} a_k z^k$ and set $P_n(z)=\sum_{k=0}^{n} a_k z^k$. Then $g(z)=(f(z)-P_n(z))/z^{n+1}$ is a transcendental entire function of the same order as f, so that $m(r,g)$ is unbounded. This easily implies that $m(r,f)/r^{n+1}$ is unbounded.

The following example illustrates what Theorem 4.1 actually implies. When $\rho=\rho(f)<1/2$, set

$$\alpha=(1+2\rho)/4, \qquad \beta=\cos\left(\pi\,\frac{1+2\rho}{4}\right), \qquad \sigma=\frac{2}{1-2\rho}. \tag{8}$$

Now it is easily seen that there is a number R_0 such that for each $R\ge R_0$, we have

$$\log m(r,f)>\beta\,\log M(r,f)$$

for some r with $R\le r\le R^{\sigma}$.

The following observation will be useful for us.

Lemma 4.2. *Let f be a transcendental entire function such that*

$$\limsup_{r\to\infty} m(r,f)=+\infty.$$

If U is an unbounded component of $\mathcal{F}(f)$ and if U_n is the component of $\mathcal{F}(f)$ containing $f^n(U)$, then U_n is unbounded.

Proof of Lemma 4.2. If we pick $w\in U$ and $R>0$, there exists $r>|w|$ such that $m(r,f)>R$. Since U is unbounded, there exists $z\in U$ with $|z|>r$. Since U is connected, we may join z to w by a path in U and find a point $\zeta\in U$ with $|\zeta|=r$. Thus $|f(\zeta)|>R$. Since R was arbitrary, we conclude that U_1 is unbounded. The same argument combined with induction now shows that each U_n is unbounded. This proves Lemma 4.2.

5. Baker's original results

In the paper [2] where he stated his problem, Baker himself obtained the first results on it. Apart from posing the problem in the first place, the importance of his original contributions extends in two directions. Firstly, he obtained such a good result on components of the Fatou set where the iterates of the function have a finite limit function, that this result cannot be improved in terms of the rate of growth of $M(r,f)$ alone. Secondly, he obtained the first results based on the growth of the maximum modulus of the function alone and not on any other considerations. In this connection, he formulated and proved a lemma ([2], Lemma 9, p. 492) that has since been used in almost all papers to obtain results on the basis of growth.

Further, Baker [2] dealt with the case of what are now known as Baker domains, but only of period one. Since this requires somewhat different techniques from the case of domains where there is a finite limit function, we consider these cases separately.

We begin with the case of finite limit functions ([2], Theorem 1, p. 484).

Theorem 5.1. *Let f be a transcendental entire function whose growth is at most order* $1/2$ *minimal type. Let U be a component of the Fatou set of f such that every convergent subsequence f^{n_k} of the iterates of f converges locally uniformly in U to a finite limit function. Then U is bounded.*

About ten years earlier, Baker's student Bhattacharyya [9] had proved more specialized results in this direction. For example, he proved that if f is a transcendental entire function whose growth is at most order $1/2$ minimal type, and if U is an invariant attracting or superattracting component of $\mathcal{F}(f)$, then U is bounded.

For Baker domains, Baker [2] obtained the following result ([2], Theorem 1, p. 484).

Theorem 5.2. *Let f be a transcendental entire function whose growth is at most order* $1/2$ *minimal type. Then there does not exist a Baker domain of period one for f, that is, there is no component U of the Fatou set of f such that $f(U) \subset U$ and f^n converges locally uniformly in U to the constant infinity.*

Concerning growth, Baker proved the following result ([2], Theorem 2, p. 484).

Theorem 5.3. *Let f be a transcendental entire function such that for some real p with $1 < p < 3$, we have*

$$\log M(r, f) = O((\log r)^p) \tag{9}$$

as $r \to \infty$. Then every component of the Fatou set of f is bounded.

We now proceed to give Baker's proofs of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Let f be a transcendental entire function whose growth is at most order $1/2$ minimal type. Let U be a component of the Fatou set of f such that a subsequence f^{n_k} of the iterates of f converges locally uniformly in U to a finite limit function φ. Then there is a component V of $\mathcal{F}(f)$ such that for some integers $m \geq 0$ and $p \geq 1$, we have $f^m(U) \subset V$ and $f^p(V) \subset V$. By Theorem 4.1 and Lemma 4.2, U is bounded if, and only if, V is bounded. So it suffices to show that V is bounded. Note that our assumption that in U, every convergent subsequence of iterates of f converges locally uniformly in U to a finite limit function, implies that the same is true in V.

To get a contradiction, we assume that V is unbounded.

Choose $\alpha \in V$. Then the sequence $f^n(\alpha)$ is bounded, for otherwise there is a subsequence $f^{m_j}(\alpha) \to \infty$, and by passing to a subsequence without changing notation, we may assume that $f^{m_j} \to \psi$ locally uniformly in V. Now either ψ is analytic or $\psi \equiv \infty$. Since $f^{m_j}(\alpha) \to \infty$, we have $\psi \equiv \infty$, which contradicts our assumption that ψ is finite.

The next observation contains the seed of the generalization of Theorem 5.1 by Zheng (Theorem 6.2 in Section 6). Let A be a constant such that $A > 1$ and such that $|f^n(\alpha)| < A$ for all $n \geq 0$. Let γ be a path in V joining α to a point β with $|\beta| > 2A$ and

$$m(|\beta|, f) > |\beta|^2.$$

This is possible ([2], p. 490, line 12 from the bottom, says "This is impossible" but obviously means "This is possible") since $m(r, f)/r^2$ is unbounded by Theorem 4.1 and since V is unbounded. We may and will replace γ, without changing notation, by its subpath from α to its first point of modulus $= |\beta|$. Hence $|z| \leq |\beta|$ for all $z \in \gamma$.

The path $f(\gamma)$ joins $f(\alpha)$ to a point of modulus $> |\beta|$. By induction on n, we see that the path $f^n(\gamma)$ joins the point $f^n(\alpha)$ of modulus $< A$ to a point of modulus $> |\beta|^2 > |\beta| > 2A$. Since the convergence of any convergent subsequence f^{n_k} to its limit function in V would be uniform of γ, it follows that no limit function can be constant. This implies that V belongs to a Siegel disk cycle of domains. Hence the identity mapping is among the limit functions, that is, $f^{n_k}(z) \to z$ locally uniformly in V for some sequence f^{n_k}. But now $|z| \leq |\beta|$ for all $z \in \gamma$ while every $f^n(\gamma)$ contains a point of modulus $> |\beta|^2$. Since $|\beta|^2$ is a fixed number $> |\beta|$, we get a contradiction. This completes the proof of Theorem 5.1.

Remark. Analysing the proof of Theorem 5.1 we see that it depended only on two things: that a point has bounded orbit under f, and that $m(r, f)/r^2$ is unbounded, or just infinitely often > 1. A closer look reveals that instead of making $m(r, f)/r^2$ large, it would have been enough to make $m(r, f)/r$ large. The theorem of Zheng (Theorem 6.2 in Section 6) is based on this observation.

Proof of Theorem 5.2. Let f be a transcendental entire function whose growth is at most order 1/2 minimal type. Suppose that U is a component of the Fatou set of f such that $f(U) \subset U$ and f^n converges locally uniformly in U to the constant infinity. Then U is necessarily unbounded. Now [2], Theorem 6, p. 485, shows that there is a path γ in U tending to infinity on which $|f(z)| = O(|z|^k)$ for some constant

k. Hence $m(r, f) \leq Ar^k$ for some positive constant A, for all large r. This contradicts Theorem 4.1. The proof of Theorem 5.2 is complete.

Remark. By referring to [2], Theorem 6, p. 485 above, we have followed the historically accurate path. A better result, namely, that $|f(z)| = O(|z|)$ along a path going to infinity in a Baker domain was established later by Baker himself in [4]. The proof of [2], Theorem 6, p. 485, is based on Theorem 3.1. After Baker proved that all multiply connected components of the Fatou set of a transcendental entire function are wandering domains, it became clear that Baker domains are simply connected, so that a better estimate of the hyperbolic distance became available, to be used in place of Theorem 3.1. This then lead to the sharper result $|f(z)| = O(|z|)$.

6. Further results for periodic components of the Fatou set

Instead of asking for which functions f all the components of the Fatou set of f are bounded, one may ask for which functions f particular types of components of $\mathcal{F}(f)$ are bounded. Experience shows that wandering domains are the most difficult to deal with, followed by Baker domains, while components of $\mathcal{F}(f)$ where all limit functions of the iterates are finite are the easiest to handle.

If U and V are components of $\mathcal{F}(f)$ with $f(U) \subset V$ and if $m(r, f)$ is unbounded (for example, if the growth of f is at most of order $1/2$, minimal type), then by Lemma 4.2, U is unbounded if, and only if, V is unbounded. Therefore, when considering the question of whether a periodic or preperiodic component of the Fatou set of such a function f is unbounded, it is sufficient to consider periodic components.

Unlike the general study of complex dynamics, where one can further restrict one's consideration to invariant components on the grounds that a periodic component of period p for f is an invariant component of $\mathcal{F}(f^p)$ and $\mathcal{F}(f^p) = \mathcal{F}(f)$, one cannot usually perform the same reduction when studying Baker's problem. The reason is that even if something is assumed about the rate or regularity of growth of f, the iterates of f normally no longer possess those qualities.

The first improvement of Baker's original results for periodic components was obtained by Stallard [21, 22] who proved the following theorem ([21], Theorem 3A, p. 49).

Theorem 6.1. *Let f be a transcendental entire function of order $<$ $1/2$. Then there does not exist a Baker domain of any period for f, that is, there is no component U of the Fatou set of f such that $f^p(U) \subset U$*

for some $p \geq 1$ and such that f^n converges locally uniformly in U to the constant infinity.

Stallard has not published her proof of Theorem 6.1, and the result is not mentioned in [22]. A different proof of Theorem 6.1 was provided by Anderson and the author in [1]. Here we omit the proof but we note that the proof in [1] bears some similarity to the proofs of Theorems 8.3 and 9.1 below, which we will give in Section 10.

Note the small gap left by the fact that in Theorem 6.1, the order of f must be $< 1/2$, while in Theorem 5.2, the growth of f is at most order $1/2$ minimal type. This gap was filled by Zheng Jian-Hua [24] who proved the following results.

Theorem 6.2. ([24], Theorem 1, p. 355) *Let f be a transcendental entire function such that*

$$\limsup_{r\to\infty} \frac{m(r,f)}{r} = +\infty. \tag{10}$$

Let U be a periodic or preperiodic component of the Fatou set of f. Then U is bounded.

In particular, f has no Baker domains or Baker domain cycles.

This applies, in particular, if the growth of f is at most of order $1/2$, minimal type.

Theorem 6.3. ([24], Theorem 2, p. 356) *Let f be a transcendental entire function such that for all r in a sequence tending to ∞, we have*

$$m(r,f) > r. \tag{11}$$

Let U be a component of $\mathcal{F}(f)$. If there is a point $z_0 \in U$ such that the sequence $f^n(z_0)$ is bounded, then U is bounded.

Theorem 6.4. ([24], Theorem 3, p. 357) *Let f be a transcendental entire function of order $< 1/2$ and let U be a wandering domain of $\mathcal{F}(f)$. If there is a point $z_0 \in U$ such that*

$$\log^+ \log^+ |f^n(z_0)| = O(n)$$

as $n \to \infty$, then U is bounded.

We omit the proof of Theorem 6.4. We proceed to prove Theorems 6.2 and 6.3. We reproduce Zheng's arguments, beginning with his lemmas.

Lemma 6.5. ([24], Lemma 2.1, p. 357) *Let γ_1 and γ_2 be disjoint Jordan curves in $\mathbb{C}$ with $z_0 \in \operatorname{int}\gamma_2 \subset \operatorname{int}\gamma_1$. Suppose that f is analytic in $(\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1) \cup \gamma_1 \cup \gamma_2$, that $|f| > R$ on γ_1, and that $|f| \leq R$ on*

γ_2. *Then there exists a Jordan curve* $\Gamma \subset (\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1) \cup \gamma_2$ *with* $z_0 \in \operatorname{int}\Gamma$ *such that* $|f| = R$ *on* Γ.

Lemma 6.6. ([24], Lemma 2.2, p. 357) *Let f be a transcendental entire function. Suppose that $z_0 \in \mathbb{C}$, $R > 0$, and $|f^n(z_0)| < R$ for all $n \geq 0$. Suppose further that $A \geq 1$ and that $m(R, f) > AR$. Then, for each positive integer m, there exists a Jordan curve Γ_m in $\{z : |z| < R\}$ such that $z_0 \in \operatorname{int}\Gamma_m$ and*

$$|f^m(z)| \geq m(R, f) > AR$$

for all $z \in \Gamma_m$.

Proof of Lemma 6.5. We write $\alpha = f(\gamma_1)$. Since $f(\gamma_2) \subset \{w : |w| \leq R\}$ and $\alpha \subset \{w : |w| > R\}$, we have $0 \in \operatorname{int}\alpha$, for otherwise $f(\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1)$ is unbounded, which is impossible. Here we have used the fact that $\partial f(G) \subset f(\partial G)$ when f is analytic in the closure of a domain G. Hence $f(\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1) \supset \{w : |w| = R\}$. Since f takes each value in $f(\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1)$ only finitely many times in the set $\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1$, there exists at least one component β of $f^{-1}(\{w : |w| = R\})$ such that $z_0 \in \operatorname{int}\beta$, for otherwise there is a Jordan curve L from z_0 to ∞ which does not intersect $f^{-1}(\{w : |w| = R\})$, and this contradicts the fact that $|f|$ is continuous on the part of L lying in $\operatorname{ext}\gamma_2 \cap \operatorname{int}\gamma_1$. Therefore we may extract from β a Jordan curve Γ which has the required properties. This completes the proof of Lemma 6.5.

Proof of Lemma 6.6. The assumption of Lemma 6.6 implies that

$$|f| \geq m(R, f) > AR \tag{12}$$

on $\Gamma \equiv \{z : |z| = R\}$. Since $|f(z_0)| < R$ and $|z_0| < R$, there exists $R_1 > 0$ such that

$$R = \max\{|f(z)| : |z - z_0| = R_1\} > |f(z_0)|.$$

Applying the maximum modulus principle to f, in view of (12), we find that

$$\{z : |z - z_0| \leq R_1\} \subset \operatorname{int}\Gamma = \{z : |z| < R\}$$

and that $R > R_1 > 0$. Since $|f(z)| \leq R$ whenever $|z - z_0| = R$, it follows from Lemma 6.5 that there exists a Jordan curve $\Gamma_1 \subset \{z : R_1 < |z - z_0|\} \cap \{z : |z| < R\}$ such that $z_0 \in \operatorname{int}\Gamma_1$ and such that $|f| = R$ on Γ_1. Furthermore, by (12), we have

$$|f^2| \geq m(R, f) > AR$$

on Γ_1.

To continue by induction on k, suppose that for a positive integer k there exists a Jordan curve $\Gamma_{k-1} \subset \{z : |z| < R\}$ with $z_0 \in \operatorname{int} \Gamma_{k-1}$ such that

$$|f^k| \geq m(R, f) > AR \tag{13}$$

on Γ_{k-1}. Since $|f^k(z_0)| < R$, there exists $R_k > 0$ such that

$$R = \max\{|f^k(z)| : |z - z_0| = R_k\} > |f^k(z_0)|.$$

Applying the maximum modulus principle to f^k, in view of (13), we find that

$$\{z : |z - z_0| \leq R_k\} \subset \operatorname{int} \Gamma_{k-1}$$

and that $R > R_k > 0$. Thus, whenever $|z - z_0| = R_k$, we have $|f^k(z)| \leq R$. By Lemma 6.5, there exists a Jordan curve $\Gamma_k \subset \{z : R_k < |z - z_0|\} \cap \operatorname{int} \Gamma_{k-1}$ such that $z_0 \in \operatorname{int} \Gamma_k$ and such that $|f^k| = R$ on Γ_k. Furthermore, by (13), we have

$$|f^{k+1}| \geq m(R, f) > AR$$

on Γ_k.

By induction on k, we thus obtain the conclusion of Lemma 6.6. This completes the proof of Lemma 6.6.

After this preparation, we are ready to prove Zheng's theorems 6.2 and 6.3.

Proof of Theorem 6.2. Let f be as in Theorem 6.2. To get a contradiction, suppose that U is an unbounded periodic or preperiodic component of $\mathcal{F}(f)$. By (10) and Lemma 4.2, the domain $f^n(U)$ is unbounded for each $n \geq 1$. So we may assume without loss of generality that U is periodic of period $p \geq 1$, so that $f^p(U) \subset U$.

The function f^2 must have at least one fixed point in $\mathbb{C}$, so there exists $z_0 \in \mathbb{C}$ with $f^2(z_0) = z_0$, by, e.g., [8]. By (10), if $A > 1$ is given, there exists a large R such that $m(R, f) > AR$ and $R > |z_0| + |f(z_0)| \geq |f^n(z_0)|$ for all $n \geq 0$.

By Lemma 6.6, for an arbitrary positive integer j, there exists a Jordan curve Γ_j with $\Gamma_j \subset \{z : |z| \leq R\}$ with $z_0 \in \operatorname{int} \Gamma_j$ such that

$$|f^j(z)| \geq m(R, f) > AR \tag{14}$$

for all $z \in \Gamma_j$.

Suppose first that $f^n(z) \to \infty$ as $n \to \infty$ locally uniformly for $z \in U$. Then U belongs to a cycle of Baker domains. Since U is a Baker domain of period p, it follows from [4], Corollary, that there exist an arc γ_1 in U going from a finite point to infinity, and a positive constant L_1 such that

$$|f^p(z)| < L_1|z|$$

for all $z \in \gamma_1$.

Let γ_2 be a line segment connecting z_0 and the finite end point of γ_1 (so γ_2 need not be contained in U). Since f^p is entire, there exists a positive constant L_2 such that

$$|f^p(z)| \le L_2$$

for all $z \in \gamma_2$.

Thus

$$|f^p(z)| \le L_1|z| + L_2$$

for all $z \in \gamma = \gamma_1 \cup \gamma_2$.

Let z_p be a point of intersection of γ and Γ_p. Clearly $|z_p| \le R$. From (14) with $j = p$ it follows that

$$AR < |f^p(z_p)| \le L_1|z| + L_2 \le L_1 R + L_2.$$

We obtain $A < L_1 + L_2/R \le L_1 + L_2$. In this inequality, L_1 and L_2 are fixed numbers while A can be chosen to be arbitrarily large. This is a contradiction. It follows that a function f satisfying the assumptions of Theorem 6.2 cannot have a Baker domain cycle.

By the classification of dynamics, it now follows that there exists a point $\alpha \in U$ with bounded orbit, so there is a positive number B such that $|f^n(\alpha)| < B$ for all $n \ge 0$. We may apply Lemma 6.6, taking $A > 2$ and taking R, in addition to other requirements, so large that $R > B + |\alpha|$, and taking $z_0 = \alpha$. This is possible by the assumptions of Theorem 6.2. It follows that there is a Jordan arc γ in U connecting α and a point of $U \cap \{z : |z| = R\}$. It follows from (14) that for each $n \ge 0$, the set $f^n(\gamma)$ contains points in both $\{z : |z| < B\}$ and $\{z : |z| > R\}$. Hence no subsequence of f^n can have a constant limit in U. Thus U belongs to a Siegel disk cycle. Hence there exists a sequence f^{n_k} tending to the identity mapping locally uniformly in U as $k \to \infty$. Then for all large n_k, and for all $z \in \gamma$, we have $|f^{n_k}(z)| < 2|z|$. Choosing z to be a point of intersection of γ and Γ_{n_k}, we find by (14) that also $AR < |f^{n_k}(z)| < 2|z| \le 2R$. Since $A > 2$, this gives a contradiction. The proof of Theorem 6.2 is now complete.

Proof of Theorem 6.3. Let f satisfy the assumptions of Theorem 6.3. Suppose that $\mathcal{F}(f)$ has an unbounded component U such that for some $z_0 \in U$, the orbit $\{f^n(z_0) : n \ge 0\}$ is bounded. Pick $B > 1$ such that $|f^n(z_0)| < B$ for all $n \ge 0$. We apply Lemma 6.6, taking $A = 1$ and choosing R so large that in addition to all other conditions, we have $R > B + |z_0|$. It follows that for each $j \ge 1$ there is a Jordan curve Γ_j contained in $\{z : |z| \le R\}$ and containing z_0 in its interior, on which

$$|f^j(z)| \ge m(R, f) > R. \tag{15}$$

As in the proof of Theorem 6.2, when considering a point α with a bounded orbit, we deduce that there exists a positive integer q such that the component V of $\mathcal{F}(f)$ containing $f^q(U)$ belongs to a Siegel disk cycle. We may now assume that $U = V$. There exists a sequence $n_k \to \infty$ such that $f^{n_k}(z) \to z$ locally uniformly for $z \in U$ as $k \to \infty$. Let γ be a Jordan arc in $U \cap \{z : |z| \leq R\}$ joining z_0 to a point of $U \cap \{z : |z| = R\}$. Then for each j there is a point of intersection z_j of γ and Γ_j. Since $m(R, f) > R$, we can choose $\varepsilon > 0$ such that $m(R, f) > (1+\varepsilon)R$. Set $d = (m(R, f)/R) - \varepsilon > 1$. Then for sufficiently large n_k, due to the uniform convergence of f^{n_k} to the identity map on γ, we have $|f^{n_k}(z_{n_k})| < d|z_{n_k}|$, which contradicts (15). This completes the proof of Theorem 6.3.

Remark. After investigating the above proofs due to Zheng, it might be said that their essence is contained in two ideas: carefully refining Baker's proof of Theorem 5.1, and making use of the Jordan curves Γ_j on which $|f^j|$ is bounded below.

7. Results based on growth alone

A number of authors have obtained results to the effect that all components of the Fatou set of a transcendental entire function f are bounded provided that the maximum modulus of f has a given upper bound on its growth, regardless of the regularity of the growth. The first such result is Theorem 5.3 due to Baker. The next author to extend these results was Baker's student Stallard [22]. She obtained the following theorem ([22], Theorem B, p. 43).

Theorem 7.1. *Let f be a transcendental entire function such that for some $\varepsilon > 0$, we have*

$$\log\log M(r, f) = O\left(\frac{(\log r)^{1/2}}{(\log\log r)^{\varepsilon}}\right)$$

as $r \to \infty$. Then every component of the Fatou set of f is bounded.

While there have been further results involving the growth of f, they have all included conditions on the regularity of growth, in addition to the rate of growth. Therefore it seems that Theorem 7.1 still yields the best known result involving the rate of growth only, allowing one to conclude that all components, and not only all components of a particular type, of the Fatou set of f are bounded.

The methods employed by Baker, Stallard, and others to obtain results on the basis of the rate of growth have been very clever. The proof by Baker of Theorem 5.3 ([2], Theorem 2, p. 484) was based on careful estimates of P.D. Barry on the minimum modulus of slowly growing functions, a much improved version of the $\cos \pi\rho$–theorem for functions satisfying (9) for some finite p. The proof by Stallard for Theorem 7.1 is considerably more complicated, and starts with growth estimates for functions of zero order that can be found in Cartwright's monograph ([11], Theorems 52 and 53, pp. 83, 84). We omit these proofs here, since while they are ingenious, they are quite technical and lengthy.

However, we shall discuss one of the facts that forms an intellectual basis for those proofs. It is the lemma now often known as the Baker–Stallard lemma. In Baker's paper [2], it appears for all intents and purposes as Lemma 9, p. 492, restricted to the situation considered there. In its present slightly more general form, it was formulated and proved by Stallard ([22], Lemma 2.7, p. 45). For technical reasons, we prefer not to use the notation $(R_n)^{c(n)}$ as given there, where $c(n) > 1$, but denote these numbers by t_n, since they really form a new sequence of radii $> R_n$.

Lemma 7.2. *Suppose that f is a transcendental entire function and that there exist sequences R_n, ρ_n, and t_n, tending to infinity with n, such that*

(1) $R_{n+1} = M(R_n, f)$ *for all* n;
(2) $R_n \leq \rho_n \leq t_n$ *for all* n; *and*
(3) $m(\rho_n, f) > t_{n+1}$ *for all sufficiently large* n.

Then all the components of the Fatou set of f are bounded.

Proof of Lemma 7.2. Let f satisfy the assumptions of Lemma 7.2. To get a contradiction, suppose that U is an unbounded component of $\mathcal{F}(f)$. Choose distinct points α and β in $\mathcal{J}(f)$ such that $f(\alpha) = \beta$.

Choose a positive integer m_1 such that assumption (3) of Lemma 7.2 holds for all $n \geq m_1$. Since U is unbounded and connected, there is $m_2 \geq m_1$ such that U intersects all circles $\gamma_n = \{z : |z| = R_n\}$, $\gamma'_n = \{z : |z| = t_n\}$, and $\gamma''_n = \{z : |z| = \rho_n\}$ for all $n \geq m_2$. We choose an integer $N \geq m_2$ and note that U contains a Jordan arc C joining a point $w_N \in \gamma_N$ to a point $w'_{N+1} \in \gamma'_{N+1}$. It is clear by assumption (2) of Lemma 7.2 that C must contain a point $w''_{N+1} \in \gamma''_{N+1}$.

Now $f(U)$ is contained in a component of $\mathcal{F}(f)$ containing $f(C)$. By assumption (1) of Lemma 7.2, we have $R_{N+1} = M(R_N, f) \geq |f(w_N)|$. Since $m(\rho_{N+1}, f) > t_{N+2}$, we have $|f(w''_{N+1})| > t_{N+2}$. Hence $f(C)$ must contain an arc joining a point $w_{N+1} \in \gamma_{N+1}$ to a point $w'_{N+2} \in \gamma'_{N+2}$.

We continue this process inductively and find that $f^k(U)$ is contained in a component of $\mathcal{F}(f)$ containing an arc of $f^k(C)$ which joins a point $w_{N+k} \in \gamma_{N+k}$ to a point $w'_{N+k+1} \in \gamma'_{N+k+1}$.

Thus, on C, the function f^k takes a value of modulus at least R_{N+k}. Since $R_{N+k} \to \infty$ as $k \to \infty$, and since U is a component of $\mathcal{F}(f)$, we deduce that $f^k \to \infty$ as $k \to \infty$, locally uniformly in U. Since each f^k omits α and β in U, it follows that there exists a positive integer L such that for all $k \geq L$ and for all $z \in C$, we have $|f^k(z)| > 2\max\{|\alpha|, |\beta|\}$. It now follows from Theorem 3.1 that there exist constants A and B such that

$$|f^k(z)| < A|f^k(w)|^B$$

whenever $z, w \in C$ and $k \geq L$.

For each $k \geq L$, we can choose $z_k, \zeta_k \in C$ such that $f^k(z_k) = w_{N+k} \in \gamma_{N+k}$ and $f^k(\zeta_k) = w'_{N+k+1} \in \gamma'_{N+k+1}$. Hence for each $k \geq L$, we have

$$M(R_{N+k}, f) = R_{N+k+1} \leq t_{N+k+1} = |f^k(\zeta_k)| < A|f^k(z_k)|^B = A(R_{N+k})^B.$$

Since $R_{N+k} \to \infty$ as $k \to \infty$, we obtain a contradiction with the assumption that f is transcendental, so its maximum modulus cannot grow polynomially in any sequence. This completes the proof of Lemma 7.2.

We see that Schottky's theorem is the fundamental reason why Lemma 7.2 works. Other than that, the assumptions of Lemma 7.2 are geared towards assuring that the image of an arc, which itself covers a sufficiently long radial variation, under f stretches so far in the radial direction that this stretch is retained under iteration. Such long stretches are then incompatible with Schottky's theorem.

8. Survey of results based on the regularity of growth

The first results based on the regularity of the growth of the maximum modulus of f were obtained by Stallard [22]. She proved the following theorem ([22], Theorem C, p. 44).

Theorem 8.1. *Let f be a transcendental entire function of order $\rho < 1/2$. Suppose that there exists a real number $c \in [1, \infty)$ such that*

$$\lim_{r\to\infty} \frac{\log M(2r, f)}{\log M(r, f)} = c. \tag{16}$$

Then every component of the Fatou set of f is bounded.

Stallard ([22], p. 44) notes that as Baker observed in [2], p. 489, if

$$f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} + z + a,$$

where a is a sufficiently large positive constant, then $\mathcal{F}(f)$ has an unbounded component while f is of order $1/2$ with

$$\lim_{r\to\infty} \frac{\log M(2r, f)}{\log M(r, f)} = \sqrt{2}.$$

This shows that the condition $\rho < 1/2$ in Theorem 8.1 cannot be relaxed to include $\rho = 1/2$.

To prove her result, Theorem 8.1, Stallard showed that there exist sequences of radii required by the assumptions of Lemma 7.2. For functions of positive order of such regular growth, this turned out to be reasonably easy. For functions of zero order, she again made use of the estimates found in Cartwright's monograph mentioned before. We omit the details as they are quite technical.

Hua and Yang also studied the question of what can be proved on the basis of the regularity of growth. Theorem 1 in [18], p. 1283 states the following.

Theorem 8.2. *Let f be a transcendental entire function. If the order of f is $< 1/2$, then all components of the Fatou set of f are bounded provided that there exist real numbers C_1 and C_2 with $C_2 > 1$ and $C_1 > C_2^2$ such that for all sufficiently large r, we have*

$$\log M(C_1 r, f) < C_2 \log M(r, f).$$

This means that the increasing function $M(r, f)$ does not jump higher too fast anywhere, for large r.

We omit the proof of Theorem 8.2. Unfortunately, the author is not able to follow certain arguments in the proof of Lemma 2 in [18], which is required for the proof of [18], Theorem 1 (that is, our Theorem 8.2), particularly equation (13) in [18], p. 1287, where the assumed regularity condition on $M(r, f)$ is applied to an auxiliary function $G(r)$, which is the maximum modulus function of an entire function related to but different from f.

Theorems 2 and 3 in the paper by Hua and Yang ([18], pp. 1283–1284) contain the following claims:

Let f be a transcendental entire function. If the lower order $\mu(f) < 1/2$ then every preperiodic component of $\mathcal{F}(f)$ is bounded. If the lower order $\mu(f) < 1/2$ and if for each real number $m > 1$, we have

$$\log M(r^m, f) \geq m^2 \log M(r, f)$$

for all sufficiently large r, then every component of $\mathcal{F}(f)$ is bounded.

The proofs of both theorems are eventually based on [18], Lemma 4, p. 1288 in an essential way. Here again the author is not able to follow the details of the proof of [18], Lemma 4, which seems to require an inequality for the lower logarithmic density of the set E where the minimum modulus of f is sufficiently large, rather than for the upper logarithmic density, which is what the assumption that the lower order of f is less than 1/2 would yield.

J.M. Anderson and the author introduced another regularity condition. We set $\varphi(x) = \log M(e^x, f)$ so that, by the Hadamard three-circles theorem, $\varphi(x)$ is an increasing convex function of x. The function $\varphi'(x)$ may fail to exist at a countable set of points x. At such points $\varphi'(x)$ is defined to be the right-hand derivative. We proved the following result ([1], Theorem 2, p. 3245).

Theorem 8.3. *Let f be a transcendental entire function of order $\rho < 1/2$ such that for some positive constant c*

$$\frac{\varphi'(x)}{\varphi(x)} \geq \frac{1+c}{x} \tag{17}$$

for all sufficiently large x, where $\varphi(x) = \log M(e^x, f)$. Then every component of $\mathcal{F}(f)$ is bounded.

The one-sided condition (17) may be compared to Stallard's condition (16), which is essentially the same as

$$\varphi(x+1) \sim c\varphi(x) \qquad \text{as } x \to \infty.$$

The assumption (17) is related to growth as well as regularity. It is equivalent to the condition

$$\liminf_{r\to\infty} \frac{d \log\log M(r,f)}{d(\log\log r)} > 1$$

which implies, in particular, that

$$\log M(r,f) > (\log r)^{1+\delta}$$

for some $\delta > 0$ and all $r > r_0(\delta)$. This guarantees a certain minimal growth rate for $M(r,f)$ from some point onwards. It prevents the existence of large annuli which are almost zero free, where the growth of the function f becomes like that of a polynomial. If the radial spread in such an annulus is large, then a function of order $< 1/2$ (or even of order < 1), which is given by an infinite product, will eventually behave like a polynomial in the annulus. If we have a sequence of such annuli, and if the degree of the polynomial that f resembles in the $n-$th annulus is p_n, then $p_n \to \infty$ as $n \to \infty$. However, regardless of how large p_n is, the condition (17) will eventually be violated, for any fixed $c > 0$, if the annuli get large enough.

We will prove Theorem 8.3 in Section 10.

Remark. Since the condition (17) essentially amounts to the requirement for a lower bound for the rate of growth of the maximum modulus of f, it is perhaps not surprising that it is satisfied by any function of finite order and positive lower order. To see this, suppose that f has finite order $< \tau$ and lower order $> \delta > 0$. Then for all sufficiently large r we have

$$0 < \delta < \frac{\log\log M(r,f)}{\log r} < \tau < +\infty,$$

that is,

$$e^{\delta x} < \varphi(x) < e^{\tau x} \tag{18}$$

for all sufficiently large x. Choose ε with $0 < \varepsilon = \delta/(2\tau)$ and then choose $c > 0$ so small that $(1+c)(1-\varepsilon) < 1$. It follows that for all sufficiently large x we have

$$(1+c)(1-\varepsilon) \le 1 - e^{-\tau x} = 1 - \exp(x(\tau - \delta/\varepsilon)). \tag{19}$$

Suppose that $x_0 > 1$ is so large that (18) and (19) are both valid for all $x \ge x_0$. Then pick any $x_2 > x_0/\varepsilon$ and set $x_1 = \varepsilon x_2$, so that $x_0 < x_1 < x_2$. Since $\varphi'(x)$ is increasing, we have

$$\varphi'(x_2)(x_2 - x_1) \ge \int_{x_1}^{x_2} \varphi'(t)\,dt = \varphi(x_2) - \varphi(x_1)$$

and hence

$$x_2\frac{\varphi'(x_2)}{\varphi(x_2)} \ge \frac{1-\varphi(x_1)/\varphi(x_2)}{1-x_1/x_2} \ge \frac{1-e^{\tau x_1 - \delta x_2}}{1-\varepsilon} = \frac{1-e^{x_1(\tau-\delta/\varepsilon)}}{1-\varepsilon} \ge 1+c.$$

This yields (17). Of course, even though this argument only required f to be of finite order, in Theorem 8.3 the order of f must be $< 1/2$ for other reasons.

9. A minimum modulus problem for functions of order $< 1/2$

Let us summarize the results that have been obtained concerning Baker's problem; for this purpose, we limit ourselves to the growth considered by Baker in his conjecture. If the growth of the transcendental entire function f does not exceed order 1/2, minimal type, then every periodic and preperiodic component of the Fatou set of f is known to be bounded (Theorem 6.2). So it remains to study wandering domains, which may be assumed to be simply connected since any multiply connected component of $\mathcal{F}(f)$ is known to be bounded (and, incidentally, a wandering domain) by a result of Baker [3]. If f is exactly of order 1/2, minimal type, then it appears that nothing is known. If the order of f is $< 1/2$, then wandering domains are bounded if $M(r,f)$ has sufficiently regular growth, as is seen from Theorems 8.1, 8.2, and 8.3. As we have seen, the case where a problem remains can at least heuristically be characterized as the case where there is a sequence of annuli of increasing radial spread in which f has few or no zeros, so that in each of those annuli, f behaves eventually like a polynomial, of degree that tends to infinity in the sequence of annuli.

It does not seem plausible that any fixed regularity condition of the type (17) could capture all transcendental entire functions of order $< 1/2$. Therefore, instead of referring to a prescribed sort of regularity, one might try to develop a condition that simply refers back to the function itself. Equivalently, one can try to distill from the known proofs the property of f minimally required for the proofs to work out. In this connection, the author proved a theorem involving such a condition, and asked whether the condition holds for all transcendental entire functions of order $< 1/2$.

The result is the following ([17], Theorem 1).

Theorem 9.1. *Let f be a transcendental entire function of order $<$ $1/2$. Suppose that there exist positive numbers R_0, L, δ, and C with $R_0 > e$, $M(R_0, f) > e$, $L > 1$, and $0 < \delta \leq 1$ such that for every $r > R_0$ there exists $t \in (r, r^L]$ with*

$$\frac{\log m(t,f)}{\log M(r,f)} \geq L\left(1 - \frac{C}{(\log r)^{\delta}}\right). \tag{20}$$

Then all the components of the Fatou set of f are bounded.

We will prove Theorem 9.1 in Section 10. The corresponding question is then the following ([17]).

Minimum modulus problem. *Does every transcendental entire function of order $< 1/2$ satisfy the assumptions of Theorem 9.1 ?*

As a motivation for this question, the author noted the following in [17]. As we have discussed, the issue is to guarantee that if there is at least a certain radial spread in a component U of $\mathcal{F}(f)$, in the sense that the component contains points z and w such that $\log|w|/\log|z|$ is large, then the same spread is retained in the image components containing $f^n(U)$. The novelty of the condition of Theorem 9.1 is that, unlike any of its predecessors, it allows the radial spread to be reduced, due to the presence of the factor $1 - \frac{C}{(\log r)^{\delta}}$ on the right hand side of (20). "Nonetheless, it turns out that if we start with a sufficiently large spread, and in an unbounded domain we may start with as large a spread as we like, we are able to keep the propagated spreads large enough even if they are reduced from the original value.

Even though we are allowed to use any $L > 1$ in Theorem 9.1, and in particular we may take L close to 1 if that works, it seems likely that, particularly if the order of f is close to $1/2$, we may need to take L very large, depending on $\rho(f)$. If we are not close to or inside an annulus containing very few zeros of f, it would seem plausible that the condition (20) should be easy to satisfy, with a wide margin, by taking t to be a value arising from the consequences of Theorem 4.1. This is because then $\log m(t, f)/\log M(t, f)$ is greater than a fixed constant while $\log M(t, f)/\log M(r, f)$ should be quite large. So there should be a potential problem at most if we are in an annulus where f behaves like a polynomial. But in that case we should be able to take t close to r^L, and then the three numbers $\log m(t, f)$, $\log M(t, f)$, and $L \log M(r, f)$, should be close together. There may be some error term required to estimate $\log m(t, f)/(L \log M(r, f))$ from below, but (20) allows for such a term. For this reason, it seems sensible to ask whether every transcendental entire function f of order $< 1/2$ satisfies the assumptions of Theorem 9.1, and perhaps even conjecture that this is so." [17]

It might be of some interest to study whether the question can be answered for those functions f for which Baker and Stallard have already proved that all components of the Fatou set are bounded. For example, to start with the strongest reasonable assumptions, does f satisfy the assumptions of Theorem 9.1 provided that f satisfies (9) for some real $p > 1$, for example, for $p = 2$? One can ask whether the careful estimates used by Baker and by Stallard for these functions might be helpful also in the study of this minimum modulus problem.

10. Proofs of results based on the regularity of growth: self-sustaining spread

In this section we give a unified proof for Theorems 8.3 and 9.1. Both are based on what Anderson and the author called self-sustaining spread in [1]. Only the precise cause of the spread differs: in Theorem 8.3 it arises from the assumption (17), and in Theorem 9.1 it arises from the assumption (20).

Thus we assume throughout that f is a transcendental entire function of order $< 1/2$. To get a contradiction, we assume that the Fatou set of f contains an unbounded component D, which is a simply connected wandering domain. As mentioned, all other cases have already been settled.

Roughly speaking, the proof is divided into two very different parts. The first part, and the only part that uses dynamics, amounts to showing that since we are dealing with a component of the Fatou set, the radial spreads that we get cannot get too large in a certain sense. The second part uses (17) or (20), and shows that the spread must be larger than that after all. These two facts are incompatible, so we get a contradiction, which then completes the proofs of these theorems.

Suppose that K is a compact subset of D. In this first part of the proof, our aim is to show that for a certain complex constant a depending only on D and for a possibly large positive number $C > 1$ depending on K, we have

$$\frac{1}{C} \leq \frac{|f^j(z) - a|}{|f^j(w) - a|} \leq C \tag{21}$$

for all $z, w \in K$ and for all $j \geq 0$. This is basically a consequence of standard estimates for the hyperbolic metric in simply connected domains.

To find a, note that since D is a wandering domain, it is disjoint from any of its inverse images. Thus there is a disk $B(a, \rho)$ such that

$$B(a, \rho) \cap \cup_{j=0}^{\infty} f^j(D) = \emptyset. \tag{22}$$

Let D_j be the component of the Fatou set of f containing $f^j(D)$. Note that also D_j is an unbounded wandering domain of f and hence is simply connected.

Let $L > 1$ be a large constant, to be determined soon. Pick $j \geq 0$ and $z, w \in K$. Suppose that $|f^j(z) - a|/|f^j(w) - a| > L$. Let $\zeta \in \partial D_j$ be the point closest to a, so that in particular, $|\zeta - a| < |f^j(w) - a|$. Recall that $h_\Omega(z_1, z_2)$ denotes the hyperbolic distance between the points z_1, z_2 of the domain Ω, and that $\lambda_\Omega(z)$ denotes the density of the hyperbolic metric of Ω at $z \in \Omega$. Thus

$$h_{D_j}(f^j(z), f^j(w)) \leq h_D(z, w) \leq L_0 \equiv \max\{h_D(z_1, z_2) : z_1, z_2 \in K\}.$$

Since D_j is simply connected, it follows from Koebe's one-quarter theorem that

$$\lambda_{D_j}(z) \geq \frac{1}{4\,\mathrm{dist}\,(z, \partial D_j)} \geq \frac{1}{4|z - \zeta|} \geq \frac{1}{4(|z - a| + |\zeta - a|)}$$

for all $z \in D_j$, where $\operatorname{dist}(z, \partial D_j)$ denotes the Euclidean distance of z from ∂D_j. Hence

$$\begin{aligned} L_0 \quad &\geq h_{D_j}(f^j(z), f^j(w)) \geq \int_{|f^j(w)-a|}^{|f^j(z)-a|} \frac{dr}{4(r+|\zeta-a|)} \\ &= \frac{1}{4} \log \frac{|f^j(z)-a|+|\zeta-a|}{|f^j(w)-a|+|\zeta-a|} \\ &\geq \frac{1}{4} \log \frac{|f^j(z)-a|+|f^j(w)-a|}{2|f^j(w)-a|} \\ &= \frac{1}{4}\left(\log\left(1+\frac{|f^j(z)-a|}{|f^j(w)-a|}\right)-\log 2\right) \\ &\geq \frac{1}{4}\left(\log(1+L)-\log 2\right), \end{aligned}$$

which gives a contradiction if L is sufficiently large if compared to L_0. This proves (21).

Next we observe that even though the constant C depends on K and may be large, we can control the radial spread of the set $f^j(K)$ better by using the logarithmic scale.

Suppose that C_0 is a preassigned constant subject only to $C_0 > 1$. Next we show that by (21) and (22), we have

$$\frac{1}{C_0} \leq \frac{\log(2|f^j(z)-a|/\rho)}{\log(2|f^j(w)-a|/\rho)} \leq C_0 \tag{23}$$

for all $z, w \in K$ and for all sufficiently large $j \geq j_0$, say. Having to restrict ourselves to $j \geq j_0$ is one cost that we pay in order to get an estimate involving an arbitrary $C_0 > 1$. For if (23) does not hold, then there are sequences $z_j, w_j \in K$ and integers $n_j \to \infty$ such that

$$\frac{\log(2|f^{n_j}(z_j)-a|/\rho)}{\log(2|f^{n_j}(w_j)-a|/\rho)} > C_0,$$

that is,

$$\frac{2|f^{n_j}(z_j)-a|}{\rho} > \left(\frac{2|f^{n_j}(w_j)-a|}{\rho}\right)^{C_0}. \tag{24}$$

By passing to a subsequence, we may assume that $|f^{n_j}(z_j)-a| \to R_2$ and $|f^{n_j}(w_j)-a| \to R_1$, say, where $\rho \leq R_1 < R_2 < \infty$ or $R_1 = R_2 = \infty$. In the former case, we do not have $f^{n_j} \to \infty$ locally uniformly in D, so that by passing to a further subsequence, we may assume that $f^{n_j} \to \omega$ locally uniformly in D, where ω is a complex number with $|\omega - a| \geq \rho$ (by (22)). Hence $f^{n_j}(z_j) \to \omega$ and $f^{n_j}(w_j) \to \omega$ as $j \to \infty$, which contradicts (24). Thus $R_1 = R_2 = \infty$. But now, by (21),

$|f^{n_j}(z_j)-a| \le C|f^{n_j}(w_j)-a| < (2/\rho)^{C_0-1}|f^{n_j}(w_j)-a|^{C_0}$ when $|f^{n_j}(w_j)|$ is large enough, which is a contradiction. This completes the proof of (23). This also finishes off the first part of the proofs of Theorems 8.3 and 9.1. We have now established an upper bound for radial spread, which is effective since the number $C_0 > 1$ is still at our disposal and so we may choose C_0 to be very close to 1.

We proceed to show that if we choose K to be of large radial spread, as we may since we are choosing a compact subset of an unbounded domain D, then the large radial spread will have to persist, to the extent that we will end up contradicting (23). This contradiction then shows that the domain D with its defining properties could not exist at all, and the proofs of Theorems 8.3 and 9.1 will be complete.

First we deal with the situation in Theorem 8.3, so we assume that (17) holds.

We will explain in a moment how K is to be chosen. Before that, we make a number of general observations. We choose a real number C_0 with $1 < C_0 < 4$. We may assume that (23) holds for $j \ge j_0$, say, for all $z, w \in K$. If $\beta > 0$ is given, we have $\beta \log M(r, f) > \log r$ for all $r \ge R_0$, say. Suppose that (17) holds for all $x \ge x_0 > 1$, with a certain positive constant c. Then, if $x_0 < x_1 < x_2$, we have

$$\log \frac{\varphi(x_2)}{\varphi(x_1)} = \int_{x_1}^{x_2} \frac{\varphi'(x)}{\varphi(x)}\, dx \ge \int_{x_1}^{x_2} \frac{1+c}{x}\, dx = (1+c) \log \frac{x_2}{x_1}$$

and so

$$\frac{\varphi(x_2)}{\varphi(x_1)} \ge \left(\frac{x_2}{x_1}\right)^{1+c}.$$

Recall the definition (8) of σ and β, in terms of the order $\rho = \rho(f) < 1/2$ of f. Let $L > \sigma^2 > 4$ be a large constant satisfying also $\beta L^c > \sigma^{2(1+c)}$. Now β determines R_0 as above. We further choose R_0 so large that in addition, we have

$$(25)\quad \log(2|w-a|/\rho) \ge \sqrt{4/L}\log|w|,\ \log(2|z-a|/\rho) \le \sqrt{L/4}\log|z|$$

whenever $|z| \ge R_0$ and $|w| \ge R_0$. Suppose that the compact connected subset K of D is chosen so that $\log|\zeta| > \max\{x_0, \log R_0\}$ for all $\zeta \in K$ and so that there are $z, w \in K$ with $\log|w| > L\log|z|$. This is possible since D is unbounded. Write $K_j = f^j(K)$ so that K_j is a compact connected subset of D_j.

It follows from Theorem 4.1 that there are r_1, r_2 with $|z| \le r_1 \le |z|^\sigma$ and $|w|^{1/\sigma} \le r_2 \le |w|$ such that $\log m(r_j, f) > \beta \log M(r_j, f)$ for $j = 1, 2$. Since K is connected, there are points $\zeta_1, \zeta_2 \in K$ with $|\zeta_j| = r_j$ for $j = 1, 2$. Write $x_j = \log r_j$ for $j = 1, 2$, so that $\log|z| \le x_1 \le \sigma\log|z|$

and $\sigma^{-1}\log|w| \le x_2 \le \log|w|$. We obtain

$$\begin{aligned}\frac{\log|f(\zeta_2)|}{\log|f(\zeta_1)|} &\ge \frac{\log m(r_2,f)}{\log M(r_1,f)} \ge \beta\,\frac{\log M(r_2,f)}{\log M(r_1,f)}\\ &= \beta\,\frac{\varphi(x_2)}{\varphi(x_1)} \ge \beta\left(\frac{x_2}{x_1}\right)^{1+c} \ge \beta\left(\frac{\sigma^{-1}\log|w|}{\sigma\log|z|}\right)^{1+c}\\ &> \beta\left(\frac{L}{\sigma^2}\right)^{1+c} > L.\end{aligned}$$

Also for $j = 1, 2$, we have

$$\begin{aligned}\log|f(\zeta_j)| &\ge \log m(r_j,f) > \beta\log M(r_j,f)\\ &> \log r_j = \log|\zeta_j| > \max\{x_0, \log R_0\}.\end{aligned}$$

Note that $\log r_1 = x_1 \ge \log|z| > \max\{x_0, \log R_0\}$ while, since $L > \sigma^2$, we have

$$\begin{aligned}\log r_2 &= x_2 \ge \sigma^{-1}\log|w| > L\sigma^{-1}\log|z|\\ &> \log|z| > \max\{x_0, \log R_0\}.\end{aligned}$$

We find that there are points $\zeta_3 = f(\zeta_1)$ and $\zeta_4 = f(\zeta_2)$ in $f(K)$ with $\log|\zeta_j| > \max\{x_0, \log R_0\}$ for $j = 3, 4$ and with

$$\frac{\log|\zeta_4|}{\log|\zeta_3|} > L.$$

Now we may repeat the above argument and deduce that for all $j \ge 1$, there are $z, w \in f^j(K)$ with $\log|w| > L\log|z|$. Since $C_0 < 4 < L$, we get a contradiction with (23) when $j \ge j_0$, using also (25). This completes the proof of Theorem 8.3.

Under the assumptions of Theorem 9.1, we proceed in much the same way. Only certain technicalities differ, but a close examination of the argument shows that the underlying ideas are quite similar to what was done above.

Suppose that (20) holds for the values of the parameters as given in the statement of Theorem 9.1. Note that the symbols C and L now mean the quantities in (20). Set

$$C_1 = C + \log^+ M(1,f) \ge C > 0.$$

Choose $C_2 > 2$ so large that

$$\frac{1}{L} < \prod_{n=1}^{\infty}\left(1 - \frac{1}{C_2^{n\delta}}\right).$$

Next pick R' so that

$$r < r^{C_2} < M(r,f) \tag{26}$$

for all $r \geq R'$. Choose $R_1 = \max\{R_0, R', 2\exp(C_1^{1/\delta} C_2)\}$. Let K be a compact connected subset of D containing points z_0 and w_0 with

$$|w_0| > |z_0| > R_1$$

and

$$\frac{\log |w_0|}{\log |z_0|} > L^2. \tag{27}$$

Set $K_n = f^n(K)$. We seek to prove that for each $n \geq 1$, there are points $z_n, w_n \in K_n$ with

$$|w_n| > |z_n| > R_1$$

and

$$\frac{\log |w_n|}{\log |z_n|} > L^2 \prod_{k=1}^{n} \left(1 - \frac{1}{C_2^{k\delta}}\right) > L.$$

Since K is connected and (27) holds, there is $\zeta_0 \in K$ with $|w_0| = |\zeta_0|^L$. Thus $|\zeta_0| > |z_0|$. By (20), there is $t \in (|\zeta_0|, |w_0|]$ with

$$\frac{\log m(t,f)}{\log M(|\zeta_0|, f)} \geq L\left(1 - \frac{C}{(\log |\zeta_0|)^\delta}\right).$$

We have

$$|f(z_0)| \leq M(|z_0|, f).$$

Take any point $u_0 \in K$ with $|u_0| = t$. This is possible since K is connected. We have

$$\frac{\log |f(u_0)|}{\log |f(z_0)|} \geq \frac{\log m(t,f)}{\log M(|z_0|, f)} = \frac{\log m(t,f)}{\log M(|\zeta_0|, f)} \frac{\log M(|\zeta_0|, f)}{\log M(|z_0|, f)}.$$

We next find a lower bound for

$$\frac{\log M(|\zeta_0|, f)}{\log M(|z_0|, f)}.$$

If $1 < r_1 < r_2$ and $x_j = \log r_j$ for $j = 1, 2$, and if $r_1 = |z_0|$ and $r_2 = |\zeta_0|$, we have

$$\frac{\log M(|\zeta_0|, f)}{\log M(|z_0|, f)} = \frac{\varphi(x_2)}{\varphi(x_1)}.$$

Since φ is convex, we have

$$\varphi(x_1) \leq \frac{x_2 - x_1}{x_2}\varphi(0) + \frac{x_1}{x_2}\varphi(x_2)$$

so that

$$\varphi(x_2) \geq \frac{x_2}{x_1}\varphi(x_1) - \frac{x_2 - x_1}{x_1}\varphi(0),$$

hence

$$\frac{\varphi(x_2)}{\varphi(x_1)} \geq \frac{x_2}{x_1} - \left(\frac{x_2}{x_1} - 1\right)\frac{\varphi(0)}{\varphi(x_1)}.$$

If $\varphi(0) \leq 0$, we get

$$\frac{\varphi(x_2)}{\varphi(x_1)} \geq \frac{x_2}{x_1}.$$

In general, if $\varphi(0) > 0$,

$$\frac{\varphi(x_2)}{\varphi(x_1)} \geq \frac{x_2}{x_1}\left\{1 - \left(1 - \frac{x_1}{x_2}\right)\frac{\varphi(0)}{\varphi(x_1)}\right\} \geq \frac{x_2}{x_1}\left\{1 - \frac{\varphi(0)}{\varphi(x_1)}\right\}.$$

In particular,

$$\frac{\log M(|\zeta_0|, f)}{\log M(|z_0|, f)} = \frac{\varphi(x_2)}{\varphi(x_1)} \geq \frac{\log|\zeta_0|}{\log|z_0|}\left\{1 - \frac{\log M(1, f)}{\log M(|z_0|, f)}\right\}.$$

We conclude that

$$\begin{aligned}
\frac{\log|f(u_0)|}{\log M(|z_0|, f)} &\geq L\frac{\log|\zeta_0|}{\log|z_0|}\left(1 - \frac{C}{(\log|\zeta_0|)^\delta}\right)\left(1 - \frac{\log M(1, f)}{\log M(|z_0|, f)}\right) \\
&= \frac{\log|w_0|}{\log|z_0|}\left(1 - \frac{C}{(\log|\zeta_0|)^\delta}\right)\left(1 - \frac{\log M(1, f)}{\log M(|z_0|, f)}\right) \\
&\geq \frac{\log|w_0|}{\log|z_0|}\left(1 - \frac{C}{(\log|\zeta_0|)^\delta} - \frac{\log M(1, f)}{\log M(|z_0|, f)}\right) \\
&\geq \frac{\log|w_0|}{\log|z_0|}\left(1 - \frac{C}{(\log|z_0|)^\delta} - \frac{\log M(1, f)}{\log M(|z_0|, f)}\right) \\
&\geq \frac{\log|w_0|}{\log|z_0|}\left(1 - \frac{C + \log^+ M(1, f)}{(\log|z_0|)^\delta}\right)
\end{aligned}$$

since $\log M(|z_0|, f) \geq \log|z_0| \geq (\log|z_0|)^\delta$. So, whether $\varphi(0) \leq 0$ or $\varphi(0) > 0$, we have

$$\frac{\log|f(u_0)|}{\log M(|z_0|, f)} \geq L^2\left(1 - \frac{C_1}{(\log|z_0|)^\delta}\right) > L^2\left(1 - \frac{1}{C_2^\delta}\right) > L > 1$$

since $(\log|z_0|)^\delta > (\log R_0)^\delta \geq C_1 C_2^\delta$.

Since $|f(z_0)| \leq M(|z_0|, f)$ and $f(K) = K_1$ is connected, the set K_1 contains a point z_1 with $|z_1| = M(|z_0|, f)$. We set $f(u_0) = w_1$ and note that

$$\frac{\log|w_1|}{\log|z_1|} > L^2\left(1 - \frac{1}{C_2^\delta}\right) > L.$$

Lemma 10.1. *Suppose that $n \geq 1$ and that for all m with $1 \leq m \leq n$, there exist $z_m, w_m \in K_m$ with*

$$|w_m| > |z_m| > R_1,$$

$$|z_m| \geq |z_{m-1}|^{C_2} \geq |z_0|^{C_2^m}$$

for $1 \leq m \leq n$, *and*

$$\kappa_m \equiv \frac{\log |w_m|}{\log |z_m|} > L^2 \prod_{k=1}^{m} \left(1 - \frac{1}{C_2^{k\delta}}\right) > L.$$

Then K_{n+1} *contains points* z_{n+1} *and* w_{n+1} *such that*

$$|w_{n+1}| > |z_{n+1}| > R_1,$$

$$|z_{n+1}| \geq |z_n|^{C_2} \geq |z_0|^{C_2^{n+1}},$$

and

$$\kappa_{n+1} \equiv \frac{\log |w_{n+1}|}{\log |z_{n+1}|} > L^2 \prod_{k=1}^{n+1} \left(1 - \frac{1}{C_2^{k\delta}}\right) > L.$$

Proof of Lemma 10.1. Since K_n is connected and

$$\frac{\log |w_n|}{\log |z_n|} > L,$$

there is $\zeta_n \in K_n$ with

$$\frac{\log |w_n|}{\log |\zeta_n|} = L.$$

Now by (20), find $t \in (|\zeta_n|, |w_n|] = (|\zeta_n|, |\zeta_n|^L]$ with

$$\frac{\log m(t,f)}{\log M(|\zeta_n|, f)} \geq L\left(1 - \frac{C}{(\log |\zeta_n|)^{\delta}}\right) > 1. \tag{28}$$

Then choose $u_n \in K_n$ with $|u_n| = t$. Note that $|z_n| < |\zeta_n|$. We have, as before,

$$\begin{aligned}
\frac{\log |f(u_n)|}{\log M(|z_n|, f)} &\geq \frac{\log m(t,f)}{\log M(|\zeta_n|, f)} \frac{\log M(|\zeta_n|, f)}{\log M(|z_n|, f)} \\
&> L\left(1 - \frac{C}{(\log |\zeta_n|)^{\delta}}\right) \frac{\log |\zeta_n|}{\log |z_n|} \left(1 - \frac{\log^+ M(1,f)}{\log M(|z_n|, f)}\right) \\
&\geq \frac{\log |w_n|}{\log |z_n|} \left(1 - \frac{C}{(\log |\zeta_n|)^{\delta}} - \frac{\log^+ M(1,f)}{\log M(|z_n|, f)}\right) \\
&\geq \frac{\log |w_n|}{\log |z_n|} \left(1 - \frac{C_1}{(\log |z_n|)^{\delta}}\right).
\end{aligned}$$

Now

$$\log |z_n| \geq C_2^n \log |z_0|,$$

so

$$\frac{C_1}{(\log |z_n|)^{\delta}} \leq \frac{1}{C_2^{n\delta}} \frac{C_1}{(\log |z_0|)^{\delta}} \leq \frac{1}{C_2^{(n+1)\delta}}.$$

Choose $z_{n+1} \in K_{n+1}$ with $|z_{n+1}| = M(|z_n|, f)$. This is possible since K_{n+1} is connected, $f(z_n) \in K_{n+1}$, $|f(z_n)| \leq M(|z_n|, f)$, and $f(u_n) \in K_{n+1}$ while by (28),

$$|f(u_n)| \geq m(t, f) \geq M(|\zeta_n|, f) > M(|z_n|, f).$$

We get, with $w_{n+1} = f(u_n)$, that

$$\kappa_{n+1} = \frac{\log |w_{n+1}|}{\log |z_{n+1}|} > \kappa_n \left(1 - \frac{1}{C_2^{(n+1)\delta}}\right) > L^2 \prod_{k=1}^{n+1} \left(1 - \frac{1}{C_2^{k\delta}}\right) > L.$$

Also $|z_{n+1}| = M(|z_n|, f) \geq |z_n|^{C_2} > |z_n| > R_1$ by (26). This completes the proof of Lemma 10.1.

We continue with the proof of Theorem 9.1. We have previously shown that the hypothesis of Lemma 10.1 holds for $n = 1$. Hence induction on n together with Lemma 10.1 shows that for every $n \geq 1$ there are $z_n, w_n \in K_n$ with

$$\frac{\log |w_n|}{\log |z_n|} > L.$$

As in the proof of Theorem 8.3 above, the distinction between $\log |w_n|$ and $\log(2|w_n - a|/\rho)$ is immaterial and is easily handled by taking R_1 even larger; we omit the details. Taking C_0 in (23) with $1 < C_0 < L$ and choosing the appropriate j_0, we obtain a contradiction as soon as $n \geq j_0$. This completes the proof of Theorem 9.1.

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University of Illinois, Department of Mathematics, 273 Altgeld Hall, 1409 West Green Street, Urbana, IL 61801 - USA
E-mail address: aimo@uiuc.edu