ON SOLUTIONS OF THE BELTRAMI EQUATION

By

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Abstract. In this paper we study the existence and uniqueness of solutions of the Beltrami equation $f_{\bar{z}}(z) = \mu(z)f_z(z)$, where $\mu(z)$ is a measurable function defined almost everywhere in a plane domain Δ with $\|\mu\|_\infty = 1$. Here the partials $f_{\bar{z}}$ and f_z of a complex valued function $f(z)$ exist almost everywhere. In case $\|\mu\|_\infty \leq q < 1$, it is well-known that homeomorphic solutions of the Beltrami equation are quasiconformal mappings. In case $\|\mu\|_\infty = 1$, much less is known. We give sufficient conditions on $\mu(z)$ which imply the existence of a homeomorphic solution of the Beltrami equation, which is ACL and whose partial derivatives $f_{\bar{z}}$ and f_z are locally in L^q for any $q < 2$. We also give uniqueness results. The conditions we consider improve already known results.

1 Introduction

In the Beltrami equation

$$(\beta) \qquad f_{\bar{z}}(z) = \mu(z)f_z(z),$$

$\mu(z)$ is to be a measurable function defined almost everywhere in a plane domain Δ with ess.l.u.b.$\|\mu\|_\infty = 1$. Here the partials $f_{\bar{z}}$ and f_z of a complex valued function $f(z)$ exist almost everywhere. In case $\|\mu\|_\infty \leq q < 1$, it is well-known that homeomorphic solutions of the Beltrami equation are quasiconformal mappings with maximum dilatation

$$D(z) \leq K = \frac{1+q}{1-q}.$$

In case $\|\mu\|_\infty = 1$, much less is known. The only significant results known to the authors are due to O. Lehto [4] and [5] and G. David [2].

In [4] Lehto treats the case of the plane with the following two stringent restrictions on $\mu(z)$:

(Λ_1) in the complement of a compact set of measure 0, $|\mu|$ is bounded away from 1 on every compact subset;

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 76 (1998)

(Λ_2) for any complex z and $0 < r_1 < r_2 < \infty$,

$$\int_{r_1}^{r_2} \left(1 + 2\pi \int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu(z + re^{i\theta})|^2}{1 - |\mu(z + re^{i\theta})|^2} d\theta\right)^{-1} \frac{dr}{r}$$

is strictly positive and tends to ∞ as $r_1 \to 0$ or $r_2 \to \infty$.

Under these conditions, he proves the existence of a homeomorphic solution to the Beltrami equation. Condition (Λ_2) essentially supposes the pointwise equicontinuity of the approximating functions to be discussed below, thus bypassing the most difficult step in the existence proof. In [4] and [5] Lehto does not study the question of uniqueness.

In [2] David follows through the proof of existence in Ahlfors' monograph [1], giving some very detailed and complicated estimates. He considers the case when μ is defined in the plane and assumes that

(Δ) there exist constants $\alpha > 0$ and $C > 0$ such that for $\varepsilon > 0$ sufficiently small

$$\text{measure}\{z : |\mu(z)| > 1 - \epsilon\} \leq Ce^{-\alpha/\varepsilon}.$$

Under this condition he proves the existence of a homeomorphic solution and shows that under suitable normalizations it satisfies a uniqueness result.

The main results as well as various auxiliary results in this paper are obtained under conditions of the form

$$\iint_B F\left(\frac{1}{1 - |\mu(z)|}\right) dA < \Phi_B,$$

where B is a bounded measurable set, $\Phi_B > 0$ is a constant which depends on B, and $F(x)$, defined for $x \geq 1$, is either the identity function, or $F(x) = x^\lambda$, $\lambda > 1$, or

$$F(x) = \exp \frac{x}{1 + \log x}.$$

With the choice

$$F(x) = \exp \frac{x}{1 + \log x},$$

i.e., condition (A) below, we prove the existence of a homeomorphic solution of the Beltrami equation having properties detailed in the statement of our Theorem 1 (Existence Theorem). We also give uniqueness results, which are stated as Theorems 2 and 2′.

In the Appendices we compare our results with those already known. We show that David's results are not subsumed by Lehto's by providing an example where (Δ) holds while (Λ_1) does not. We also show that (Δ) implies conditions (A) and (B) of Theorem 1 but that condition (A) does not imply condition (Δ).

2 Statements of the main results

Theorem 1 (*Existence Theorem*). *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ with $\|\mu\|_\infty \leq 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\text{(A)} \qquad \iint_B \exp \frac{\dfrac{1}{1-|\mu|}}{1+\log\left(\dfrac{1}{1-|\mu|}\right)} dA < \Phi_B,$$

and

$$\text{(B)} \qquad \iint_{\{|z|<R\}\cap\Delta} \frac{1}{1-|\mu|} dA = O(R^2), \quad R \to \infty.$$

Then there exists a homeomorphic mapping $f(z)$ of Δ into the plane, which is ACL and whose partial derivatives f_z and $f_{\bar{z}}$ are in L^q on every compact subset of Δ for every $q < 2$ and which satisfies the Beltrami equation (β) a.e. The partials f_z and $f_{\bar{z}}$ are also distributional derivatives.

Theorem 2 (*Uniqueness Theorem*). *Let $\mu(z)$ and $f(z)$ be as in Theorem 1, with Δ being the plane. Let $\hat{f}(z)$ be a homemorphism of the plane onto itself which has a.e. partial derivatives $\hat{f}_z(z)$ and $\hat{f}_{\bar{z}}(z)$ locally in L^2. If $\hat{f}$ satisfies the Beltrami equation (β) a.e., then*

$$\hat{f}(z) \equiv af(z) + b,$$

where a and b are constants, $a \neq 0$.

Theorem 2′. *If $\hat{f}$ is a homeomorphism of a domain Δ onto a domain Θ and has the same properties as in the above Uniqueness Theorem, then*

$$\hat{f}(z) = \xi(f(z)),$$

where ξ is a conformal mapping of $f(\Delta)$ onto Θ.

3 Construction of the function $f(z)$

We can assume that $\mu(z)$ is defined in the plane by assigning the value 0 in the complement of Δ. For the construction of the function $f(z)$ we use only (A). Condition (B) is used to prove that $f(z)$ maps the plane onto itself.

Now we define μ_n, $n = 1, 2, \ldots$, so that

$$\mu_n(z) = \mu(z) \quad \text{if } |\mu(z)| \le 1 - \frac{1}{n},$$
$$\mu_n(z) = 0 \quad \text{if } |\mu(z)| > 1 - \frac{1}{n}.$$

From the theory of quasiconformal mappings, there exist q.c. mappings f_n, $n = 1, 2, \ldots$, of the plane onto itself with complex dilatations μ_n, $n = 1, 2, \ldots$.

Let z_0 be a fixed point in the plane. For $r_2 > r_1 > 0$ denote by A the circular ring

$$A = \{z : r_1 < |z - z_0| < r_2\},$$

and by $M_n(r_1, r_2)$ the module of its image under f_n. The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation μ_n, where $\mu_n = \mu_n(z) = \mu_n(z_0 + re^{i\theta})$, in the following manner (see [6]):

$$M_n(r_1, r_2) \ge \int_{r_1}^{r_2} \frac{1}{\int\limits_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1 - |\mu_n|^2} d\theta} \frac{dr}{r}.$$

Proposition 1. *For any point z_0 and circular ring $A = \{r_1 < |z - z_0| < r_2\}$, the module $M_n(r_1, r_2)$ of the image of A under f_n tends uniformly to ∞ as $r_1 \to 0$, for all n, r_2 fixed and z_0 in some compact set.*

Proof. Using the lower estimate for the module of the image domain introduced earlier, we obtain

$$M_n(r_1, r_2) \ge \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int\limits_0^{2\pi} \frac{1}{1 - |\mu|} d\theta} \frac{dr}{r}.$$

For any z_0 in a compact subset T of the plane containing the disc $|z - z_0| < r_2$,

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} \exp \frac{\frac{1}{1 - |\mu|}}{1 + \log \frac{1}{1 - |\mu|}} d\theta \frac{dr}{r} \le C,$$

where C depends only on the compact subset T and the choice of r_2. Now we have

$$r^2 \int_0^{2\pi} \exp \frac{\frac{1}{1 - |\mu|}}{1 + \log \frac{1}{1 - |\mu|}} d\theta < \frac{2C}{\log \frac{r_2}{r_1}}$$

on a set E of logarithmic measure $\frac{1}{2}\log\frac{r_2}{r_1}$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\frac{\frac{1}{1-|\mu|}}{1+\log\frac{1}{1-|\mu|}}\,d\theta<\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\quad\text{on } E.$$

The function

$$h(x)=\exp\frac{x}{1+\log x},$$

defined for $x\geq 1$, is convex and increasing and $\lim_{x\to\infty}h(x)=\infty$. Let $l(y)=\log y\log\log y$ for $y>1$ and let $h^{-1}(y)$ be the inverse of $h(x)$. One can show that

$$\lim_{x\to\infty}\frac{h^{-1}(h(x))}{l(h(x))}=1;$$

so, for some constant $c>0$ and sufficiently large y, $h^{-1}(y)<cl(y)$. Therefore, using the convexity of $h(x)$, we have

$$h\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta\right)<\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\quad\text{on } E$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta<h^{-1}\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)\quad\text{on } E.$$

Combined with the asymptotic behavior of $l(y)$, this implies

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta<cl\left(\frac{C}{\pi r^2\log\frac{r_2}{r_1}}\right)\quad\text{on } E.$$

Now we fix R, $0<R<1$ and evaluate $M_n(R^{2N},R^N)$ for any fixed function f_n:

$$\begin{aligned}M_n(R^{2N},R^N)&\geq\frac{1}{4}\int_{R^{2N}}^{R^N}\frac{1}{\int_0^{2\pi}\frac{1}{1-|\mu|}d\theta}\frac{dr}{r}>\frac{1}{8\pi c}\int_E\frac{1}{l\left(\frac{C}{\pi r^2\log(1/R)^N}\right)}\frac{dr}{r}\\&\geq\frac{1}{8\pi c}\int_{R^{2N}}^{R^{3N/2}}\frac{1}{l\left(\frac{C}{\pi r^2\log(1/R)^N}\right)}\frac{dr}{r}.\end{aligned}$$

Using the properties of $l(y)$, the logarithmic function and explicit integration, one can prove that

$$\frac{1}{8\pi c}\int_{R^{2N}}^{R^{3N/2}}\frac{1}{l\left(\frac{C}{\pi r^2\log(1/R)^N}\right)}\frac{dr}{r}\sim\frac{\log(4/3)}{16\pi c\log N},\quad\text{as } N\to\infty.$$

Thus, for some $L > 0$,

$$\sum_{N=L}^{\infty} M_n(R^{2N}, R^N) > \sum_{N=L}^{\infty} \frac{\log(4/3)}{16\pi c \log N}.$$

By the superadditive property of the module it follows that

$$\lim_{N\to\infty} M_n(R^N, R) = \infty,$$

independently of the choice of z_0 and n.

Now let $0 < R < \min\{1, r_2\}$. For any fixed N there is r_1, sufficiently small, such that

$$\{R^N < |z - z_0| < R\} \subset \{r_1 < |z - z_0| < r_2\};$$

and therefore $M_n(r_1, r_2) \geq M_n(R^N, R)$ or $\lim_{r_1\to 0} M_n(r_1, r_2) \geq M_n(R^N, R)$ for any fixed N. Thus $\lim_{r_1\to 0} M_n(r_1, r_2) = \infty$, independent of the choice of z_0 and n. □

From now on we assume that the quasiconformal mappings $\{f_n(z)\}, n = 1, 2, \ldots$, have two fixed points a_1 and a_2, with $d = |a_1 - a_2| > 0$.

Proposition 2. *The family of quasiconformal mappings $\{f_n(z)\}$, $n = 1, 2, \ldots$, is uniformly equicontinuous on each compact subset T of the plane.*

Proof. Suppose that this is not the case. Then for some $\epsilon > 0$ there exist indices $j_n \uparrow \infty$, positive numbers $\delta_n \downarrow 0$ and points $z_1^{(n)}$ and $z_2^{(n)}$ in T, such that $|z_1^{(n)} - z_2^{(n)}| < \delta_n$ and $|f_{j_n}(z_1^{(n)}) - f_{j_n}(z_2^{(n)})| \geq \epsilon$.

Denote by z_0 an accumulation point of the sequence $\{z_1^{(n)}\}$. The distance between z_0 and at least one of the points a_1 and a_2 is at least $d/2$. We assume that this point is a_1. The image under f_{j_n} of the ring $\delta < |z - z_0| < d/2$ for sufficiently small $\delta > 0$ and sufficiently large n is a doubly connected domain for which one complementary component has a diameter $\geq \epsilon$, and the other contains a_1 and ∞; thus its module is bounded independent of δ, which contradicts Proposition 1. □

Proposition 3. *For the sequence $f_n(z)$ there exists a subsequence of functions which converges uniformly to a function $f(z)$ on compact subsets.*

Proof. The result follows from Proposition 2 and the Theorem of Arzela-Ascoli. □

We denote the convergent subsequence again by $f_n(z)$.

Proposition 4. *If*

$$\iint_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2) \quad \textit{as } R \to \infty,$$

then $f_n(z)$ converges uniformly in n to ∞, as $z \to \infty$.

Proof. There exists a constant P, such that

$$\iint_{R \le |z| \le R^2} \frac{1}{1-|\mu|} dA \le PR^4.$$

We claim that for $R \ge R_0 > 0$,

$$\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta \le \frac{3PR^2}{r}$$

on a set E, $E \subset (R, R^2)$, of length at least $(R^2 - R)/2$. Otherwise, there would be a set X of length equal to $(R^2 - R)/2$ such that

$$\int_R^{R^2} \int_0^{2\pi} \frac{1}{1-|\mu|} r dr d\theta \ge \int_X \frac{3PR^2}{r} r dr \ge \tfrac{3}{2} PR^2 (R^2 - R),$$

which contradicts our assumption. Therefore,

$$\int_R^{R^2} \frac{1}{\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta} \frac{dr}{r} \ge \frac{R^2 - R}{6PR^2}.$$

Now fix $R > R_0 > 1$. There exists P such that

$$\iint_{R^{2k}<|z|<R^{4k}} \frac{dA}{1-|\mu|} \le PR^{8k}, \quad \text{for } k = 1, 2, \dots .$$

Let A_N be the annulus $\{z : R^2 < |z| < R^{4N}\}$ and M_n^N the module of its image under f_n, $n = 1, 2, \dots$. Then

$$\begin{aligned} M_n^N &\ge \int_{R^2}^{R^{4N}} \frac{1}{\int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1-|\mu_n|^2} d\theta} \frac{dr}{r} \ge \frac{1}{4} \sum_{k=1}^{N} \int_{R^{2k}}^{R^{4k}} \frac{1}{\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta} \frac{dr}{r} \\ &\ge \frac{1}{4} \sum_{k=1}^{N} \frac{R^{4k} - R^{2k}}{6PR^{4k}} \ge \frac{N}{24P}. \end{aligned}$$

Therefore M_n^N tends uniformly in n to ∞, as $N \to \infty$. Since $f_n(z)$ tends uniformly in n on compact subsets to $f(z)$ as $n \to \infty$, we conclude

$$f(z) \to \infty, \quad \text{as } z \to \infty. \qquad \square$$

4 $f(z)$ is a homeomorphism

Lemma 1. *Let h_n, $n = 1, 2, \ldots$, be a sequence of homeomorphisms of the plane onto itself with two fixed points a_1 and a_2. If h_n converges uniformly on every compact subset of the plane to a function $h(z)$, and if for every annulus A there is a $q > 0$ such that for all n*

$$M(h_n(A)) \geq q, \tag{1}$$

then h is a homeomorphism.

Proof. Assume that h is not a homeomorphism. Then there exist two points $z_1 \neq z_2$ such that $h(z_1) = h(z_2) = w_0$. We can assume that none of the points z_1, z_2 and w_0 coincides with a_2 and that z_1 does not lie on the segment connecting z_2 and a_2. Construct a line t through z_1 which does not meet that segment, a circle C_1 passing through a_2 and z_2, which does not meet t, and a concentric circle C_2, outside of C_1, which also does not meet t. Let D be the doubly connected domain bounded by C_1 and t, and A the ring domain bounded by C_1 and C_2. Then $A \subset D$ and

$$M(h_n(D)) > M(h_n(A)).$$

On the other hand, the spherical diameters of $h_n(C_1)$ and $h_n(t)$ are bounded away from zero, while the spherical distance between them tends to 0 as $n \to \infty$. This implies that $M(h_n(D)) \to 0$ as $n \to \infty$, which contradicts (1). $\square$

Lemma 2. *Let μ^* be a measurable complex valued function with $\|\mu^*\|_\infty = 1$ and let μ_n^* be a sequence of measurable functions, constructed in the same way as in Section 3. If*

(i) *h_n are the corresponding quasiconformal mappings of the complex plane onto itself with two distinct fixed points;*

(ii) *h_n converge uniformly on compact subsets of the plane to h; and*

(iii) *for each compact subset L in the plane, and some positive constant Φ_L,*

$$\iint\limits_L \frac{1}{1 - |\mu^*|} dA < \Phi_L, \tag{2}$$

then

1. *there exists* $q > 0$ *such that* $M(h_n(A)) \geq q$, *and*
2. h *is a homemorphism.*

Proof. Fix a point z_0 in the plane, and for $0 < r_1 < r_2$ let S be the annulus $\{z : r_1 \leq |z - z_0| \leq r_2\}$. According to (2), there exists a constant Φ_S such that

$$\iint_S \frac{1}{1 - |\mu^*|} dA < \Phi_S. \tag{3}$$

As in Proposition 1, using the appropriate estimate for the module of the image, we have

$$M(h_n(S)) \geq \frac{1}{4} \int_{r_1}^{r_2} \frac{dr/r}{\int_0^{2\pi} \frac{1}{1-|\mu^*|} d\theta}.$$

There is a measurable subset X of $[r_1, r_2]$ of measure $(r_2 - r_1)/2$ on which

$$\int_0^{2\pi} \frac{1}{1 - |\mu^*|} d\theta \leq \frac{8\Phi_S}{(r_2 - r_1)(r_2 + 3r_1)}.$$

Otherwise, we would have

$$\int_{r_1}^{r_2} \int_0^{2\pi} \frac{1}{1 - |\mu^*|} d\theta r dr \geq \frac{8\Phi_S}{(r_2 - r_1)(r_2 + 3r_1)} \int_{r_1}^{(r_1+r_2)/2} r dr = \Phi_S,$$

which contradicts (3).

Therefore, for each A there exists q, as defined below, such that for all n

$$M(h_n(A)) \geq \frac{1}{4} \int_X \frac{dr/r}{\int_0^{2\pi} \frac{1}{1-|\mu^*|} d\theta} \geq \frac{\pi(r_2 + 3r_1)(r_2 - r_1)}{32\Phi_S} \int_{(r_1+r_2)/2}^{r_2} \frac{dr}{r} = q > 0.$$

By Lemma 1, $h(z)$ is a homeomorphism. □

Proposition 5. *The function f(z) constructed in Proposition 3 is a homeomorphism and maps the plane onto the plane.*

Proof. Inequality (2) for μ follows from (A). This together with the properties of $f_n(z)$ implies that $f(z)$ is a homeomorphism. □

5 Differentiability properties of $f(z)$

Lemma 3. *If $h_n(z)$ are quasiconformal mappings of the plane onto itself with complex dilatation μ_n^*, satisfying $|\mu_n^*| \le |\mu^*| \le 1$ a.e., if $h_n(z)$ converge uniformly on compact subsets of the plane to a homeomorphism $h(z)$, and if for every rectangle R there is a positive constant Φ_R such that*

$$\iint_R \frac{1}{1-|\mu^*|} dA < \Phi_R, \tag{4}$$

then $h(z)$ is ACL.

Proof. Let $R = \{(x, y) : a \le x \le b, c \le y \le d\}$. Denote by I_y the segment with ordinate y and $a \le x \le b$, and by R_y the rectangle which is the subset of R below that segment. Let $A(y)$ be the area of the image of R_y under $h(z)$. Then $A(y)$ is an increasing function of y and thus has a finite derivative $A'(y)$ for almost all y in the interval $[c, d]$. Moreover, by Fubini's Theorem,

$$\int_{I_y} \frac{1}{1-|\mu^*|} dx$$

is finite for almost all y in $[c, d]$. Further, for any system of infinitely many nonoverlapping subintervals (ξ_k, ξ_k^*), $k = 1, 2, \dots, m$, of (a, b) and any $\delta > 0$ sufficiently small (i.e., $y + \delta < d$), consider

$$\frac{1}{\delta} \sum_{k=1}^{m} \int_{\xi_k}^{\xi_{k+1}} \int_{y}^{y+\delta} \frac{1}{1-|\mu^*|} dA = \frac{1}{\delta} \int_{y}^{y+\delta} \left(\sum_{k=1}^{m} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|} dx \right) dy.$$

By a theorem of Lebesgue on differentiation, for almost all y in (c, d) this expression has a limit as $\delta \to 0$, which is

$$\sum_{k=1}^{m} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|} dx.$$

Allowing all possible rational pairs of ξ_k, ξ_k^*, this will hold simutaneously for almost all y in (c, d). Choose y_0 for which all of this applies. Now let (x_k, x_k^*) be an arbitrary system of nonoverlapping subintervals of (a, b). Let ξ_k, ξ_k^* be rational numbers satisfying $x_k < \xi_k < \xi_k^* < x_k^*$, and set

$$w_k = h(x_k + iy_0), \qquad w_k^* = h(x_k^* + iy_0),$$
$$\omega_k = h(\xi_k + iy_0), \qquad \omega_k^* = h(\xi_k^* + iy_0).$$

To prove that f is absolutely continuous on I_{y_0} it is enough to show that $\sum_{k=1}^{m} |w_k^* - w_k|$ has an upper bound which tends to zero with $\sum_{k=1}^{m}(x_k^* - x_k)$.

We choose $\delta > 0$ with $y_0 + \delta < d$ and denote by $R_k(\delta)$ the rectangle $\{(x,y) : \xi_k < x < \xi_k^*,\ y_0 < y < y_0 + \delta\}$. Taking its corners as vertices, it becomes a quadrangle whose image under $h(z)$ we denote by $Q_k(\delta)$. Let $m(Q_k(\delta))$ be the area of $Q_k(\delta)$ and $\beta_k(\delta)$ the distance between the images of the vertical sides of $R_k(\delta)$. Let $M(Q_k(\delta))$ be the module of $Q_k(\delta)$ for curves joining the images of horizontal sides of $R_k(\delta)$. Then, as in Theorem 4.1 of [3],

$$M(Q_k(\delta)) \geq \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))}.$$

Denote the corresponding entities under $h_j(z)$ by $R_k^j(\delta)$, $Q_k^j(\delta)$ and $M(Q_k^j(\delta))$. Using an upper estimate for the module of a quadrangle from [6], we have, after applying the Schwarz Inequality,

$$M(Q_k^j(\delta)) \leq \frac{1}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{(1+|\mu_j^*|)^2}{1-|\mu_j^*|^2} dA \leq \frac{2}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|} dA.$$

Letting $j \to \infty$, we obtain

$$M(Q_k(\delta)) \leq \frac{2}{\delta^2} \int_y^{y+\delta} \int_{\xi_k}^{\xi_{k+1}} \frac{1}{1-|\mu^*|} dA,$$

and therefore

$$\frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \leq \frac{2}{\delta^2} \iint_{R_k(\delta)} \frac{1}{1-|\mu^*|} dA.$$

Thus

$$\sum_{k=1}^{m} \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \leq \frac{2}{\delta^2} \iint_{\cup R_k(\delta)} \frac{1}{1-|\mu^*|} dA.$$

The Schwarz Inequality then gives

$$\begin{aligned}\left(\sum_{k=1}^{m} \beta_k(\delta)\right)^2 &\leq \sum_{k=1}^{m} \frac{(\beta_k(\delta))^2}{m(Q_k(\delta))} \cdot \sum_{k=1}^{m} m(Q_k(\delta)) \\ &\leq \frac{2}{\delta^2} \iint_{\cup R_k(\delta)} \frac{1}{1-|\mu^*|} dA \cdot (A(y_0+\delta) - A(y_0)).\end{aligned}$$

Letting $\delta \to 0$, we have

$$\left(\sum_{k=1}^{m} |\omega_k^* - \omega_k|\right)^2 \leq 2A'(y_0) \sum_{k=1}^{m} \int_{\xi_k}^{\xi_k^*} \frac{1}{1-|\mu^*|} dx$$
$$\leq 2A'(y_0) \sum_{k=1}^{m} \int_{x_k}^{x_k^*} \frac{1}{1-|\mu^*|} dx.$$

Letting $\xi_k \to x_k, \xi_k^* \to x_k^*$, we have finally

$$\left(\sum_{k=1}^{m} |w_k^* - w_k|\right)^2 \leq 2A'(y_0) \sum_{k=1}^{m} \int_{x_k}^{x_k^*} \frac{1}{1-|\mu^*|} dx.$$

Since the integral is absolutely continuous as a set function, this completes the proof. □

Proposition 6. *The function f(z) is ACL.*

Proof. Since (A) holds for μ, so does (4); and μ_n, μ, f_n, f satisfy the conditions of Lemma 3. This implies that f is ACL. □

Lemma 4. *If*

(i) *h_n are quasiconformal mappings of the plane onto itself with complex dilatation μ_n^* satisfying $|\mu_n^*| \leq |\mu^*| \leq 1$ a.e.;*

(ii) *$h_n(z)$ converge uniformly on compact subsets of the plane to $h(z)$;*

(iii) *for some $\lambda > 1$ and every compact subset L of the plane there exists a positive constant Φ_L such that*

$$\iint_L \left(\frac{1}{1-|\mu^*|}\right)^\lambda dA \leq \Phi_L, \tag{5}$$

then $h(z)$ has partial derivatives h_z and $h_{\bar{z}}$ which are in L^q on every compact subset of the plane, where

$$q \leq \frac{2\lambda}{1+\lambda}.$$

Proof. Let L be a compact subset of the plane and Θ a Jordan domain containing L. By condition (ii), h_n converge uniformly on $\bar{\Theta}$ to h. Let J_n denote the Jacobian of h_n. We use the fact that a.e.

$$|(h_n)_z| = \frac{(J_n)^{1/2}}{(1-|\mu_n^*|^2)^{1/2}}.$$

By Hölder's Inequality it follows that

$$\iint_{\bar\Theta} |(h_n)_z|^q dA = \iint_{\bar\Theta} \frac{(J_n)^{q/2}}{(1-|\mu_n^*|^2)^{q/2}} dA$$
$$\le \left(\iint_{\bar\Theta} (J_n)^{\frac{qp'}{2}} dA\right)^{1/p'} \left(\iint_{\bar\Theta} \frac{1}{(1-|\mu_n^*|^2)^{\frac{qq'}{2}}} dA\right)^{1/q'},$$

where $1/p' + 1/q' = 1$. We choose $p' = 2/q$, which implies that q is at most $2\lambda/(1+\lambda)$. Then the first integral becomes $\iint_{\bar\Theta} J_n \, dA$, which is the area of the image of $\bar\Theta$ under $h_n(z)$ and is therefore finite. Thus the first integral is uniformly bounded for all n. The second integral is also uniformly bounded because of (5). Therefore,

$$\iint_{\bar\Theta} |(h_n)_z|^q dA$$

are uniformly bounded in n. For $q > 1$, $L^q(\bar\Theta)$ is a reflexive Banach space. Therefore the sequence of functions $(h_n)_z$ has a weakly convergent subsequence $(h_{n_k})_z$. The same is true for $(h_n)_{\bar z}$, because the h_n are sense preserving and thus $|(h_n)_{\bar z}| < |(h_n)_z|$. The same is true for the real partials $(h_{n_k})_x$ and $(h_{n_k})_y$, which converge to $\hat h$ and $\hat{\hat h}$ in $L^q(\bar\Theta)$. Since the h_n are quasiconformal, $(h_{n_k})_x$ and $(h_{n_k})_y$ are distributional derivatives (see [1] p. 28). For any test function $t(x,y)$ with support in Θ,

$$\iint_{\bar\Theta} (h_{n_k})_x t(x,y) dA = -\iint_{\bar\Theta} h_{n_k} t(x,y)_x dA.$$

Passing to the limits as $k \to \infty$, we have, by the weak convergence of $(h_{n_k})_x$ and the uniform convergence of h_{n_k},

$$\iint_{\bar\Theta} \hat h t \, dA = -\iint_{\bar\Theta} h t_x \, dA.$$

Similarly,

$$\iint_{\bar\Theta} \hat{\hat h} t \, dA = -\iint_{\bar\Theta} h t_y \, dA.$$

Thus $h(z)$ has distributional derivatives in $L^q(\Theta)$. By Theorem 2.1.4 of [7], h (being continuous) coincides with its representative in the context of this theorem and $h_z \in L^q(\Theta)$. The same is true for $h_{\bar z}$, and therefore both partials belong to L^q (L).

Note that this also proves that h is ACL, but under stronger conditions than in Lemma 3. □

Proposition 7. *The partials f_z and $f_{\bar z}$ of $f(z)$ are in L^q on compact subsets of the plane for every $q < 2$.*

Proof. Since (A) holds for μ, so does (5) for every $\lambda > 1$. This, together with the properties of f_n, f, μ_n and μ, implies that f_z and $f_{\bar z}$ are in $L^q(L)$ for every compact set L in the plane and $1 < q < 2$, and thus for any q, $0 < q < 2$. □

6 $f(z)$ satisfies the Beltrami equation

Lemma 5. *Let h_n be quasiconformal mappings of the plane onto itself with complex dilatations μ_n^* satisfying $|\mu_n^*| < |\mu^*| \leq 1, \lim_{n\to\infty} \mu_n^* = \mu^*$, a.e. Suppose that the functions h_n converge uniformly on compact subsets of the plane to h and let $\lambda > 1$. If, for any rectangle R with sides parallel to the axes, there exists a positive constant Φ_R such that*

$$\iint\limits_R \left(\frac{1}{1-|\mu^*|}\right)^\lambda dA \leq \Phi_R, \tag{6}$$

then

$$h_{\bar z}(z) = \mu^* h_z(z) \qquad a.e.$$

Proof. Denote $\zeta(z) = h_{\bar z}(z) - \mu^*(z) h_z(z)$. Then $\zeta(z)$ is defined a.e. Now we can write

$$\zeta = [h_{\bar z} - (h_n)_{\bar z}] + [(h_n)_{\bar z} - \mu_n^*(h_n)_z] + [\mu_n^*(h_n)_z - \mu^*(h_n)_z] + [\mu^*(h_n)_z - \mu^* h_z].$$

Denote

$$\begin{aligned} I_{1,n}(z) &= h_{\bar z} - (h_n)_{\bar z}, \\ I_{2,n}(z) &= (h_n)_{\bar z} - \mu_n^*(h_n)_z, \\ I_{3,n}(z) &= \mu_n^*(h_n)_z - \mu^*(h_n)_z, \\ I_{4,n}(z) &= \mu^*(h_n)_z - \mu^* h_z. \end{aligned}$$

Then

$$\iint\limits_R \zeta\, dA = \iint\limits_R I_{1,n}\, dA + \iint\limits_R I_{2,n}\, dA + \iint\limits_R I_{3,n}\, dA + \iint\limits_R I_{4,n}\, dA.$$

Now $\iint_R I_{2,n} dA = 0$ because $I_{2,n} = 0$ a.e. According to Lemma 4, h has L^1 derivatives in R, and so do h_n as quasiconformal mappings. By formula (6.17) on p. 50 of [3],

$$\iint\limits_R \left[(h_n)_{\bar z} - h_{\bar z}\right] dA = \frac{1}{2i} \int\limits_{\partial R} \left[h_n - h\right] dz.$$

Since h_n tend uniformly to h on the boundary ∂R, the above integrals converge to 0 as $n \to \infty$. So we have

$$\lim_{n\to\infty} \iint_R I_{1,n}\, dA = 0. \tag{7}$$

Further, from Hölder's Inequality it follows that

$$\begin{aligned}\left|\iint_R I_{3,n}\, dA\right| &= \left|\iint_R (\mu_n^* - \mu^*)(h_n)_z dA\right| \\ &\le \left(\iint_R |\mu_n^* - \mu^*|^p dA\right)^{1/p} \left(\iint_R |(h_n)_z|^q dA\right)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1,\ p, q > 0$, and

$$q = \frac{2\lambda}{1+\lambda},$$

thus $1 < q < 2$. As in the proof of Lemma 4, one can show that $\iint_R |(h_n)_z|^q\, dA$ are uniformly bounded in n. Since $\mu_n^* \to \mu^*$ as $n \to \infty$ a.e. and $|\mu_n^* - \mu^*|^p \le 2^p$, by Lebesgue's Theorem

$$\iint_R |\mu_n^* - \mu^*|^p dA \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\iint_R I_{3,n}\, dA \to 0 \quad \text{as } n \to \infty.$$

By p. 138 of [3], μ^* can be approximated by uniformly bounded step functions ϕ_k with constant values on squares with sides parallel to the axes, so that $\lim_{k\to\infty} \phi_k = \mu^*$ a.e. in R. Using this, we have

$$\iint_R I_{4,n}\, dA = \iint_R (\mu^* - \phi_k)\big((h_n)_z - h_z\big) dA + \iint_R \phi_k\big((h_n)_z - h_z\big) dA.$$

By Hölder's Inequality,

$$\begin{aligned}&\iint_R \left|(\mu^* - \phi_k)\big((h_n)_z - h_z\big)\right| dA \\ &\le \left(\iint_R |\mu^* - \phi_k|^p dA\right)^{1/p} \left(\iint_R |(h_n)_z - h_z|^q dA\right)^{1/q},\end{aligned}$$

with q as above. Since $\iint_R |(h_n)_z|^q dA$ is uniformly bounded and $h_z \in L^q(R)$, $\iint_R |(h_n)_z - h_z|^q dA$ is uniformly bounded. Since $\lim_{k\to\infty} \phi_k = \mu^*$ a.e., by Lebesgue's Theorem we can choose k_0 so that, for $k \geq k_0$ and every n,

$$\iint\limits_R |\mu^* - \phi_k| |(h_n)_z - h_z| dA \leq \varepsilon.$$

On the other hand, since ϕ_{k_0} has a constant value on the finitely many squares or parts of squares covering R and by the argument used to prove (7), we have

$$\iint\limits_R \phi_{k_0}((h_n)_z - h_z) dA \to 0 \quad \text{as } n \to \infty.$$

Therefore, there exists N_0 such that for $n \geq N_0$

$$\left| \iint\limits_R \phi_{k_0}((h_n)_z - h_z) dA \right| < \varepsilon$$

or

$$\left| \iint\limits_R I_{4,n} dA \right| \leq 2\varepsilon.$$

This shows

$$\iint\limits_R I_{4,n} dA \to 0 \quad \text{as } n \to \infty,$$

and completes the proof that

$$\iint\limits_R \zeta dA = 0.$$

Using the measure-theoretic argument of [3] p. 189, one can show that $\zeta = 0$ a.e. Thus we have

$$h_{\bar{z}}(z) - \mu^*(z) h_z(z) = 0 \quad \text{a.e.}$$

□

Proposition 8. *The function $f(z)$ satisfies the Beltrami equation (β).*

Proof. Since (A) holds for μ, so does (6). This, together with the properties of f_n, f, μ_n and μ, implies that f satisfies the Beltrami equation (β). □

7 The inverse function $g(w)$ of $f(z)$

Lemma 6. *Let*

(i) h_n, $n = 1, 2, \ldots$, *be homeomorphisms of the plane onto itself with fixed points* $a_1 \neq a_2$ *such that*

(ii) h_n *converge uniformly to a homeomorphism* h *on compact subsets of the plane and*

(iii) h_n *converge uniformly to* ∞, *as* $z \to \infty$ *and* $n \to \infty$.

(iv) *Suppose that for any fixed annulus* A, *the module* $M(h_n(A))$ *is uniformly bounded away from zero; and let*

(v) l_n, $n = 1, 2, \ldots$, *and* l *be the inverse homeomorphisms to* h_n, $n = 1, 2, \ldots$, *and* h, *respectively.*

Then the sequence l_n, $n = 1, 2, \ldots$, *is pointwise equicontinuous; and* l_n *converges uniformly to* l *on compact subsets of the plane.*

Proof. Assume that the sequence l_n is not pointwise equicontinuous. Then there exist a point p, a number $\varepsilon > 0$, a sequence $\delta_k \downarrow 0$, and sequences $\{w_k\}_{k=1}^{\infty}$ and $\{l_{n_k}\}_{k=1}^{\infty}$ such that $|w_k - p| < \delta_k$ and

$$|l_{n_k}(w_k) - l_{n_k}(p)| \geq \varepsilon. \tag{8}$$

The sequence $l_{n_k}(w_k)$ has a finite accumulation point because h_n converges uniformly to ∞ as $z \to \infty$ and $n \to \infty$. We can find a subsequence which tends to this accumulation point. We denote this subsequence again by $\{l_{n_k}\}_{k=1}^{\infty}$ and the accumulation point by z_0. If necessary, after taking another subsequence, we can assume that there is a finite point q such that $l_{n_k}(p) \to q$, as $k \to \infty$. Because of (8) we have $z_0 \neq q$.

Assume that $z_0 \neq a_1$ and that q does not lie on the segment joining z_0 and a_1. Draw a circle Γ_1^0 through z_0 and a_1 such that q is outside of Γ_1^0 and a line Γ_2^0 through q not meeting Γ_1^0. Draw a second circle Γ_1 tangent to Γ_1^0 at a_1 of larger radius which does not meet Γ_2^0 and a line Γ_2 parallel to Γ_2^0 which lies between Γ_1 and Γ_2^0. There is an integer k_0 such that for $k \geq k_0$, $l_k(w_k)$ is inside Γ_1 and $l_{n_k}(p)$ is in the half plane determined by Γ_2 not containing Γ_1. For each $k \geq k_0$, we construct a circle $\Gamma_1^{(k)}$ passing through a_1 and $l_{n_k}(w_k)$, tangent to Γ_1 at a_1, and a line $\Gamma_2^{(k)}$ parallel to Γ_2 and passing through $l_{n_k}(p)$.

For each $k \geq k_0$, $h_{n_k}(\Gamma_1^{(k)})$ passes through a_1 and w_k; and $h_{n_k}(\Gamma_2^{(k)})$ passes through p and ∞. Therefore, the spherical diameter of each of the families of curves $h_{n_k}(\Gamma_1^{(k)})$ and $h_{n_k}(\Gamma_2^{(k)})$ is bounded away from 0. However, the spherical distance between $h_{n_k}(\Gamma_1^{(k)})$ and $h_{n_k}(\Gamma_2^{(k)})$ is less than δ_k. If we denote by A_k the doubly-connected domain bounded by $\Gamma_1^{(k)}$ and $\Gamma_2^{(k)}$ and by $M\big(h_{n_k}(A_k)\big)$ the module of its image under h_{n_k}, it follows from Lemma 2 of [4] that $M\big(h_{n_k}(A_k)\big) \to 0$ as $k \to \infty$.

Now consider A to be the doubly connected domain bounded by the curves Γ_1 and Γ_2 and let $M\big(h_{n_k}(A)\big)$ the module of its image under h_{n_k}. Since $A \subset A_k$ for $k \geq k_0$,

$$M\big(h_{n_k}(A)\big) \leq M\big(h_{n_k}(A_k)\big);$$

and thus $M\big(h_{n_k}(A)\big) \to 0$ as $k \to \infty$. This contradicts our assumption (iv).

If q lies on the segment joining z_0 and a_1 and $q \neq a_1$, we reverse the roles of z_0 and q in the above proof. If $q = a_1$ and $a_2 \neq z_0$, then the proof goes as above, interchanging the roles of a_1 and a_2. Finally, if $q = a_1$ and $z_0 = a_2$, instead of taking circles tangent to a_1 we take circles centered at a_1.

This proves the pointwise equicontinuity of the sequence $\{l_n\}$. Therefore, by Lemma 5.1, page 71 of [3], there exists a subsequence of $\{l_{n_k}\}_{k=1}^{\infty}$ converging uniformly on compact subsets of the plane to a continuous function l_0. For any z and k, $h_{n_k}\big(l_{n_k}(z)\big) = z$ and $l_{n_k}\big(h_{n_k}(w)\big) = w$. Since h_{n_k} converges uniformly to h and l_{n_k} converges uniformly to l_0, we have

$$h\big(l_0(z)\big) = z \qquad \text{and} \qquad l_0\big(h(w)\big) = w.$$

Therefore, $l_0(z)$ is identical to $l(z)$; and the sequence $l_n(z)$ converges to $l(z)$ uniformly on compact subsets of the plane. □

Let $g(z)$ be the inverse of $f(z)$ and let $g_n(z)$ be the inverse of $f_n(z)$, for $n = 1, 2, \dots$.

Proposition 9. *The sequence g_n converges uniformly to g on compact subsets of the plane.*

Proof. Now, since (A) holds for f_n, (2) holds; and, by Lemma 2, $M\big(f_n(A)\big)$ are uniformly bounded away from 0. This, together with the properties of f_n, f, the definition of g_n and g, and Lemma 6, implies the statement of the above proposition. □

Proposition 10. *The function g is ACL and g_w is locally in L^2.*

Proof. Let B be a compact set in the plane and U a compact disk such that B lies in the interior of U. Denote by J_n denote the Jacobian of f_n. The functions g_n, $n = 1, 2, \dots$, are quasiconformal with Jacobian $1/J_n\big(g_n(w)\big)$. We denote their complex dilatations by ν_n, $n = 1, 2, \dots$. Thus

$$\iint_U |(g_n)_w(w)|^2 dA_w = \iint_U \frac{1}{J_n(g_n(w))\left(1 - |\nu_n(w)|^2\right)} dA_w$$
$$= \iint_{g_n(U)} \frac{J_n(z) dA_z}{J_n(z)(1 - |\mu_n(z)|^2)} = \iint_{g_n(U)} \frac{1}{1 - |\mu_n(z)|^2} dA_z.$$

By the uniform convergence of g_n, all $g_n(U)$ lie in a compact set V, so

$$\iint_{g_n(U)} \frac{1}{1 - |\mu_n(z)|} dA_z \leq \iint_V \frac{1}{1 - |\mu(z)|} dA_z;$$

and all the terms $\iint_U \left|\left(g_n(w)\right)_w\right|^2 dA_w$ are uniformly bounded because (A) holds for $\mu(z)$. Since g_n, $n = 1, 2, \dots$, is sense-preserving, $\iint_U \left|\left(g_n(w)\right)_{\bar{w}}\right|^2 dA_w$ are also uniformly bounded. Thus the sequences $(g_n)_w$ and $(g_n)_{\bar{w}}$ are uniformly bounded sequences in the Hilbert space $L^2(U)$, so there are subsequences $(g_{n_k})_w$ and $(g_{n_k})_{\bar{w}}$ which converge weakly in $L^2(U)$. The same is true for the real partials $(g_{n_k})_u$ and $(g_{n_k})_v$, $w = u + iv$, which converge weakly to $\hat{g}$ and $\hat{\hat{g}}$ in $L^2(U)$. Since g_n are quasiconformal, $(g_{n_k})_u$ and $(g_{n_k})_v$ are distributional derivatives (see p. 28 of [1]). Then, for any test function $t(u, v)$,

$$\iint_U (g_{n_k})_u t dA_w = -\iint_U (g_{n_k}) t_u dA_w,$$
$$\iint_U (g_{n_k})_v t dA_w = -\iint_U (g_{n_k}) t_v dA_w.$$

Passing to the limit as $k \to \infty$, we have, by the weak convergence of $(g_{n_k})_u$ and $(g_{n_k})_v$ and the uniform convergence of g_{n_k},

$$\iint_U \hat{g} t dA_w = -\iint_U g t_u dA_w \quad \text{and} \quad \iint_U \hat{\hat{g}} t dA_w = -\iint_U g t_v dA_w.$$

Thus g has distributional derivatives in L^2; and so, by Lemma 2 of [1] or Theorem 2.1.4 of [7], g is ACL. (Note that since g is continuous, it coincides with its representative in the sense used in Theorem 2.1.4 of [7].) Moreover, g_w is in $L^2(U)$ and thus in $L^2(B)$. □

8 Proof of the main results

Proof of Theorem 1. As mentioned at the beginning of Section 3, we can extend $\mu(z)$ to a measurable function in the whole plane which satisfies condition

(A) for any bounded measurable set B. The existence of the homeomorphism $f(z)$ of the plane onto itself is proved in Section 4. By the statements in Section 5, $f(z)$ is ACL and its partial derivatives f_z and $f_{\bar{z}}$ are in $L^q(K)$, for any compact subset K of the plane and any $q < 2$. Moreover, they are distributional derivatives. By the results in Section 6, they satisfy the Beltrami equation. Restricting f to Δ completes the proof of Theorem 1. □

Proof of Theorem 2. Since $\hat{f}$ and g have partials locally in L^2, $\hat{f}(g(w))$ is absolutely continuous in the sense of Tonelli (see p. 143 of [3]) and a.e.

$$\frac{\partial}{\partial \bar{w}} \hat{f}\big(g(w)\big) = \hat{f}_z\big(g(w)\big) g_{\bar{w}}(w) + \hat{f}_{\bar{z}}\big(g(w)\big) g_w(w).$$

Except on a set E of measure 0, $\hat{f}_{\bar{z}}(z) = \mu(z)\hat{f}_z(z)$; and except on the set $f(E)$, $\hat{f}_{\bar{z}}\big(g(w)\big) = \mu\big(g(w)\big)\hat{f}_z\big(g(w)\big)$. Therefore,

$$\hat{f}_z\big(g(w)\big) g_{\bar{w}}(w) + \hat{f}_{\bar{z}}\big(g(w)\big) g_w(w) = \hat{f}_z\big(g(w)\big)\left[g_{\bar{w}}(w) + \mu\big(g(w)\big) g_w(w)\right]$$

except on the set $f(E)$.

Now we show that $g_w = 0$ and $g_{\bar{w}} = 0$ a.e. on $f(E)$. Obviously, we are done if $f(E)$ is a set of measure 0. If $f(E)$ has a positive measure, then

$$0 = \iint_E \frac{1}{1 - |\mu(z)|^2} dA = \iint_{f(E)} \frac{J_w(w)}{1 - \big|\mu\big(g(w)\big)\big|^2} dA_w = \iint_{f(E)} \big|g_w(w)\big|^2 dA_w.$$

Here J_w is the Jacobian of g, and the change of variables is justified by Lemma 2.1, Chapter III [3]. Therefore $g_w = 0$ a.e. on $f(E)$. Since g is sense-preserving, $g_{\bar{w}} = 0$ a.e. on $f(E)$.

Now let F and G be the sets where $f(z)$ and $g(z)$ are not differentiable, respectively. Let H be the set where f does not satisfy (β). By the same argument as above, it follows that since F and H have measure 0, $g_w = 0$ and $g_{\bar{w}} = 0$ a.e. on $f(F) \cup f(H)$.

One can show, using differentiability a.e., that except on the set $f(F) \cup G$,

$$g_w(w) = \frac{f_z\big(g(w)\big)}{J_z\big(g(w)\big)} \quad \text{and} \quad g_{\bar{w}} = -\frac{f_{\bar{z}}\big(g(w)\big)}{J_z\big(g(w)\big)},$$

where J_z is the Jacobian of f. Substituting these expressions above, we have, except on the set $f(E) \cup f(F) \cup G$,

$$\hat{f}_z\big(g(w)\big)\left[g_{\bar{w}}(w) + \mu\big(g(w)\big) g_w(w)\right] = \hat{f}_z\big(g(w)\big)\left[-\frac{f_{\bar{z}}\big(g(w)\big)}{J_z\big(g(w)\big)} + \mu\big(g(w)\big)\frac{f_z\big(g(w)\big)}{J_z\big(g(w)\big)}\right].$$

Also, except on the set $f(H)$,

$$-f_{\bar{z}}\big(g(w)\big)+\mu\big(g(w)\big)f_z\big(g(w)\big)=0.$$

Since G has measure 0 and g_w and $g_{\bar{w}}$ are 0 a.e. on $f(E)\cup f(F)\cup f(H)$, we conclude that

$$\hat{f}_z\big(g(w)\big)\big[g_{\bar{w}}(w)+\mu\big(g(w)\big)g_w(w)\big]=0 \quad \text{a.e.}$$

This proves that

$$\frac{\partial}{\partial\bar{w}}\hat{f}\big(g(w)\big)=0 \quad \text{a.e.}$$

By Corollary 1, [1] p. 31, $\hat{f}\big(g(w)\big)$ is a conformal mapping of the plane onto itself. Thus

$$\hat{f}\big(g(w)\big)=aw+b \quad \text{with constants } a\neq 0 \text{ and } b.$$

Therefore

$$\hat{f}(z)=af(z)+b.$$

□

Proof of Theorem 2′. We extend $\mu(z)$ to the whole plane, construct the homeomorphisms $f(z)$ and $g(z)$, and restrict $g(z)$ to $f(\Delta)$. Then $\hat{f}\big(g(w)\big)$ defined on $f(\Delta)$ satisfies the condition

$$\frac{\partial}{\partial\bar{w}}\hat{f}\big(g(w)\big)=0 \quad \text{a.e. in } f(\Delta).$$

Thus $\hat{f}\big(g(w)\big)$ is a conformal mapping of $f(\Delta)$ onto $\Theta=\hat{f}(\Delta)$, which we call $\xi(w)$.

□

9 Appendices

Appendix 1. Here we are going to construct an example of a function $\mu(z)$ which satisfies David's condition (Δ) but does not satisfy Lehto's condition (Λ_1). To do so, we construct two sequences $\{I_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ of subsets of the open unit interval I_0 using the following procedure by induction. Denote by C_1 a Cantor set in I_0 of measure t, $0<t<1$. Then $I_1=I_0\setminus C_1$ is a set of countably many open intervals. Assume that I_{n-1} is constructed and is a union of countably many open intervals. Then C_n is defined as the union of Cantor sets on the open subintervals of I_{n-1}, each with a measure t times the measure of the corresponding subinterval. Then we define I_n as $I_n=I_{n-1}\setminus C_n$, thus I_n is an union of countably many open intervals.

From now on, depending on the context, we denote by $l(A)$ the linear measure or by $m(A)$ the area measure of a measurable set A. Thus we have

$$
\begin{aligned}
&I_1 = I_0 \setminus C_1, && l(C_1) = t, && l(I_1) = 1-t,\\
&I_2 = I_1 \setminus C_2, && l(C_2) = t(1-t), && l(I_2) = (1-t)^2,\\
&\vdots\\
&I_n = I_{n-1} \setminus C_n, && l(C_n) = t(1-t)^{n-1}, && l(I_n) = (1-t)^n.
\end{aligned}
$$

We define also $E_n = I_n \times I_n$ and $G_n = C_n \times C_n$. Then $m(E_n) = (1-t)^{2n}$. Also $G_n \subset E_{n-1}$, $E_n = E_{n-1} \setminus G_n$ and $E_{n-1} \supset G_l$, for $l \geq n$. Denote by $E_\infty = \bigcap_{n=1}^{\infty} E_n$ and let $\mu(z)$ be a measurable function defined as

$$
\mu(z) = \begin{cases} 1 - e_n, & \text{on } G_{n+1},\\ 1, & \text{on } E_\infty,\\ 0, & \text{everywhere else.} \end{cases}
$$

Here e_n is a decreasing sequence of positive numbers defined as

$$
e_n = \frac{\alpha}{2n \ln(1-t)},
$$

where α is a positive constant.

To show that $\mu(z)$ satisfies David's condition (Δ), we choose $\varepsilon > 0$ to be a sufficiently small number and define $A = \{z : |\mu(z)| > 1 - \varepsilon\}$. There exist numbers e_n and e_{n+1} such that $e_{n+1} \leq \varepsilon < e_n$. Therefore, $A \subset \{z : |\mu(z)| > 1 - e_n\} \subset E_{n+1}$. Thus $m(A) < m(E_{n+1}) = (1-t)^{2(n+1)} = e^{-\alpha/e_{n+1}} \leq e^{-\alpha/\varepsilon}$. Therefore, $\mu(z)$ satisfies David's condition (Δ). However, $\mu(z) = 1$ on the set E_∞, which is not compact and whose closure is the closed unit square Q. Therefore, μ is not less than 1 outside of a compact set of measure 0 and thus does not satisfy Lehto's condition (Λ_1).

Appendix 2. In this section we relate conditions (A) and (B) of Theorem 1 to David's condition (Δ). We show that (Δ) implies (A) and (B) but (A) does not imply (Δ).

Result 1. *Let K be a bounded measurable subset of the plane and μ a complex-valued measurable function on K with $\|\mu\|_\infty \leq 1$. Then the following two conditions are equivalent.*

(C) *There exists $\beta > 0$ such that*

$$
\iint_K \exp\left\{\frac{\beta}{1-|\mu|}\right\} dA < \Phi,
$$

where Φ is a positive constant.

(D) *There exist $\alpha > 0$ and $C > 0$ such that for sufficiently small $\varepsilon > 0$*

$$\text{measure}\{z : |\mu(z)| > 1 - \varepsilon\} \leq Ce^{-\alpha/\varepsilon}.$$

Proof. We first show that (D) implies (C). Let

$$S_n = \left\{ z : \frac{1}{2^{n+1}} \leq 1 - |\mu| < \frac{1}{2^n} \right\}.$$

Then measure$\{S_n\} \leq Ce^{-\alpha 2^n}$ for $n > n_0$. Thus

$$\begin{aligned} \iint_K \exp\left(\frac{\beta}{1-|\mu|}\right) dA &= \iint_{K \setminus \bigcup_{n>n_0} S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) dA + \iint_{\bigcup_{n>n_0} S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) dA \\ &\leq A_1 + \sum_{n=n_0}^{\infty} \iint_{S_n} \exp\left(\frac{\beta}{1-|\mu|}\right) dA \\ &\leq A_1 + \sum_{n=1}^{\infty} C \exp\{2\beta - \alpha\} 2^n, \end{aligned}$$

where A_1 is an appropriate constant. If $\alpha > 2\beta$ this series converges, which proves that (D) implies (C). Now we show that (C) implies (D).

Let M be any positive number, $M > 1$. The set where

$$\exp\left\{\frac{\beta}{1-|\mu|}\right\} > M$$

has measure less than Φ/M. Therefore, the set where

$$1 - |\mu| < \frac{\beta}{\log M}$$

has measure less than Φ/M. Thus, with $\varepsilon = \beta/\log M$, the set where $|\mu| > 1 - \varepsilon$ has measure less than $\Phi e^{-\beta/\varepsilon}$. This completes the proof that (C) implies (D). □

Result 2. *Let $\mu(z)$ be defined in the plane. David's condition (Δ) implies* (D) *for every bounded measurable set K. However, condition* (D) *for every bounded measurable set K does not imply* (Δ).

Proof. The statement that (Δ) implies (D) for every bounded measurable set K is obvious. To prove the second part of Result 2, let

$$\mu(z) = \mu(x+iy) = \begin{cases} 1 - \dfrac{1}{\log(1/y)}, & 0 < y < 1,\ x \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

Then for some small $\varepsilon > 0$, the set where $1 - |\mu| < \varepsilon$ is the set where $y \leq e^{-1/\varepsilon}$. Let K be a bounded measurable set with $|x| < l$ on K. The measure of the set $1 - |\mu| < \varepsilon$ is less than $2le^{-1/\varepsilon}$. Therefore, condition (D) is satisfied for any K. However, the measure of the set in the whole plane where $1 - |\mu| < \varepsilon$ is infinite, and therefore condition (Δ) does not hold. □

Result 3. *Condition* (D) *for every bounded measurable set in the plane implies condition* (A). *However, condition* (A) *does not imply condition* (D) *for every bounded measurable set* K.

Proof. To prove the first part of the statement, we use Result 1 and the fact that (C) for every bounded measurable set implies (A).

To prove the second part of the statement, we give an example of a function μ for which (A) holds but (D) does not hold for some bounded measurable set.

Let $\{S_n\}$ be a set of disjoint disks in the plane contained in a finite disk S with

$$\text{measure}\{S_n\} = e^{-\frac{2^{n+1}}{n\log(1.5)}} - e^{-\frac{2^{n+2}}{(n+1)\log(1.5)}}.$$

Let $\mu(z) = 0$ in the complement of $\bigcup_{n=1}^{\infty} S_n$, and $\mu(z) = 1 - 1/2^{n+1}$ on S_n. Let K be any bounded measurable set. Then

$$\iint_K \exp \frac{\frac{1}{1-|\mu|}}{1 + \log\left(\frac{1}{1-|\mu|}\right)} dA \leq C + \sum_{n=1}^{\infty} \iint_{S_n} \exp \frac{2^{n+1}}{1 + (n+1)\log 2} dA$$
$$\leq C + \sum_{n=1}^{\infty} \exp\{-2^{(n+1)/2}\} < \infty.$$

Thus (A) holds for μ.

Assume that condition (D) holds for S and the above constructed $\mu(z)$. Then the measure of the set where $\{z : 1 - |\mu| < 1/2^n\}$ is $\leq Ce^{-\alpha 2^n}$ for some constant α. On the other hand, the set $\{z : 1 - |\mu| < 1/2^n\}$ has measure $e^{\frac{-2^{n+1}}{n\log(1.5)}}$. Therefore,

$$e^{\frac{-2^{n+1}}{n\log(1.5)}} \leq Ce^{-\alpha 2^n};$$

thus

$$e^{2^n\left(-\frac{2}{(n\log(1.5))}+\alpha\right)} \leq C \quad \text{as } n \to \infty,$$

which is a contradiction. Therefore, (D) does not hold for $\mu(z)$ on the set S. □

Result 4. *If* $\mu(z)$ *satisfies condition* (Δ), *then*

$$\iint_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2).$$

Proof. Since (Δ) implies (D) for every bounded measurable set, we can choose a suitable $\varepsilon > 0$ such that (D) holds for $|z| < R$ and for some constants C and α. Using the same argument as in Result 1, one can show that for every $R > 0$

$$\iint\limits_{\substack{1-|\mu|<\varepsilon \\ |z|<R}} \frac{1}{1-|\mu|} dA < \Phi,$$

where Φ is a constant independent of R, while

$$\iint\limits_{\substack{1-|\mu|\geq\varepsilon \\ |z|<R}} \frac{1}{1-|\mu|} dA \leq \frac{\pi R^2}{\varepsilon}.$$

This shows that

$$\iint\limits_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2) \quad \text{as } R \to \infty. \qquad \square$$

Appendix 3.

Result 5. *No condition of the form*

$$\iint\limits_{K} \left(\frac{1}{1-|\mu|}\right)^{\lambda} dA < \Phi$$

is sufficient for the conclusion of Proposition 1, where K is a bounded measurable set, Φ is a constant that depends on K, and $\lambda \geq 1$.

Proof. We define

$$f(\rho e^{i\theta}) = e^{\rho^{\tau}} e^{i\theta},$$

where $0 < \tau < 1$ and $0 < \rho < 1$. The complex dilatation of the mapping is

$$\mu(\rho e^{i\theta}) = e^{2i\theta}\left(\frac{\tau\rho^{\tau}-1}{\tau\rho^{\tau}+1}\right);$$

thus

$$\frac{1}{1-|\mu|} = \frac{1}{2} + \frac{1}{2\tau\rho^{\tau}}.$$

Therefore,

$$\iint\limits_{\rho<1} \frac{1}{(1-|\mu|)^{\lambda}} dA = \iint\limits_{|\rho|<1} \left(\frac{1}{2} + \frac{1}{2\tau\rho^{\tau}}\right)^{\lambda} dA.$$

This is finite if and only if

$$\int_0^1 \frac{1}{\rho^{\tau\lambda-1}} d\rho < \infty.$$

Given λ, we can always choose $\tau < 2/\lambda$. But f maps the punctured disc $0 < |z| < 1$ onto a proper annulus $1 < |z| < e$, and therefore we do not get the conclusion of Proposition 1 for this f. □

In [2], p. 69, G. David gave a similar but more complicated example.

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(Received December 23, 1997 and in revised form July 2, 1998)