EXAMPLES OF ENTIRE FUNCTIONS WITH PATHOLOGICAL DYNAMICS

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1. *Introduction*

The basic results of the iteration theory of rational and entire functions used in this paper are contained in the classical papers [**10, 11**], in the survey [**5**] and in [**15**, Appendix III]. Let f be a rational or entire function and let $f^n = f \circ \ldots \circ f$ be its nth iterate. Denote by $\mathcal{N}(f)$ the set of normality of f, that is, the maximal open set on which the family of iterates is normal in the sense of Montel. The complement of $\mathcal{N}(f)$ is called the Julia set $\mathcal{J}(f)$. The Julia set is non-empty, perfect and completely invariant, that is, it coincides with its inverse image.

A domain $\mathcal{D}$ is said to be wandering if $f^m\mathcal{D} \cap f^n\mathcal{D} = \varnothing$ for all $m > n \geqslant 0$. D. Sullivan [**19**] proved that the set of normality of a rational function has no wandering components. On the other hand I. N. Baker [**3**] constructed an entire function f such that $\mathcal{N}(f)$ has a wandering component. This component $\mathcal{D}$ in Baker's example is multiply connected, $f^m\,\mathcal{D} \to \infty$ and $\deg\,(f^m \,|\, \mathcal{D}) \to \infty$ as $m \to \infty$. We construct several entire functions which have wandering domains with quite different properties.

EXAMPLE 1. There is an entire function f which has a wandering component $\mathcal{D}$ of the set of normality such that every orbit $\{f^n z\}_{n=0}^{\infty}$ originating from $\mathcal{D}$ has an infinite limit set.

In this example all limit functions of the family $\{f^n\}$ in $\mathcal{D}$ are constants and the set of these constants is infinite.

EXAMPLE 2. There is an entire function f which has a simply connected wandering component of $\mathcal{N}(f)$ in which all iterates f^n are univalent.

Note that M. Herman [**14**] has also given an example of a simply connected wandering domain $\mathcal{D}$, but in his example $f^n\,\mathcal{D} \to \infty$ and $\deg\,(f^n \,|\, \mathcal{D}) \to \infty$ as $n \to \infty$.

The following example deals with the dynamics of an entire function f in the invariant component of $\mathcal{N}(f)$ (a domain $\mathcal{D}$ is called invariant if $f\mathcal{D} \subset \mathcal{D}$). It is known that there are the following two types of dynamics in such a component.

(i) There exists a point $a \in \bar{\mathbb{C}}$ such that $f^n z \to a$ as $n \to \infty$ uniformly on the compact subsets of $\mathcal{D}$. If f is rational, then $fa = a$. If f is transcendental, then $fa = a$ or $a = \infty$.

(ii) The map $f\colon \mathcal{D} \to \mathcal{D}$ is conformally conjugate with an irrational rotation of a disk or a ring. For entire functions the case of a ring is impossible.

One of Fatou's main results states that in case (i) the rational function necessarily has a critical point in $\mathcal{D}$. In particular, f is not univalent in $\mathcal{D}$. A similar fact is valid

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for transcendental functions: if $f^n z \to a \neq \infty$ in an invariant component $\mathscr{D}$ of the set of normality then $f\colon \mathscr{D} \to \mathscr{D}$ cannot be a covering map [2]. The following example shows that this is not true in the case when $a = \infty$.

EXAMPLE 3. There is an entire function f which has an invariant component $\mathscr{D}$ such that $f^n z \to \infty$ as $n \to \infty$, $z \in \mathscr{D}$ and f is univalent in $\mathscr{D}$.

Examples 1 to 3 are pathological from the point of view of the iteration theory of rational functions. Let us note, however, that there exists a rather wide class S of entire functions having dynamical properties similar to those of rational functions. Define S to be the class of functions f such that there exists a finite set A, depending on f, such that $f\colon \mathbb{C}\backslash f^{-1}A \to \mathbb{C}\backslash A$ is a non-ramified covering map. It was established in [6, 7] that a function $f \in S$ has no wandering components of $\mathcal{N}(f)$. We recently learned that this result was independently obtained by L. Goldberg and L. Keen [13]. In [4] I. N. Baker proved the absence of wandering domains for a class of entire functions which is contained in S. Every orbit in $\mathcal{N}(f)$, $f \in S$, is bounded [6, 7].

It is well known that the following alternative holds: the set $\mathcal{N}(f)$ is either dense or empty. A natural question arises therefore: is it true that in the first case the Julia set $\mathscr{J}(f)$ has Lebesgue measure zero? An affirmative answer to this question for rational f would lead to a description of the dynamics of a generic rational function. For entire functions the answer is negative.

EXAMPLE 4. There is an entire function f such that the Julia set $\mathscr{J}(f)$ is nowhere dense but mes $\mathscr{J}(f) > 0$.

In the theory of quasiconformal deformations and structural stability of analytic dynamical systems [8, 9, 16, 17, 19] the following important question arises: does there exist an invariant field of straight lines in $\mathscr{J}(f)$? We show that such a field does exist in Example 4. Slightly modifying this example we obtain the following.

EXAMPLE 5. There is an entire function f which has an infinite-dimensional family of measurable invariant fields of straight lines in $\mathscr{J}(f)$.

The construction of our Examples 1 to 5 is based on the theory of approximation by entire functions. The necessary results from this theory are contained in §2. Examples 1 to 3 are constructed in §3 and Examples 4 and 5 are in §4. The results of the paper were announced in [6], their detailed exposition in Russian was given in the preprint [7].

2. *Results on approximation by entire functions*

We shall make use of the following classical result.

RUNGE'S THEOREM. *Let $K \subset \mathbb{C}$ be compact with connected complement. Then any function analytic on K can be uniformly approximated by polynomials. (By a function analytic on K we mean a function analytic in a neighbourhood of K.)*

Using this theorem we prove a result on simultaneous approximation and interpolation.

MAIN LEMMA. *Let $\{G_k\}_{k=1}^{\infty}$ be a sequence of compact subsets of $\mathbb{C}$ with the following properties:*

(i) $\mathbb{C}\backslash G_k$ *is connected for every* k;

(ii) $G_k \cap G_m = \varnothing$ *for* $k \neq m$;

(iii) $\min\{|z|: z \in G_k\} \to \infty$.

Let $z_k \in G_k$, $\varepsilon_k > 0$ *and the function* ϕ *be analytic on* $G = \bigcup_{k\geqslant 1} G_k$. *Then there exists an entire function* f *satisfying*

$$|f(z) - \phi(z)| < \varepsilon_k, \quad z \in G_k; \tag{1}$$

$$f(z_k) = \phi(z_k), \quad f'(z_k) = \phi'(z_k), \quad k \in \mathbb{N}. \tag{2}$$

The original proof of the main lemma due to the authors was considerably simplified by Ju. I. Ljubich who suggested that we apply the following geometrical statement.

LEMMA 1. *Let A be a locally convex linear topological space, let V be a domain in A, let W be a convex dense subset in V and let S be an affine subspace of A of finite codimension, such that $S \cap V \neq \varnothing$. Then $S \cap W$ is dense in $S \cap V$.*

Proof. By induction on codimension the proof is reduced to the case of S having real codimension 1. Then S is defined by the equation $F(g) = \mathscr{C}$ where F is a continuous linear functional. Let $g \in V \cap S$ and let B be a convex neighbourhood of g. Since W is dense in V, we can find $f_1, f_2 \in W \cap B$ such that $F(f_1) > \mathscr{C}$, $F(f_2) < \mathscr{C}$. Set $f = tf_1 + (1-t)f_2$, where

$$t = \frac{\mathscr{C} - F(f_2)}{F(f_1) - F(f_2)}.$$

We have $f \in W \cap B$ and $F(f) = \mathscr{C}$. Lemma 1 follows.

Proof of the main lemma. Let $\mathscr{U}$ be a simply connected neighbourhood of G_1 such that ϕ is analytic in $\mathscr{U}$ and $\mathscr{U} \cap G_k = \varnothing$ for $k \geqslant 2$. Consider the space $\mathscr{A}$ of all functions analytic in $\mathscr{U}$ with the topolgy of uniform convergence on compact sets. Consider the convex domain in A:

$$V = \{g : |g(z) - \phi(z)| < \tfrac{1}{2}\varepsilon_1, z \in G_1\}.$$

Let W be the subset of V consisting of polynomials. By Runge's theorem W is dense in V. Clearly W is also convex. Now consider the affine subspace

$$S = \{g \in A : g(z_1) = \phi(z_1),\ g'(z_1) = \phi'(z_1)\}.$$

By Lemma 1 there exists $f_1 \in W \cap S$; that is, f_1 is a polynomial satisfying

$$|f_1(z) - \phi(z)| < \tfrac{1}{2}\varepsilon_1, \qquad z \in G_1,$$

$$F_1(z_1) = \phi(z_1), \qquad f_1'(z_1) = \phi'(z_1).$$

By a similar argument we find a sequence of polynomials f_m with the following properties

$$\left|\phi(z)-\sum_{k=1}^{m} f_k(z)\right| < \tfrac{1}{2}\varepsilon_m, \qquad z\in G_m; \tag{3}$$

$$|f_m(z)| < 2^{-m+k}\varepsilon_k, \qquad z\in G_k, k<m; \tag{4}$$

$$|f_m(z)| < 2^{-m}, \qquad |z| < \tfrac{1}{2}\min\{|\zeta|:\zeta\in G_m\}; \tag{5}$$

$$\sum_{k=1}^{m} f_k(z_i) = \phi(z_i), \qquad \sum_{k=1}^{m} f_k'(z_i) = \phi'(z_i), \qquad 1\leqslant i\leqslant m. \tag{6}$$

It follows from (5) that the series $f=\sum_{m=1}^{\infty} f_m$ converges uniformly on the compact subsets of $\mathbb{C}$ and defines an entire function. Finally (3), (4) imply (1) and (6) implies (2). The main lemma is proved.

Furthermore, we need two technical lemmas.

LEMMA 2. *Let $f(z)=z+g(z)$ be an analytic function in the disk $\{z:|z|<R\}$ such that $g(0)=g'(0)=0$ and $|g(z)|<\varepsilon R$ for $|z|<R$ and some $\varepsilon<\frac{1}{2}$. Then*

$$|z|\left(1-\frac{\varepsilon}{R}|z|\right)\leqslant |f(z)|\leqslant |z|\left(1+\frac{\varepsilon}{R}|z|\right); \tag{7}$$

$$|\arg f(z)-\arg z| < 2\frac{\varepsilon}{R}|z|, \qquad |z|<R. \tag{8}$$

Proof. The function $g(z)/z^2$ is analytic in $\{z:|z|<R\}$ and by the maximum principle its modulus does not exceed ε/R. This immediately implies (7). Furthermore,

$$|\arg f(z)-\arg z|\leqslant |\log(f(z)/z)| = |\log(1+g(z)/z)|\leqslant 2|g(z)/z|\leqslant 2\frac{\varepsilon}{R}|z|,$$

and (8) is also proved.

LEMMA 3. *Let $q>1$. Then there exists a number $s(q)$ such that the estimates*

$$r_0\sum_{k=0}^{m-1}\varepsilon_k < s(q), \qquad r_0>0, \varepsilon_k>0$$

and

$$r_k(1-\varepsilon_k r_k)\leqslant r_{k+1}\leqslant r_k(1+\varepsilon_k r_k), \qquad 0\leqslant k\leqslant m-1$$

imply that

$$\frac{1}{q}r_0\leqslant r_k\leqslant q r_0, \qquad 0\leqslant k\leqslant m.$$

Proof. Let

$$r_0\sum_{k=0}^{m-1}\varepsilon_k < \frac{1}{q}\log q. \tag{9}$$

Suppose inductively that $r_i\leqslant qr_0$, $0\leqslant i\leqslant k-1$. Then we obtain

$$r_k\leqslant r_0\prod_{i=0}^{k-1}(1+\varepsilon_i r_i)\leqslant r_0\exp\left(\sum_{i=0}^{k-1}\varepsilon_i r_i\right)\leqslant r_0\exp\left(q r_0\sum_{i=0}^{k-1}\varepsilon_i\right).$$

By (9) the last expression does not exceed qr_0.

Now choose $\gamma > 1$ such that the estimate $1-x \geqslant \exp(-\gamma x)$ holds for $0 < x < \frac{1}{2}$. Suppose that

$$r_0 \sum_{k=0}^{m-1} \varepsilon_k < \min\left\{\frac{1}{q\gamma}\log q, \frac{1}{2q}\right\} := s(q).$$

We have already proved that under this condition the estimates $r_i < qr_0$, $0 \leqslant i \leqslant m$ hold. Thus we have

$$r_k \geqslant r_0 \prod_{i=0}^{k-1}(1-\varepsilon_i r_i) \geqslant r_0 \exp\left(-\gamma \sum_{i=0}^{k-1} \varepsilon_i r_i\right) \geqslant r_0 \exp\left(-\gamma r_0 q \sum_{i=0}^{k-1} \varepsilon_i\right) \geqslant r_0/q.$$

Lemma 3 is proved.

3. *Pathological dynamics on the set of normality*

EXAMPLE 1. There is an entire function f having a wandering component $\mathscr{D}$ of the set of normality such that every orbit $\{f^n z\}_{n=0}^{\infty}$ originating from $\mathscr{D}$ has an infinite limit set.

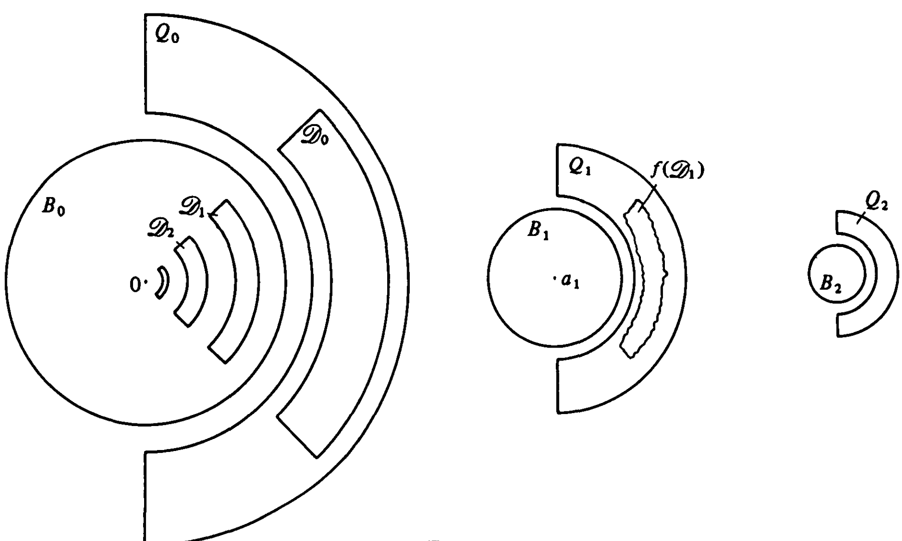

FIG. 1

Fix ε, q such that $0 < \varepsilon < \frac{1}{2}$, $1 < q < 2^{\frac{1}{3}}$, and consider a sequence $\{R_m\}_{m=0}^{\infty}$ satisfying

$$0 < R_m < \tfrac{1}{2}R_{m-1}, \qquad m = 1, 2, \ldots; \tag{10}$$

$$\varepsilon \sum_{k=0}^{m-1} \frac{R_m}{R_k} < \min\{s, \tfrac{1}{8}\pi\}, \qquad m = 1, 2, \ldots, \tag{11}$$

where $s = s(q)$ is chosen by Lemma 3. Define a sequence $\{a_m\}_{m=0}^{\infty}$ such that $a_0 = 0$, $a_{m+1} - a_m > 2R_0$. Let

$$B_m = \{z : |z-a_m| \leqslant \tfrac{1}{2}R_m\},$$

$$\mathscr{D}_m = \{z : q^{-2}R_m < |z| < q^{-1}R_m, |\arg z| < \tfrac{1}{4}\pi\},$$

$$Q_m = \{z : q^{-3}R_m < |z-a_m| < R_m, |\arg z| < \tfrac{1}{2}\pi\}$$

(see Figure 1), and define the function ϕ on the set $\bigcup_{m=0}^{\infty}(B_m \cup Q_m)$ in the following way:

$$\phi(z) = z + a_{m+1} - a_m, \qquad z \in B_m, m \in \mathbb{Z}_+;$$

$$\phi(z) = q^{-\frac{3}{2}} R_{m+1}, \qquad z \in Q_m, m \in \mathbb{Z}_+.$$

By the main lemma there exists an entire function f satisfying

$$f(a_m) = a_{m+1}, \qquad f'(a_m) = 1, \qquad m \in \mathbb{Z}_+; \tag{12}$$

$$|f(z) - \phi(z)| < \tfrac{1}{2}\varepsilon R_m, \qquad z \in B_m, m \in \mathbb{Z}_+; \tag{13}$$

$$f(Q_m) \subset \mathcal{D}_{m+1}, \qquad m \in \mathbb{Z}_+. \tag{14}$$

We show that $\mathcal{D}_0 \subset \mathcal{N}(f)$ and the iterates $f^n D_0$ have constant limit functions a_m.

Let $|z| < \frac{1}{2}R_m$, $r_k = |f^k z - a_k|$, $k \in \mathbb{Z}_+$. We shall prove that

$$q^{-1} r_0 \leqslant r_k \leqslant q r_0, \qquad 0 \leqslant k \leqslant m. \tag{15}$$

Suppose inductively that

$$q^{-1} r_0 \leqslant r_i \leqslant q r_0, \qquad 0 \leqslant i \leqslant k-1 < m.$$

As $q r_0 < (2^{\frac{1}{3}}/2) R_m < R_m < \frac{1}{2}R_i$, it follows that $f^i z \in B_i$, $0 \leqslant i \leqslant k-1$. Then, from (12) and (13) and Lemma 2, we have

$$r_i\left(1 - \varepsilon\frac{r_i}{R_i}\right) \leqslant r_{i+1} \leqslant r_i\left(1 + \varepsilon\frac{r_i}{R_i}\right), \qquad 0 \leqslant i \leqslant k-1.$$

By (11),

$$r_o \sum_{i=0}^{m-1} \frac{\varepsilon}{R_i} \leqslant \varepsilon \sum_{i=0}^{m-1} \frac{R_m}{R_i} < s.$$

Using Lemma 3 we obtain

$$q^{-1} r_0 \leqslant r_k \leqslant q r_0.$$

Thus (15) is proved by induction, and we have

$$f^k z \in B_k, \qquad 0 \leqslant k \leqslant m-1.$$

Using Lemma 2 once more, we obtain

$$|\arg(f^{k+1}z - a_{k+1}) - \arg(f^k z - a_k)| \leqslant 2\varepsilon\frac{r_k}{R_k} \leqslant 2\varepsilon\frac{R_m}{R_k}, \qquad 0 \leqslant k \leqslant m-1.$$

Therefore, by (11),

$$|\arg(f^m z - a_m) - \arg z| \leqslant 2\varepsilon \sum_{k=0}^{m-1} \frac{R_m}{R_k} < \frac{\pi}{4}.$$

It follows from this and (15) (for $k = m$) that $f^m \mathcal{D}_m \subset Q_m$, $m \in \mathbb{Z}_+$. Since $fQ_m \subset \mathcal{D}_{m+1}$, by (14), it follows that $a_0 = 0$ is a limit point for every orbit originating from $\mathcal{D}_0$. Hence all points a_m have the same property. The required example is constructed.

PROBLEM. Does there exist an entire function f and a component $\mathcal{D}$ of $\mathcal{N}(f)$ such that the limit functions of subsequences of f^n in $\mathcal{D}$ form an infinite bounded set?

To construct Examples 2 and 3 we need the following approximation theorem (see for example **[12]**).

ARAKELJAN'S THEOREM. *Let $G \subset \mathbb{C}$ be a closed set. The following properties are equivalent:*

(i) *every function ϕ continuous on G and analytic in the interior of G can be uniformly approximated by entire functions;*

(ii) *$\bar{\mathbb{C}} \backslash G$ is connected and locally connected at ∞.*

EXAMPLE 2. There is an entire function f having a simply connected wandering component of $\mathcal{N}(f)$ in which all iterates are univalent. Let us consider the following vertical strips symmetric with respect to the lines $\{z\colon \operatorname{Re} z = 3m\}$, $m = 10, 11, \ldots$:

$$\Pi_m = \{z : |\operatorname{Re} z - 3m| < 1\},$$

$$\Pi'_m = \{z : |\operatorname{Re} z - 3m| < 1 - 2^{-m-2}\},$$

$$\Pi''_m = \{z : |\operatorname{Re} z - 3m| < 1 - 2^{-m}\}.$$

Let ϕ_m be a linear function which maps Π''_m onto Π_{m+1}. Obviously

$$|\operatorname{Re}(\phi_m(z) - 3(m+1)) - \operatorname{Re}(z - 3m)| < 2^{-m}, \qquad z \in \Pi''_m. \tag{16}$$

Now consider the strip

$$\Pi^* = \{z : |\operatorname{Re} z - 30| < \tfrac{1}{2}\}.$$

It follows from (16) that

$$\phi_m \circ \phi_{m-1} \circ \ldots \circ \phi_{10} \Pi^* \subset \Pi''_{m+1}, \qquad m = 10, 11, \ldots.$$

Using Arakeljan's theorem construct the function f satisfying

$$\left.\begin{array}{ll} |f(z) - \phi_m(z)| < 2^{-3m}, & z \in \Pi'_m; \\ |f(z)| < 1, & z \in \partial\Pi_m, m = 10, 11, \ldots; \\ |f(z)| < 1, & z \in \mathbb{U} = \{z : |z| < 1\}. \end{array}\right\} \tag{17}$$

The unit disk is invariant and hence it is contained in $\mathcal{N}(f)$. Further, it is easy to see that $f^m \Pi^* \subset \Pi''_{m+10}$. Consequently Π^* is contained in $\mathcal{N}(f)$. Let $\mathcal{D}$ be the component of $\mathcal{N}(f)$ containing Π^*. Then $f^m z \to \infty$ in $\mathcal{D}$. However if $z \in \partial\Pi_m$ then by (17) the orbit $\{f^m z\}_{m=0}^{\infty}$ is bounded. Thus $f^m \mathcal{D} \in \Pi_{m+10}$ and $\mathcal{D}$ is a wandering component.

Now we check that the iterates of f are univalent in $\mathcal{D}$. Consider the strips

$$\mathcal{L}_m = \{z : |\operatorname{Re} z - 3m| < 1 - 2^{-m-1}\}.$$

It is easy to see that $f \mathcal{L}_m \supset \Pi_{m+1}$ and hence that $f^m \mathcal{D} \subset \mathcal{L}_{m+10}$. Let us show that f is univalent in $\mathcal{L}_m$. The Cauchy formula for derivatives implies that

$$f(z) = z + 3 + g(z), \qquad |g'(z)| < \tfrac{1}{2}$$

for $z \in \mathcal{L}_m$. Consequently, if $f(z_1) = f(z_2)$; $z_1, z_2 \in \mathcal{L}_m$, then

$$|z_1 - z_2| = |g(z_1) - gz_2)| \leqslant \tfrac{1}{2}|z_1 - z_2|,$$

and hence $z_1 = z_2$. Thus univalence is proved.

To check that $\mathcal{D}$ is simply connected we apply the following lemma (see [**4**, Theorem 3.1, Corollary] or [**7**, Proposition 3]).

LEMMA 4. *Let f be an entire function bounded on a curve tending to ∞. Then all the components of $\mathcal{N}(f)$ are simply connected.*

This lemma completes the investigation of Example 2.

EXAMPLE 3. There is an entire function f which has an invariant component $\mathscr{D}$ of the set $\mathcal{N}(f)$ such that $f^n z \to \infty$ as $n \to \infty$, $z \in \mathscr{D}$, and f is univalent in $\mathscr{D}$.

Let us consider the half-planes

$$P_1 = \{z : \operatorname{Re} z > -4\}, \qquad P_2 = \{z : \operatorname{Re} z < -5\}.$$

Define the functions $f_1(z) = 2z$ in P_1 and $f_2(z) = \exp z - 6$ in P_2. By Arakeljan's theorem there exists an entire function f such that

$$|f(z) - f_i(z)| < \tfrac{1}{2}, \quad z \in P_i, i = 1, 2.$$

This implies that $\operatorname{Re} f(z) > \operatorname{Re} z + \frac{1}{2}$ for $z \in \mathscr{D}_0 = \{z : \operatorname{Re} z > 1\}$. Hence all orbits in $\mathscr{D}_0$ tend to ∞. Furthermore we have

$$\sup\{|f(z) + 6| : z \in P_2\} < 1.$$

Hence all orbits in P_2 tend to an attractive fixed point. Thus $\mathscr{D}_0$ and P_2 are contained in different components of $\mathcal{N}(f)$. Let $\mathscr{D}$ be the component of $\mathcal{N}(f)$ containing $\mathscr{D}_0$. Consider the line $\mathscr{L} = \{z : \operatorname{Re} z = -3\} \subset P_1$. We have $f\mathscr{L} \subset P_2$ and hence

$$\mathscr{D} \subset \mathscr{Q} = \{z : \operatorname{Re} z > -3\}.$$

Let us show that f is univalent in $\mathscr{Q}$. Let $f(z) = 2z + g(z)$. Then $|g(z)| < \frac{1}{2}$ in each disk of radius 1 centred at $z \in \mathscr{Q}$. Hence $|g'(z)| < \frac{1}{2}$, $z \in \mathscr{Q}$. If $f(z_1) = f(z_2)$; $z_1, z_2 \in \mathscr{Q}$, then

$$2|z_1 - z_2| = |g(z_1) - g(z_2)| \leqslant \tfrac{1}{2}|z_1 - z_2|,$$

which implies that $z_1 = z_2$. Thus f is univalent in the half-plane $\mathscr{Q}$ and, moreover, in $\mathscr{D}$.

REMARK. M. Herman **[15]** gave an elementary example with the same properties as those in Example 3.

4. *Julia sets of positive area*

EXAMPLE 4. There is an entire function f such that the Julia set $\mathscr{J}(f)$ is nowhere dense but $\operatorname{mes} \mathscr{J}(f) > 0$.

Consider a sequence $\varepsilon_k > 0$ such that

$$36 \sum_{k=0}^{\infty} \varepsilon_k < 1, \tag{18}$$

and the sequences of squares

$$Q_k = \{x + iy : |x - 4k| < 1, |y| < 1\},$$

$$Q_k^1 = \{x + iy : |x - 4k| < 1 + \varepsilon_k, |y| < 1 + \varepsilon_k\}, \qquad k = 0, 1, \ldots.$$

Delete the following two strips from each Q_k:

$$\mathscr{L}_k = \{x + iy : x - 4k| < \varepsilon_k\}, \qquad \mathscr{M}_k = \{x + iy : |y| < \varepsilon_k\}.$$

The set $Q_4 \backslash (\mathscr{L}_k \cup \mathscr{M}_k)$ is the union of four squares $Q_{k,j}$, $1 \leqslant j \leqslant 4$. Consider the affine surjective map $\phi_{k,j} \colon Q_{k,j} \to Q_{k+1}^1$.

Using the main lemma (or Arakeljan's theorem) construct an entire function f satisfying

(i) $|f(z)-\phi_{k,j}(z)| < \varepsilon_{k+1}$, $z \in Q_{k,j}$;

(ii) f is univalent in $Q_{k,j}$ and $|f'(z)| > 2$, $z \in Q_{k,j}$;

(iii) f has an attractive fixed point $\alpha = -2$.

To verify condition (ii) we may apply the same reasoning as in Examples 2 and 3. It follows from (iii) that $\mathscr{J}(f)$ is nowhere dense. We have

$$Q_{k+1} \subset fQ_{k,j} \subset Q''_{k+1},$$

where

$$Q''_{k+1} = \{x+iy: |x-4(k+1)| < 1+2\varepsilon_{k+1}, |y| < 1+2\varepsilon_{k+1}\}.$$

Consider the set

$$K^{(n)} = \{z: f^k z \in \bigcup_{j=1}^{4} \bar{Q}_{k,j}, k = 0, 1, \ldots, n\}.$$

It is the union of 4^{n+1} disjoint closed sets $K^{(n)}_{i_0, i_1, \ldots, i_n}$, $1 \leqslant i_k \leqslant 4$ on each of which f^n is univalent. Since diam $K^{(n)}_{i_0, i_1, \ldots, i_n} \leqslant 2^{-n}$ it follows that $K^{(\infty)} = \bigcap_{n=0}^{\infty} K^{(n)}$ is a Cantor set.

We show that mes $K^{(\infty)} > 0$. Denote $Q''_n \setminus \bigcup_{j=1}^{4} Q_{n,j}$ by E_n. We have mes $E_n \leqslant 36\,\varepsilon_n$. As $|(f^n)'| > 2^n$, $z \in K^{(n)}$, it follows that

$$\operatorname{mes}(K^{(n)}_{i_0, \ldots, i_n} \cap f^{-n-1} E_{n+1}) \leqslant 36\,\varepsilon_{n+1} \cdot 4^{-n-1}.$$

Hence $\operatorname{mes}(K^{(n)} \cap f^{-n-1} E_{n+1}) \leqslant 36\,\varepsilon_{n+1}$. As $K^{(n+1)} = K^{(n)} \setminus f^{-n-1} E_{n+1}$, the required statement follows from (18).

Furthermore if $z \in K^{(\infty)}$ then $|f^n(z)| = O(n)$, $n \to \infty$, and $|(f^n)'(z)| > 2^n$. Thus the spherical derivative

$$|(f^n)'(z)|/(1+|f^n(z)|^2) \to \infty, \qquad n \to \infty, z \in K^{(\infty)},$$

and so $K^{(\infty)} \subset \mathscr{J}(f)$.

REMARK. In Example 4 as constructed,

$$\operatorname{mes}\{z \in \mathscr{J}(f): |f^m z| \to \infty, m \to \infty\} > 0.$$

Combining the ideas of this example and of Example 1 we may construct an entire function f for which

$$\operatorname{mes}\{z \in \mathscr{J}(f): \underline{\lim}_{n \to \infty} |f^n z| < \infty\} > 0.$$

PROBLEM. Construct an entire function f with the property that

$$\operatorname{mes}\{z \in \mathscr{J}(f): \sup_n |f^n z| < \infty\} > 0.$$

EXAMPLE 5. There is an entire function f having an infinite-dimensional family of measurable invariant fields of straight lines on $\mathscr{J}(f)$.

Given a field of straight lines on a set $X \subset \mathbb{C}$, let $\theta(z)$ be the angle between the line corresponding to $z \in X$ and the positive ray of the real axis. As $\theta(z)$ is defined only modulo πz it is convenient to determine the field of lines by the function

$$\mu(z) = \begin{cases} \exp 2i\theta(z), & z \in X, \\ 0, & z \in \mathbb{C} \setminus X. \end{cases}$$

A field is called measurable if the function μ is measurable. Such a field is defined almost everywhere and is considered as non-trivial if $\operatorname{mes} X > 0$. We mean that μ defines the field not only on X but on all sets $Y \supset X$. The field is called invariant (under f) if

$$\mu(fx) = (f'(x)/|f'(x)|)^2 \mu(x).$$

We show that in Example 4 there exists a non-trivial invariant field of lines on $\mathscr{J}(f)$. Set $c_k = \phi'_{k,j}(z) > 2, z \in Q_{k,j}$. Let $\delta_k > 0$, $\sum_{k=0}^{\infty} \delta_k < \infty$ and the approximation of $\phi_{k,j}$ by f is so close that $|f'(z) - c_k| < \delta_k$, $z \in Q_{k,j}$. Then

$$|\arg f'(z)| = |\operatorname{Im} \log(f'(z))/c_k)| \leqslant A\,|f'(z)/c_k - 1| \leqslant A\,\delta_k, \qquad z \in Q_{k,j}.$$

Thus the series $\theta(z) = -\sum_{k=0}^{\infty} \arg f'(f^k z)$ converges absolutely and uniformly on the set $\mathscr{L} = \bigcup_{n=0}^{\infty} f^n K^{(\infty)}$ and defines there a continuous function. Moreover,

$$\theta(fz) = \theta(z) + \arg f'(z)$$

and consequently θ defines a non-trivial invariant field of lines on $\mathscr{L}$. This field has the natural extension to an invariant field on the completely invariant set

$$\bigcup_{n=0}^{\infty} f^{-n} \mathscr{L} = \mathscr{L}^*.$$

To construct the entire function required in Example 5, one has to carry out the construction of Example 4 simultaneously on an infinite set of sequences of squares. One obtains infinitely many disjoint completely invariant sets $\mathscr{L}_k^*$ having positive measure. Moreover, there exists an invariant field of lines μ_k on each $\mathscr{L}_k^*$. These fields generate an infinite-dimensional family of invariant fields $\sum_{k=1}^{\infty} \lambda_k \mu_k$, $\lambda_k \in \mathbb{C}$, $|\lambda_k| = 1$.

REMARK C. McMullen [18] proved that $\operatorname{mes} \mathscr{J}(f) > 0$ for $f(z) = \sin z$. This Julia set cannot carry an infinite-dimensional family of invariant fields of lines because $f \in S$.

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