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# ON THE ITERATION OF ENTIRE FUNCTIONS

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The iteration theory of entire functions originated in the paper by Fatou [1] and was developed mainly in the works of Baker [2–7]. The recent progress in study of iteration of polynomials and rational functions stimulated the increasing interest in the iteration theory of entire functions. Many modern works in this subject are devoted to special classes of entire functions and to concrete functions, e.g. the exponential function. At the same time the study of dynamical properties of general entire functions initiated by Fatou and Baker is itself of a considerable interest.

Denote by $f^m$ the $m$th iterate of an entire function $f$. Let $F(f)$ be the maximal open set where the family of iterates $\{f^m\}$ is normal in the sense of Montel;

$$J(f) = \boldsymbol{C}\backslash F(f).$$

In what follows we consider only non-linear entire functions $f$. The Julia set $J(f)$ is then a non-empty perfect completely invariant set. If $f$ is transcendental then $J(f)$ is unbounded. The following result is due to Baker [2].

THEOREM B1. *The Julia set coincides with the closure of the set of repulsive periodic points.*

Define

$$I(f) = \{z \in \boldsymbol{C}\colon f^n z \to \infty,\ n \to \infty\}.$$

If $f$ is a polynomial then $I(f)$ is a domain containing $\infty$. In this case we have

$$J(f) = \partial I(f). \tag{1}$$

We shall study the set $I(f)$ for transcendental entire functions.

THEOREM 1. *For every entire function $f$ the set $I(f)$ is non-empty.*

It follows that $I(f)$ is infinite (consider a trajectory $\{f^n z\colon n = 1, 2, \ldots\}$, $z \in I(f)$). Furthermore for every entire function the equality (1) is satisfied. To prove this take $z \in J(f)$ and a neighbourhood $V$ of $z$. Consider an arbitrary point $z_1 \in I(f)$ and let $z_2 = fz_1 \neq z_1$. The family $\{f^n\}$ is not normal in $V$, hence there exists a pre-image $z^* \in V$ of one of the points $z_1$, $z_2$. We have $z^* \in I(f)$. On the other hand it is easily seen that $\operatorname{Int} I(f) \subset F(f)$. Thus $J(f) \subset \partial I(f)$. The opposite inclusion is evident.

For the function $f(z) = e^{-z} + z + 1$, which occurs as an example in [1] the set $I(f)$ contains the right half-plane, thus

$$\operatorname{Int} I(f) \neq \emptyset.$$

We shall use the following result of Wiman–Valiron theory [8, 9].

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire transcendental function. The sequence $|c_n| r^n$ tends to zero for every $r > 0$, thus it contains a maximal term. Denote the index of this term by $N(r)$. If there are several maximal terms, take the largest of their indices. The function $N(r)$ is called the *central index*. Evidently, $N(r)$ is increasing and $N(r) \to \infty$ if $r \to \infty$. Denote

$$M(r) = \max \{|f(z)|\colon |z| = r\}, \quad r > 0.$$

For every $r > 0$, take a point $w(r)$ with the following properties:

$$M(r) = |f(w(r))|, \qquad |w(r)| = r.$$

THEOREM WV. *Let $f$ be a transcendental entire function, $\alpha > \frac{1}{2}$. If*

$$|z - w(r)| < r(N(r))^{-\alpha}, \tag{2}$$

*then*

$$f(z) = \left(\frac{z}{w(r)}\right)^{N(r)} f(w(r))(1 + \varepsilon_1), \tag{3}$$

$$f'(z) = N(r)\left(\frac{z}{w(r)}\right)^{N(r)} f(w(r))(w(r))^{-1}(1 + \varepsilon_2) \tag{4}$$

*where $\varepsilon_i = \varepsilon_i(r, z) \to 0$ uniformly with respect to $z$ if $r \to \infty$, $r \notin E$. The exceptional set $E$ depending of $f$ and $\alpha$ has a finite logarithmic measure, i.e.*

$$\operatorname{lm} E = \int_E \frac{dt}{t} < \infty.$$

*Proof of Theorem* 1. The result is known for polynomials so we may suppose that $f$ is transcendental. Choose an $r_1 > 2$, $r_1 \notin E$, so large that

$$M(r) > 4r, \quad r \geqslant r_1;$$

(5) $$|\log(1+\varepsilon_i)| < 1, \quad r \geqslant r_1,\ r \notin E;$$

(6) $$\operatorname{lm}(E \cap [r_1, \infty)) < 1;$$

$$N(r_1) > 10^4.$$

Denote $w(r_1)$ by $w_1$ and consider the sector

$$C_1' = \left\{z\colon \left|\log\left|\frac{z}{w_1}\right|\right| < \frac{5}{N(r_1)},\ \left|\arg\frac{z}{w_1}\right| < \frac{5}{N(r_1)}\right\}.$$

The function

$$z \mapsto N(r_1)(\log z - \log w_1) + \log f(w_1)$$

maps univalently the sector $C_1'$ onto the square

$$\{\zeta = \xi + i\eta\colon |\xi - \log|f(w_1)|| < 5,\ |\eta - \arg f(w_1)| < 5\}.$$

Using (3), (5) and Rouche's theorem we prove that there exists a domain $C_1 \subset C_1'$ which is mapped univalently by the function $\log f$ onto the square

$$Q = \{\zeta\colon |\xi - \log|f(w_1)|| < 4,\ |\eta - \arg f(w_1)| < 4\}.$$

Thus the image $f(C_1)$ contains the annulus

$$A_2 = \{z\colon e^{-4}M(r_1) < |z| < e^4 M(r_1)\}.$$

In view of (6) we can take $r_2 \notin E$ such that

$$\frac{1}{2}M(r_1) < r_2 < 2M(r_1).$$

Write $w_2 = w(r_2)$. Then the sector

$$C_2' = \left\{z\colon \left|\log\left|\frac{z}{w_2}\right|\right| < \frac{5}{N(r_2)},\ \left|\arg\frac{z}{w_2}\right| < \frac{5}{N(r_2)}\right\}$$

is contained in $A_2$.

Repeating this construction, we obtain a sequence of domains $C_j \to \infty$, $C_{j+1} \subset f(C_j)$ such that there exists a uniform branch of $f^{-1}$ in $C_{j+1}$ for which $\overline{f^{-1}(C_{j+1})} \subset C_j$. Let $B_j = f^{-j}(C_{j+1})$ where $f^{-j}$ is the composition of the just mentioned branches. We have $\bar{B}_{j+1} \subset B_j$, thus $\bigcap_{j=1}^{\infty} B_j \neq \emptyset$. If $z$ belongs to this intersection then $f^j z \in C_{j+1}$. Hence, $z \in I(f)$ and the theorem is proved.

*Remark.* Using the notation from the proof of Theorem 1, we can show that $\operatorname{diam} B_j \to 0$, $j \to \infty$. In view of (4) we obtain for $z \in C_k$:

$$(7) \qquad |f'(z)| \geqslant N(r_k)\,(e^{-5/N(r_k)})^{N(r_k)} M(r_k)\, r_k^{-1} e^{-1} \geqslant \frac{1}{2} e^{-6} N(r_k)\, r_{k+1} \cdot r_k^{-1}$$

By definition, $\operatorname{diam} C_{j+1} \leqslant \text{const} \cdot r_{j+1} (N(r_{j+1}))^{-1}$. Putting in (7) $k = j$ we obtain

$$\operatorname{diam} f^{-1} C_{j+1} \leqslant \text{const} \frac{r_j \cdot 2e^6}{N(r_j)\, N(r_{j+1})}.$$

Applying (7) repeatedly for $k = j-1, \ldots, 1$, we obtain

$$\operatorname{diam} B_j = \operatorname{diam} f^{-j} C_{j+1} \leqslant \text{const} \frac{r_1 \cdot 2^j e^{6j}}{N(r_1) \ldots N(r_{j+1})} \to 0 \quad \text{as } j \to \infty.$$

It is evident that $I(f) \cap J(f) = \emptyset$ for polynomials $f$. We shall prove that $I(f) \cap J(f) \neq \emptyset$ for any transcendental entire function $f$. To do this we need two well-known lemmas which we prove for completeness.

LEMMA 1. *Let $D$ be a multi-connected component of the set $F(f)$ of an entire function $f$. Then $f^n \to \infty$ uniformly on compacts in $D$.*

*Proof.* Let $\gamma$ be a Jordan curve in $D$ whose interior domain $D_1$ contains points of $J(f)$. If for a subsequence $n_k$ we have $|f^{n_k}| \leqslant M$ on $\gamma$ then by maximum principle $|f^{n_k}| \leqslant M$ in $D_1$ and $|(f^{n_k})'| \leqslant M$ in $D_1$. This contradicts to the existence of repulsive periodic points in $D_1$ (Theorem B 1).

LEMMA 2. *Let $D$ be a multi-connected component of the set $F(f)$ of an entire function $f$ and let $\gamma$ be a Jordan curve, non-homotopic to a point in $D$. Then the index with respect to 0 satisfies $\operatorname{ind}_0 f^n \gamma \neq 0$ for all sufficiently large $n$.*

*Proof.* Let $\operatorname{ind}_0 f^{n_k} \gamma = 0$ for a subsequence $n_k$. This means that $f^{n_k}(z) \neq 0$ in the interior component $D_1$ of $C \setminus \gamma$. By Lemma 1 we have $f^{n_k} \gamma \to \infty$. Using the minimum principle we conclude that $f^{n_k} \to \infty$ in $D_1$. This contradicts the Theorem B1.

The following result is due to Baker [3].

THEOREM B2. *Let $f$ be a transcendental entire function. Then every unbounded component of the set $F(f)$ is simply-connected.*

The Lemma 1 and Theorem B 2 imply

COROLLARY (Baker [6]). *Every multi-connected component $D$ of the set $F(f)$ of a transcendental entire function $f$ wanders. This means that $f^n D \cap f^m D = \emptyset$ for every $n > m \geqslant 0$.*

In the papers [4–7] Baker constructs the examples of entire functions with multi-connected wandering components of the set of Fatou $F(f)$.

THEOREM 2. *Let $f$ be a transcendental entire function. Then* $I(f) \cap J(f) \neq \emptyset$.

*Proof.* Two cases are to be considered.

(1). The set $F(f)$ has a multi-connected component $D$. Denote $D_m = f^m D$ and let $K_m$ be the unbounded component of the complement $\overline{C}\backslash D_m$. Let $\gamma$ be a Jordan curve in $D$ non-homotopic to a point in $D$, $\gamma_m = f^m \gamma$. We have $\gamma_m \to \infty$, $\gamma_m \subset D_m$ and $\mathrm{ind}_0 \gamma_m \neq 0$ by Lemmas 1 and 2, respectively. We conclude that $\gamma_m$ separates 0 and $K_m$. Thus $K_m \to \infty$. Fix a large number $m$. We have $f^n \to \infty$ in $D$, $D_m \cap D_n = \emptyset$ for $n > m$. Hence, $D_n \subset K_m$, $\bar{D}_n \subset K_m$ for sufficiently large $n > m$. Consequently, $\partial D \subset I(f)$ and $J(f) \cap I(f) \neq \emptyset$.

(2). All components of $F(f)$ are simply-connected. Let us show that

$$\{z_0\} = \bigcap_{j=1}^{\infty} B_j \subset J(f)$$

(the sets $B_j$ were introduced in the proof of the Theorem 1). Suppose that this is not the case. Then there exists a neighbourhood $V$ of the point $z_0$, $V \subset F(f)$. Using the remark following Theorem 1, we find a number $n$ such that $B_n \subset V$. We have $f^n B_n = C_{n+1}$, consequently $f^{n+1} B_n$ contains the annulus

$$\{z:\ e^{-4} M(r_{n+1}) < |z| < e^4 M(r_{n+1})\}.$$

Thus the set $F(f)$ contains an arbitrarily large annulus. This is a contradiction to the assumption. Theorem 2 is proved.

It is plausible that the set $I(f)$ has no bounded connected components. We shall prove a weaker statement.

THEOREM 3. *The closure $\overline{I(f)}$ of $I(f)$ has no bounded components.*

*Proof.* Let $I_0$ be a bounded component of $\overline{I(f)}$. Then there exists a domain $A$ homeomorphic to an annulus which separates $I_0$ from $\infty$, $A \cap I(f) = \emptyset$. The functions $f^n$ in $A$ do not take the values from $I(f)$, thus $A \subset F(f)$ by Montel's theorem. Let $K$ be the bounded component of $C\backslash A$. In view of (1) $K \cap J(f) \neq \emptyset$ consequently $A$ is contained in a multiconnected component of $F(f)$. By Lemma 1 $A \subset I(f)$, contrary to the assumption.

Papers [10–13] are devoted to the study of a class $S$ of entire functions having many dynamical properties similar to the properties of rational functions. By definition $f \in S$ iff there exists a finite set $A \subset C$ such that

$$f:\ C\backslash f^{-1}(A) \to C\backslash A \tag{8}$$

is an unramified covering map. Some results on the class $S$ proved in [11] remain true for a larger class $B$ which we are now going to define. An entire function $f \in B$ iff there exists a bounded set $A$ such that (8) is an unramified covering. Evidently $S \subset B$. The function $f(z) = z^{-1} \sin z \in B\backslash S$.

THEOREM 4. *Let $f \in B$ be a transcendental entire function. If $f^n z \to \infty$ then $z \in J(f)$.*

COROLLARY. *If $f \in B$ is transcendental then $J(f) = \overline{I(f)}$.*

This fact was noted in [14] for a subclass of $S$.

For any entire function $f$ let

$$\theta_R(r, f) = \text{mes}\,\{\theta \in [0, 2\pi]:\ |f(re^{i\theta})| < R\}$$

THEOREM 5. *Let $f \in B$ and for some $R > 0$*

$$\varliminf_{r\to\infty} \frac{1}{r}\int_1^r \theta_R(t, f)\,dt > 0, \tag{9}$$

*then the area of $I(f)$ is zero.*

For example, area$(I(c\exp)) = 0$ and area$(J(c\exp)) = 0$ which was proved independently in [10–11] and [15]. The condition (9) cannot be removed, because area$(I(\sin)) > 0$ [15] and $\sin \in S \subset B$.

The proofs of Theorems 4 and 5 are the same as in [11] for the class $S$.

The paper [14] is devoted to the study of the structure of the sets $I(f)$ for the simplest functions of class $S$, as e.g. $a \sin z + b$. For such functions $I(f)$ is the so-called "Cantor bouquet", i.e. an uncountable union of curves tending to $\infty$. It is plausible that the set $I(f)$ always has the following property: every point $z \in I(f)$ can be joined with $\infty$ by a curve in $I(f)$.

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