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Bloch's Principle

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Abstract. A heuristic principle attributed to André Bloch says that a family of holomorphic functions is likely to be normal if there are no non-constant entire functions with this property. We discuss this principle and survey recent results that have been obtained in connection with it. We pay special attention to properties related to exceptional values of derivatives and existence of fixed points and periodic points, but we also discuss some other instances of the principle.

Keywords. Normal family, quasinormal, Zalcman Lemma, exceptional value, fixed point, periodic point.

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Contents

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1. Bloch's heuristic principle

1.1. Introduction. A family of meromorphic functions is called normal if every sequence in the family has a subsequence which converges (locally uniformly with respect to the spherical metric). The concept of a normal family was introduced already in 1907 by P. Montel [72], but there has been a lot of interest in normal families again in recent years, an important factor being their central role in complex dynamics.

One guiding principle in their study has been the heuristic principle which says that a family of functions meromorphic (or holomorphic) in a domain and possessing a certain property is likely to be normal if there is no non-constant function meromorphic (holomorphic) in the plane which has this property. This heuristic principle is usually attributed to A. Bloch, but it does not seem to have been stated explicitly by him — although his statement "*Nihil est in infinito quod non prius fuerit in finito*" made in his 1926 papers [16, p. 84] and [18, p. 311] may be interpreted this way. The first explicit formulation of the heuristic principle seems to be due to G. Valiron [112, p. 2] in 1929. However, in [113, p. 4] Valiron mentions Bloch in this context.

Here we survey some of the results that have been obtained in connection with Bloch's Principle. We concentrate on properties related to exceptional values of derivatives in Section 2 and on properties related to fixed points of iterates in Section 3. Excellent references for Bloch's Principle are [102, Ch. 4] and [123].

We mention that André Bloch's life was tragic. He murdered his brother, uncle and aunt in 1917. Judged not responsible for his actions, he was confined to a psychiatric hospital where he remained until his death in 1948. It was there that he did all his mathematical work, including the papers cited above. For a detailed account of Bloch's life and work we refer to [28, 114].

1.2. The basic examples. A simple example for Bloch's Principle is given by the property of being bounded. Liouville's Theorem says that a bounded entire function is constant; that is, if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and if there exists a constant K such that $|f(z)| \le K$ for all $z \in \mathbb{C}$, then f is constant. And in fact a family $\mathcal{F}$ of functions holomorphic in a domain D is normal if there exists a constant K such that $|f(z)| \le K$ for all $z \in D$ and all $f \in \mathcal{F}$. Note, however, that the family $\mathcal{F}$ of all functions f holomorphic in a domain D for which there exists a constant $K = K(f)$ such that $|f(z)| \le K(f)$ for all $z \in D$ need not be normal. In fact, let D be the unit disk $\mathbb{D}$, that is, $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$, where $f_n(z) := nz$. Each f_n is bounded in D (by the constant n), but it is easily seen that $\mathcal{F}$ is not normal. So in order that the property "f is bounded" leads to a normal family it is essential that the bound does not depend on the function f. This example already shows that some care has to be taken when formulating Bloch's Principle.

A key example deals with the property of omitting three points in the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Perhaps the most important theorem in the whole subject of normal families is Montel's result that this condition implies normality.

1.2.1. Montel's Theorem. *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be distinct, let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in D such that $f(z) \ne a_j$ for all $j \in \{1, 2, 3\}$, all $f \in \mathcal{F}$, and all $z \in D$. Then $\mathcal{F}$ is normal.*

The analogous statement about functions in the plane is Picard's Theorem.

1.2.2. Picard's Theorem. *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be distinct and let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic. If $f(z) \ne a_j$ for all $j \in \{1, 2, 3\}$ and all $z \in \mathbb{C}$, then f is constant.*

In Montel's Theorem it is again essential that the points a_1, a_2, a_3 do not depend on f. We mention, however, that the condition $f(z) \ne a_1(f), a_2(f), a_3(f)$ still implies normality if there exists $\varepsilon > 0$ such that $\chi(a_j(f), a_k(f)) \ge \varepsilon$ for $j \ne k$ and all f, where $\chi(\cdot, \cdot)$ denotes the spherical distance; see, e.g. [102, p. 104] or [123, p. 224].

1.3. The Theorems of Arzelà-Ascoli and Marty. One basic result in the theory of normal families is a classical theorem due to Arzelà and Ascoli [102, p. 35], which we record in the following form.

1.3.1. Arzelà-Ascoli Theorem. *A family of meromorphic functions is normal if and only if it is locally equicontinuous (with respect to the spherical metric).*

A consequence of this is the following theorem due to Marty [102, p. 75]. Denote by

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2} = \frac{1}{2} \lim_{\zeta \to z} \frac{\chi(f(\zeta), f(z))}{|\zeta - z|}$$

the spherical derivative of a meromorphic function f.

1.3.2. Marty's Criterion. *A family $\mathcal{F}$ of functions meromorphic in a domain D is normal if and only if the family $\{f^\# : f \in \mathcal{F}\}$ is locally bounded; that is, if for every $z \in D$ there exists a neighborhood U of z and a constant M such that $f^\#(z) \le M$ for all $z \in U$ and for all $f \in \mathcal{F}$.*

1.4. Zalcman's Lemma. In order to turn the heuristic principle for certain properties into a rigorous theorem, L. Zalcman [121] proved the following result.

1.4.1. Zalcman's Lemma. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in D. If $\mathcal{F}$ is not normal, then there exist a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers, a sequence (f_k) in $\mathcal{F}$, a point $z_0 \in D$ and a non-constant meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ such that $z_k \to z_0$, $\rho_k \to 0$ and $f_k(z_k + \rho_k z) \to f(z)$ locally uniformly in $\mathbb{C}$. Moreover, $f^\#(z) \le f^\#(0) = 1$ for all $z \in \mathbb{C}$.*

The corresponding result for normal *functions* rather than normal *families* had been proved earlier by A. J. Lohwater and Ch. Pommerenke [67]. We remark that the statement about the spherical derivative does not appear in [121], but it follows immediately from the proof; see also [123, p. 216f]. This observation plays an important role in some of the more recent applications; see Section 1.5 below.

Proof of Zalcman's Lemma. Suppose that $\mathcal{F}$ is not normal. By Marty's Criterion, there exists a sequence (ζ_k) in D tending to a point $\zeta_0 \in D$ and a sequence (f_k) in $\mathcal{F}$ such that $f_k^\#(\zeta_k) \to \infty$. Without loss of generality, we may assume that $\zeta_0 = 0$ and that $\{z : |z| \le 1\} \subset D$. Choose z_k satisfying $|z_k| \le 1$ such that $M_k := f_k^\#(z_k)(1 - |z_k|) = \max_{|z|\le 1} f_k^\#(z)(1 - |z|)$. Then $M_k \ge f_k^\#(\zeta_k)(1 - |\zeta_k|)$ and hence $M_k \to \infty$. Define $\rho_k := 1/f_k^\#(z_k)$. Then $\rho_k \le 1/M_k$ so that $\rho_k \to 0$. Since $|z_k + \rho_k z| < 1$ for $|z| < (1 - |z_k|)/\rho_k = M_k$ the function $g_k(z) := f_k(z_k + \rho_k z)$ is defined for $|z| < M_k$ and satisfies

$$g_k^\#(z) = \rho_k f_k^\#(z_k + \rho_k z) \le \frac{1 - |z_k|}{1 - |z_k + \rho_k z|} \le \frac{1 - |z_k|}{1 - |z_k| - \rho_k |z|} = \frac{1}{1 - \dfrac{|z|}{M_k}}$$

there. By Marty's Criterion, the sequence (g_k) is normal in $\mathbb{C}$ and thus has a subsequence which converges locally uniformly in $\mathbb{C}$. Without loss of generality, we may assume that $g_k \to f$ for some $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ and $z_k \to z_0$ for some $z_0 \in D$. Since $g_k^\#(0) = 1$ for all k, we have $f^\#(0) = 1$, so that f is non-constant. Clearly, we also have $f^\#(z) \le 1$ for all $z \in \mathbb{C}$. ■

Remark. It is not difficult to see that if z_k, ρ_k, f_k and f are as in Zalcman's Lemma, then $\mathcal{F}$ is not normal at $z_0 = \lim_{k\to\infty} z_k$. In turn, we may achieve $z_k \to z_0$ for any point $z_0 \in D$ at which $\mathcal{F}$ is not normal.

The important point in Zalcman's Lemma is that the limit function f is non-constant. Regardless of whether $\mathcal{F}$ is normal or not it is always possible to choose z_k, ρ_k and f_k such that $f_k(z_k + \rho_k z)$ tends to a constant.

1.5. Zalcman's formalization of the heuristic principle. We first note how Zalcman's Lemma can be used to deduce Montel's Theorem from Picard's Theorem. Suppose that $\mathcal{F}$ is a non-normal family of meromorphic functions which omit three values a_1, a_2, a_3. Let f_k, z_k, ρ_k and f be as in Zalcman's Lemma. Then obviously $f_k(z_k + \rho_k z) \neq a_j$ for all z, k and j. Thus $f(z) \neq a_j$ for all $z \in \mathbb{C}$ by Hurwitz's Theorem, contradicting Picard's Theorem.

To see what kind of properties this kind of argument may be used for we first need to specify what we mean by a property. To this end we simply collect all functions enjoying a certain property in a set. It turns out to be useful to display a function $f\colon D \to \widehat{\mathbb{C}}$ together with its domain D of definition. Following A. Robinson [94, §8] and L. Zalcman [121] we thus write $\langle f, D\rangle \in P$ if f has the property P on a domain D. Bloch's Principle then asserts that the following two statements should be equivalent:

(a) *if $\langle f, \mathbb{C}\rangle \in P$, then f is constant;*
(b) *the family $\{f : \langle f, D\rangle \in P\}$ is normal on D for each domain $D \subset \mathbb{C}$.*

We say that P is a *Bloch property* if (a) and (b) are equivalent. Of course, that two statements are equivalent simply means that either both are true or both are false. But we will later meet properties P where we can prove that (a) and (b) are equivalent, but where we do not know whether either is true or false! If statements (a) and (b) are actually true, then we say that P is a *Picard-Montel* property. Zalcman's Lemma now leads to the following result.

1.5.1. Zalcman's Principle. *Suppose that a property P of meromorphic functions satisfies the following three conditions:*

(i) *if $\langle f, D\rangle \in P$, then $\langle f|_{D'}, D'\rangle \in P$ for every domain $D' \subset D$;*
(ii) *if $\langle f, D\rangle \in P$ and $\varphi(z) = \rho z + c$, where $\rho, c \in \mathbb{C}$ and $\rho \neq 0$, then $\langle f \circ \varphi, \varphi^{-1}(D)\rangle \in P$;*
(iii) *suppose that $\langle f_n, D_n\rangle \in P$ for $n \in \mathbb{N}$, where $D_1 \subset D_2 \subset D_3 \subset \ldots$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{C}$; if $f_n \to f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ locally uniformly in $\mathbb{C}$, then $\langle f, \mathbb{C}\rangle \in P$.*

Then P is a Bloch property.

The proof that (a) implies (b) is a straightforward application of the Zalcman Lemma. In order to see that (b) implies (a) it suffices to consider the family $\{f(nz) : n \in \mathbb{N}\}$ and note that this family is not normal if f is meromorphic in the plane and not constant. In fact we see that conditions (i) and (ii) suffice in order to deduce (a) from (b).

The argument shows that (b) follows not only from condition (a), but in fact from the following weaker condition

(c) *if $\langle f, \mathbb{C}\rangle \in P$ and if f has bounded spherical derivative, then f is constant.*

In particular, for properties P satisfying the hypothesis of Zalcman's Principle we see that (a) and (c) are equivalent. So in order to prove a result for functions meromorphic in the plane, it sometimes suffices to prove it for functions with bounded spherical derivative. This kind of argument is due to X. Pang, and appears in writing first in [13, 30, 122], see Theorem 2.4.1 below. This argument will also appear in Sections 1.7, 2.2 and 2.3 below.

Already here we note that the Ahlfors-Shimizu form of the Nevanlinna characteristic $T(r,f)$ shows that if f is meromorphic in the plane and has bounded spherical derivative, then $T(r,f)=\mathcal{O}(r^2)$ as $r\to\infty$. In particular, f has finite order of growth. We refer to [56, 59, 79, 80] for the terminology of Nevanlinna theory, and in particular for the definitions of characteristic and order.

For entire functions we have stronger results. Recall that an entire function f is said to be of exponential type if the maximum modulus $M(r,f)$ satisfies $\log M(r,f)=\mathcal{O}(r)$ as $r\to\infty$. The following result is a special case of a result of J. Clunie and W. K. Hayman [38, Thm. 3]; see also [91, Thm. 4].

1.5.2. Clunie-Hayman Theorem. *An entire function with bounded spherical derivative is of exponential type.*

1.6. Further examples. We have seen already some examples of properties for which Bloch's Principle holds, and we will see many more examples of such properties in the following sections. But of course it is also very enlightening to consider counterexamples to the heuristic Bloch Principle. The following examples are due to L. A. Rubel [97].

1.6.1. Example. Define $\langle f,D\rangle\in P$ if $f=g''$ for some function g which is holomorphic and univalent in D. Since the only univalent entire functions g are those of the form $g(z)=\alpha z+\beta$ where $\alpha,\beta\in\mathbb{C}$, $\alpha\neq 0$, we see that $\langle f,\mathbb{C}\rangle\in P$ implies that $f(z)\equiv 0$. Thus (a) holds.

On the other hand, define

$$g_n(z):=n\left(z+\frac{1}{10}z^2+\frac{1}{10}z^3\right)$$

and $f_n:=g_n''$ for $n\in\mathbb{N}$. Then

$$\operatorname{Re} g_n'(z)=n\left(1+\frac{1}{5}\operatorname{Re} z+\frac{3}{10}\operatorname{Re} z^2\right)>\frac{1}{2}>0$$

for $z\in\mathbb{D}$, which implies that g_n is univalent in $\mathbb{D}$. So $\langle f_n,\mathbb{D}\rangle\in P$. Now $f_n(z)=n(1/5+3z/5)$ so that $f_n(-1/3)=0$ while $f_n(0)\to\infty$ as $n\to\infty$. Thus the functions f_n do not form a normal family on $\mathbb{D}$. Hence (b) fails.

It is easily seen that P satisfies conditions (i) and (ii) of Zalcman's Principle, but condition (iii) is not satisfied.

1.6.2. Example. Define $\langle f,D\rangle\in P$ if f is holomorphic in D and $f'(z)\neq -1$, $f'(z)\neq -2$ and $f'(z)\neq f(z)$ for all $z\in D$. If $\langle f,\mathbb{C}\rangle\in P$, then f' is constant by Picard's Theorem so that f is of the form $f(z)=\alpha z+\beta$ where $\alpha,\beta\in\mathbb{C}$.

Since $f'(z) \neq f(z)$ we have $\alpha z + \beta - \alpha \neq 0$. This implies that $\alpha = 0$ so that f is constant. Thus (a) holds.

On the other hand, with $f_n(z) := nz$ we find that $\langle f_n, \mathbb{D}\rangle \in P$ for all $n \in \mathbb{N}$, but the functions f_n do not form a normal family on $\mathbb{D}$ and thus (b) fails.

Again it is clear that (i) holds, and one can verify that (iii) is also satisfied. So in this case it is property (ii) that fails to hold in Zalcman's Principle.

1.7. Other applications of Zalcman's Lemma. Not only can Zalcman's Lemma be used to prove that statements (a) and (b) as above are *equivalent* for many properties P, but it also often yields easy proofs that these statements are *true*.

Sketch of a simple proof of the Theorems of Montel and Picard (due to A. Ros [123, p. 218])**.** Let $\mathcal{F}$ be as in the statement of Montel's Theorem and suppose that $\mathcal{F}$ is not normal. Without loss of generality we may assume that $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ and that D is a disk. If $f \in \mathcal{F}$ and $n \in \mathbb{N}$ there exists a function g holomorphic in D such that $g^{2^n} = f$. Let $\mathcal{F}_n$ be the family of all such functions g. Note that

$$g^\# = \frac{1}{2^n} \frac{|f|^{1/2^n - 1}|f'|}{1 + |f|^{2/2^n}} = \frac{1}{2^n} \frac{|f|^{-1} + |f|}{|f|^{-1/2^n} + |f|^{1/2^n}} f^\# \geq \frac{1}{2^n} f^\#,$$

where we have used the inequality $a^{-1} + a \geq a^{-t} + a^t$ valid for $a > 0$ and $0 < t < 1$. By Marty's Criterion, the family $\{f^\# : f \in \mathcal{F}\}$ is not locally bounded. We deduce that, for fixed $n \in \mathbb{N}$, the family $\{g^\# : g \in \mathcal{F}_n\}$ is not locally bounded. Using Marty's Criterion again we find that $\mathcal{F}_n$ is not normal, for all $n \in \mathbb{N}$.

Note that if $g \in \mathcal{F}_n$, then g omits the values $e^{2\pi i k/2^n}$ for $k \in \mathbb{Z}$. From the Zalcman Principle (and the remarks following it) we thus deduce that there exists an entire function g_n omitting the values $e^{2\pi i k/2^n}$ and satisfying $g_n^\#(z) \leq g_n^\#(0) = 1$. The g_n thus form a normal family and we have $g_{n_j} \to G$ for some subsequence (g_{n_j}) of (g_n) and some non-constant entire function G. By Hurwitz's Theorem, G omits the values $e^{2\pi i k/2^n}$ for all $k, n \in \mathbb{N}$. Since $G(\mathbb{C})$ is open this implies that $|G(z)| \neq 1$ for all $z \in \mathbb{C}$. Thus either $|G(z)| < 1$ for all $z \in \mathbb{C}$ or $|G(z)| > 1$ for all $z \in \mathbb{C}$. In the first case G is bounded and thus constant by Liouville's Theorem. In the second case $1/G$ is bounded. Again $1/G$ and thus G is constant. Thus we get a contradiction in both cases. ■

The following result generalizes the Theorems of Picard and Montel. We have named it after R. Nevanlinna (see [79, p. 102] or [80, §X.3]) although only the part concerning functions in the plane is due to him. The part concerning normal families is due to A. Bloch [17, Thm. XLIV] and G. Valiron [112, Thm. XXVI], with proofs being based, however, on Nevanlinna's theory. A proof using different ideas was given by R. M. Robinson [95]. For a proof using Zalcman's Lemma we refer to [9, §5.1].

1.7.1. Nevanlinna's Theorem. *Let $q \in \mathbb{N}$, let $a_1, \dots, a_q \in \widehat{\mathbb{C}}$ be distinct and let $m_1, \dots, m_q \in \mathbb{N}$. Suppose that*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 2. \tag{1.7.2}$$

Then the property that all a_j-points of f have multiplicity at least m_j is a Picard-Montel property.

We may also allow $m_j = \infty$ here, meaning that f has no a_j-points at all.

It is easy to see that the property P in this result satisfies conditions (i), (ii) and (iii) of the Zalcman Principle 1.5.1 so that the statement about functions in the plane is equivalent to that about normal families. But the proof that both statements are true is more awkward.

Sketch of proof of a special case of Nevanlinna's Theorem. We assume in addition that the multiplicity $m(z)$ of each a_j-point z of f satisfies not only $m(z) \geq m_j$, but that in fact $m(z)$ is a multiple of m_j; that is, $m(z) = n(z)m_j$ where $n(z) \in \mathbb{N}$. Let P' be this property. Again (i), (ii) and (iii) are satisfied so that P' is also a Bloch property. Moreover, properties (a) and (b) occurring in the definition of Bloch property are equivalent to property (c) mentioned at the end of Section 1.5. Thus it suffices to prove (c).

So let $\langle f, \mathbb{C} \rangle \in P'$ where f has bounded spherical derivative, but suppose that f is not constant. We may assume that $a_j \neq \infty$ for all j and define

$$g(z) = \frac{f'(z)^M}{\prod_{j=1}^{q} (f(z) - a_j)^{(m_j - 1)M/m_j}}, \tag{1.7.3}$$

where M is the least common multiple of the m_j. The assumption on the multiplicities of the a_j-points implies that g does not have poles; that is, g is entire. Since f is not constant there exists a sequence (u_n) tending to ∞ such that $f(u_n) \to \infty$. The denominator of g is a polynomial in f of degree

$$\ell := \sum_{j=1}^{q} \frac{(m_j - 1)M}{m_j} = M \sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 2M.$$

Thus

$$\prod_{j=1}^{q} (f(u_n) - a_j)^{(m_j - 1)M/m_j} \geq (1 - o(1))|f(u_n)|^{\ell} \geq (1 - o(1))|f(u_n)|^{2M+1}.$$

On the other hand, $|f'(u_n)| \leq \mathcal{O}(|f(u_n)|^2)$ as $n \to \infty$ since f has bounded spherical derivative. Thus $|f'(u_n)|^M \leq o(|f(u_n)|^{2M+1})$ as $n \to \infty$. Overall we see that $g(u_n) \to 0$. On the other hand, $g(z) \not\equiv 0$ since f is not constant. Thus g is not constant, which implies that there exists a sequence (v_n) such that $g(v_n) \to \infty$. It follows that $f(v_n) \not\to \infty$.

We consider $h_n(z) := f(z+v_n)$. Since f has bounded spherical derivative, the h_n form a normal family. Passing to a subsequence if necessary, we may thus assume that h_n converges locally uniformly to some meromorphic function $h\colon \mathbb{C} \to \widehat{\mathbb{C}}$. It follows that $h(z) \equiv a_k$ for some $k \in \{1, \ldots, 5\}$, because otherwise

$$g(z+v_n) \to \frac{h'(z)^M}{\prod_{j=1}^{q}(h(z)-a_j)^{(m_j-1)M/m_j}} \neq \infty,$$

which contradicts $g(v_n) \to \infty$.

Thus $h_n \to a_k$ as $n \to \infty$. For sufficiently large n the function

$$\psi_n(z) := h_n(z) - a_k = f(z+v_n) - a_k$$

is holomorphic in $\mathbb{D}$, and we have $\psi_n \to 0$ as $n \to \infty$. Since the multiplicity of all a_k-points of f is a multiple of m_k we may define a holomorphic branch ϕ_n of the m_k-th root of ψ_n; that is, $\phi_n\colon D \to \mathbb{C}$ and $\phi_n(z)^{m_k} = \psi_n(z)$. We also have $\phi_n \to 0$. Thus $\phi_n'(z) \to 0$. Now $(m_k|\phi_n'(z)|)^{m_k} = |\psi_n(z)|^{1-m_k}|\psi_n'(z)|^{m_k}$. Hence $|f'(v_n)|^{m_k}/|f(v_n)-a_k|^{m_k-1} = |\psi_n'(0)|^{m_k}/|\psi_n(0)|^{m_k-1} \to 0$. This implies that $g(v_n) \to 0$, a contradiction. ∎

We mention that for $q = 3$ the special case of Nevanlinna's Theorem above goes back to C. Carathéodory [26]. Note that this special case still gives a generalization of the Theorems of Picard and Montel.

The general version of Nevanlinna's Theorem may be proved along the same lines. In this case the functions ϕ_n occurring in the above proof may be multi-valued. A similar argument as above may still me made, however, by using a version of Schwarz's Lemma for multivalued functions due to Z. Nehari [77]; see [9] for more details.

We note that the hypothesis (1.7.2) is best possible. In fact, if we have equality in (1.7.2), then there exists an elliptic function f which satisfies the differential equation $f'(z)^M = \prod_{j=1}^{q}(f(z)-a_j)^{(m_j-1)M/m_j}$. It follows that $\langle f, \mathbb{C}\rangle \in P$ for a non-constant function f. Note that the function g defined in the above proof by (1.7.3) now satisfies $g(z) \equiv 1$.

The possible choices for the m_j are $q = 4$ and $m_j = 2$ for all j or $q = 3$ and $(m_1, m_2, m_3) = (3, 3, 3)$, $(m_1, m_2, m_3) = (2, 3, 6)$ or $(m_1, m_2, m_3) = (2, 4, 4)$, up to permutations of the m_j. One can modify the construction to allow the case that $a_j = \infty$ for some j. Moreover, one can also modify the above considerations to include the case $m_j = \infty$. In this case the resulting functions f are trigonometric functions or the exponential function, or obtained from these functions by linear transformations.

Closely related to Nevanlinna's Theorem is one of the main results from the Ahlfors theory of covering surfaces; see [2], [56, Ch. 5] or [80, Ch. XIII]. Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $f\colon D \to \widehat{\mathbb{C}}$ be a meromorphic function. Let $V \subset \widehat{\mathbb{C}}$ be a Jordan domain. A simply-connected component U of $f^{-1}(V)$ with $\overline{U} \subset D$ is called an *island* of f over V. Note that then $f|_U : U \to V$ is a proper map. The degree of this proper map is called the *multiplicity* of the island U. An island of multiplicity one is called a *simple island*.

1.7.4. Ahlfors's Theorem. *Let $q \in \mathbb{N}$, $D_1, \dots, D_q \subset \widehat{\mathbb{C}}$ Jordan domains with pairwise disjoint closures and $m_1, \dots, m_q \in \mathbb{N}$ satisfying* (1.7.2). *Then the property that all islands of f over D_j have multiplicity at least m_j is a Picard-Montel property.*

We indicate how Zalcman's Lemma can be used to deduce Ahlfors's Theorem from Nevanlinna's Theorem; see [9] for a more detailed discussion.

Sketch of proof of Ahlfors's Theorem. Fix $a_1, \dots, a_q \in \mathbb{C}$. First we show that there exists $\varepsilon > 0$ such that the conclusion of Ahlfors's Theorem is true if $D_j = D(a_j, \varepsilon)$. Here and in what follows $D(a, \varepsilon) := \{z \in \mathbb{C} : |z - a| < \varepsilon\}$ denotes the disk of radius ε around a point $a \in \mathbb{C}$. If such an ε does not exist, then we can choose a sequence (ε_n) tending to 0 and find a sequence (f_n) of non-constant meromorphic functions on $\mathbb{C}$ which have no island of multiplicity less than m_j over $D(a_j, \varepsilon_n)$. By the arguments of Section 1.5 we may assume that the f_n have bounded spherical derivative and in fact that $f_n^\#(z) \le f_n^\#(0) = 1$ for all $z \in \mathbb{C}$. It follows that the f_n form a normal family, and thus we may assume without loss of generality that $f_n \to f$ for some meromorphic function f. Since $f^\#(0) = 1$ we see that f is not constant. We also find that f has no island of multiplicity less than m_j over $D(a_j, \varepsilon)$, for any $\varepsilon > 0$. But this implies that all a_j-points of f have multiplicity at least m_j, contradicting Nevanlinna's Theorem. Thus there exists $\varepsilon > 0$ such that the conclusion of Ahlfors's Theorem holds if $D_j = D(a_j, \varepsilon)$.

In the second step we reduce the general case to this special case. To do this we use quasiconformal mappings; see [66] for an introduction to this subject. So suppose that f is non-constant and meromorphic in the plane such that every island over D_j has multiplicity at least m_j, for $j \in \{1, \dots, q\}$. We note that there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ with $\phi(D_j) \subset D(a_j, \varepsilon)$ for $j \in \{1, \dots, q\}$, and the quasiregular map $\phi \circ f$ can be factored as $\phi \circ f = g \circ \psi$ with a non-constant meromorphic function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ and a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$. It then follows that every island of g over $D(a_j, \varepsilon)$ has multiplicity at least m_j, contradicting the first step above. ■

As mentioned, the Theorems of Picard and Montel are a special case of Nevanlinna's Theorem, namely the case $q = 3$ and $m_1 = m_2 = m_3 = \infty$. Applying the arguments used in the proof above to the Theorems of Picard and Montel instead of Nevanlinna's Theorem, we obtain a direct proof of the following special case of Ahlfors's Theorem.

1.7.5. Special case of Ahlfors's Theorem. *Let $D_1, \dots, D_3 \subset \widehat{\mathbb{C}}$ be Jordan domains with pairwise disjoint closures. Then the property that f has no islands over any of the domains D_j is a Picard-Montel property.*

Another important special case of Ahlfors's Theorem 1.7.4 is the case $q = 5$ and $m_j = 2$ for all j. This case is known as the *Ahlfors Five Islands Theorem.*

1.8. A modified Bloch principle. Closely related to Montel's Theorem 1.2.1 and Picard's Theorem 1.2.2 is the following result.

1.8.1. Great Picard Theorem. *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be distinct, $D \subset \mathbb{C}$ a domain, $\zeta \in D$ and $f : D \setminus \{\zeta\} \to \widehat{\mathbb{C}}$ meromorphic. If $f(z) \neq a_j$ for all $j \in \{1, 2, 3\}$ and all $z \in D \setminus \{\zeta\}$, then ζ is not an essential singularity of f.*

This results suggests a modification of the Bloch Principle which says that for a Picard-Montel property P there should not exist a meromorphic function having the property P in the neighborhood of an essential singularity. In other words, for a property P of meromorphic functions the conditions (a), (b) and (c) discussed in Section 1.5 should imply the following condition:

(d) *if $\langle f, \mathbb{D} \setminus \{\zeta\}\rangle \in P$ for some domain D and some $\zeta \in D$, then ζ is not essential singularity of f.*

D. Minda [69] gives a discussion of this modification of the heuristic principle, and he shows that for holomorphic families this modified heuristic principle holds under the hypotheses of Zalcman's Principle 1.5.1.

1.8.2. Minda's Principle. *Suppose that a property P of holomorphic functions satisfies the conditions* (i), (ii) *and* (iii) *of Zalcman's Principle 1.5.1. Then each of the conditions* (a), (b) *and* (c) *introduced in Section 1.5 implies that* (d) *holds.*

Minda also points out that there are *meromorphic* families where Zalcman's Principle applies, but where the modified heuristic principle does not hold. However, if one adds a further condition to the ones given by Zalcman, then the modified heuristic principle holds; see Minda [69, §5] and [10, §3] for the following result.

1.8.3. Theorem. *Suppose that a property P of meromorphic functions satisfies the conditions* (i), (ii) *and* (iii) *of Zalcman's Principle 1.5.1. Suppose that P satisfies in addition the condition*

(iv) *if $\langle f, \mathbb{C} \setminus \{0\}\rangle \in P$, then $\langle f \circ \exp, \mathbb{C}\rangle \in P$.*

Then each of the conditions (a), (b) *and* (c) *implies that* (d) *holds.*

The additional condition (iv) can be compared with (ii). Both conditions are obviously satisfied for properties P which concern only the range of f. It is easily seen that the properties occurring in Nevanlinna's and Ahlfors's Theorem in Section 1.7 satisfy (iv).

The proof of these theorems uses the following results due to O. Lehto and L. V. Virtanen [63, 64, 65].

1.8.4. Lehto-Virtanen Theorem. *Suppose that a meromorphic function f has an essential singularity at ζ. Then*

$$\limsup_{z\to\zeta} |z-\zeta| f^{\#}(z) \geq \frac{1}{2}.$$

If f is holomorphic, then

$$\limsup_{z\to\zeta} |z-\zeta| f^{\#}(z) = \infty.$$

Lehto and Virtanen [65] had shown that $\limsup_{z\to\zeta} |z-\zeta| f^{\#}(z) \geq k$ for some absolute constant $k > 0$. This weaker result would suffice for our purposes. Lehto [63] later showed that one can take $k1/2$.

Proof of Minda's Principle 1.8.2. Since we assume that the hypotheses of Zalcman's Principle 1.5.1 are satisfied, the conclusion of 1.5.1 also holds. Thus conditions (a) and (b) are equivalent, and the discussion after 1.5.1 shows that these conditions are also equivalent to (c). Suppose now that one and hence all of these conditions are satisfied. We want to prove (d).

So let $\langle f, D\setminus\{\zeta\}\rangle \in P$, where D is a domain, f is holomorphic and $\zeta \in D$. Suppose that ζ is an essential singularity of f. We may assume that $\zeta = 0$. By the Lehto-Virtanen Theorem there exists a sequence (c_n) in D such that $c_n \to 0$ and $|c_n| f^{\#}(c_n) \to \infty$. For sufficiently large n the function $f_n(z) := f(c_n + c_n z)$ is then holomorphic in the unit disk $\mathbb{D}$, and $f_n^{\#}(0) = c_n f^{\#}(c_n) \to \infty$. Thus the f_n cannot form a normal family by Marty's Criterion. On the other hand, we deduce from (i) and (ii) that the f_n also satisfy P. This is a contradiction to (b). ∎

Proof of Theorem 1.8.3. We note again that (a), (b) and (c) are equivalent. We suppose that these conditions are satisfied and want to prove (d).

So let $\langle f, D\setminus\{\zeta\}\rangle \in P$ for a domain D and some $\zeta \in D$. We may assume that $\zeta = 0$. Suppose that ζ is an essential singularity. By the Lehto-Virtanen Theorem there exists sequence (c_n) in D such that $c_n \to 0$ and $|c_n| f^{\#}(c_n) \geq 1/4$. Choose $r > 0$ such that $D(0,r) \subset D$. We define $r_n := r/|c_n|$ and $g_n\colon D(0,r_n)\setminus\{0\} \to \widehat{\mathbb{C}}$ by $g_n(z) := f(c_n z)$. Since P satisfies condition (b), as well as (i) and (ii), and since $r_n \to \infty$, we see that the g_n form a normal family. Without loss of generality we may assume that $g_n \to g$ for some $g\colon \mathbb{C}\setminus\{0\} \to \widehat{\mathbb{C}}$. Since $g_n^{\#}(1) = c_n f^{\#}(c_n) \geq 1/4$ we have $g^{\#}(1) \geq 1/4$ so that g is non-constant. It follows from (iii) that $\langle g, \mathbb{C}\setminus\{0\}\rangle \in P$. By condition (iv) thus $\langle g\circ \exp, \mathbb{C}\rangle \in P$. From (a) we may deduce that $g \circ \exp$ is constant, and so is g, a contradiction. ∎

1.9. Quasinormality. We cannot expect that condition (d) of Section 1.8 implies the conditions (a), (b), and (c) introduced in Section 1.5. For example, the condition that f take three values a_1, a_2, a_3 only N times, for some fixed number

$N \in \mathbb{N}$, satisfies condition (d), but none of the conditions (a), (b), and (c). This condition does, however, imply *quasinormality*.

We say that a family $\mathcal{F}$ of functions meromorphic in a domain D is *quasinormal* (cf. [73, 102]) if for each sequence (f_k) in $\mathcal{F}$ there exists a subsequence (f_{k_j}) and a finite set $E \subset D$ such that (f_{k_j}) converges locally uniformly in $D \backslash E$. If the cardinality of the exceptional set E can be bounded independently of the sequence (f_k), and if q is the smallest such bound, then we say that $\mathcal{F}$ is quasinormal of *order* q. We mention that Chuang's [34] definition of quasinormality is slightly different: he only requires the exceptional set E to be discrete, but not necessarily finite.

Many of the results about normal families have extensions involving the concept of quasinormality. Here we only mention the corresponding generalization of Montel's Theorem, also proved by Montel [73, p. 149].

1.9.1. Montel's Theorem. *Let* $0 \leq m_1 \leq m_2 \leq m_3$*, let* $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ *be distinct, let* $D \subset \mathbb{C}$ *be a domain and let* $\mathcal{F}$ *be a family of functions meromorphic in* D*. Suppose that* f *takes the value* a_j *at most* m_j *times in* D*, for all* $j \in \{1, 2, 3\}$ *and all* $f \in \mathcal{F}$*. Then* $\mathcal{F}$ *is quasinormal of order at most* m_2*.*

Quasinormality will also be discussed in Section 3.2 below. A detailed study of quasinormality, and in fact of a more general concept called Q_m-normality, has been given by C. T. Chuang [34].

2. Exceptional values of derivatives

2.1. Introduction. We discuss some variants of the Theorems of Picard and Montel where exceptional values of f are replaced by those of a derivative. Our starting point is the following result proved by G. Pólya's student W. Saxer in 1923; see [100, Hilfssatz, p. 210] and [101].

2.1.1. Saxer's Theorem. *Let* f *be a transcendental entire function and let* $a, b \in \mathbb{C}$*. Suppose that the equations* $f(z) = a$ *and* $f'(z) = b$ *have only finitely many solutions. Then* $b = 0$*.*

Combining this with the Great Picard Theorem, applied to f', we see that if a transcendental entire function f takes a value a only finitely many times, then f' takes every non-zero value infinitely often. In 1929, E. Ullrich [109, p. 599] showed that under this hypothesis all derivatives $f^{(k)}$, $k \geq 1$, take every non-zero value infinitely often. Ullrich writes that this result has been known for several years, and he attributes it to Pólya and Saxer [100], although it can be found there only for the first derivative.

A simple discussion of the case where f is a polynomial now leads to the following result.

2.1.2. Pólya-Saxer-Ullrich Theorem. *Let f be an entire function and let $k \geq 1$. Suppose that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $z \in \mathbb{C}$. Then f is constant.*

Here and in the following theorems the conditions $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ can be replaced by $f(z) \neq a$ and $f^{(k)}(z) \neq b$ as long as $b \neq 0$.

Ullrich's result was obtained independently a few years later by F. Bureau [23, 24, 25]. Actually Bureau considered functions with an essential singularity, as in Section 1.8. Bureau also gave a normality result, but he required additional conditions besides $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$. The complete normal family analogue was obtained by C. Miranda [71] in 1935.

2.1.3. Miranda's Theorem. *Let $\mathcal{F}$ be a family of functions holomorphic in a domain D and $k \geq 1$. Suppose that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $f \in \mathcal{F}$ and $z \in D$. Then $\mathcal{F}$ is normal.*

In 1959, W. K. Hayman [55, Thm. 1] extended the Pólya-Saxer-Ullrich Theorem to meromorphic functions. We mention that the case that f is meromorphic had also been considered by Ullrich, but in this case he required additional hypotheses, e. g. that ∞ is a Borel exceptional value [109, p. 599].

2.1.4. Hayman's Theorem. *Let f be meromorphic in the plane and $k \geq 1$. Suppose that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $z \in \mathbb{C}$. Then f is constant.*

Remark. More generally, Hayman proved that if f and $f^{(k)} - 1$ have only finitely many zeros, then f is rational.

It took 20 years until Y. Gu [54] proved the normal family analogue of Hayman's Theorem.

2.1.5. Gu's Theorem. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain D and $k \geq 1$. Suppose that $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for all $f \in \mathcal{F}$ and $z \in D$. Then $\mathcal{F}$ is normal.*

2.2. A generalization of Zalcman's Lemma. Zalcman's Principle as stated in Section 1.5 does not apply to conditions such as "$f(z) \neq 0$ and $f^{(k)}(z) \neq 1$." However, it was shown by X. Pang [83, 84] that there is an extension of Zalcman's Lemma which allows us to deal with such conditions. We state this extension in its most general form, and not only in the form needed to prove that "$f(z) \neq 0$ and $f^{(k)}(z) \neq 1$" is a Bloch property.

2.2.1. Zalcman-Pang Lemma. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \mathbb{C}$ and let $m \in \mathbb{N}$, $K \geq 0$ and $\alpha \in \mathbb{R}$ with $-m \leq \alpha < 1$. Suppose that the zeros of the functions in $\mathcal{F}$ have multiplicity at least m; that is, if $f \in \mathcal{F}$ and $\xi \in D$ with $f(\xi) = 0$, then $f^{(k)}(\xi) = 0$ for $1 \leq k \leq m-1$. If $\alpha = -m$, then suppose in addition that $|f^{(m)}(\xi)| \leq K$ if $f \in \mathcal{F}$, $\xi \in D$ and $f(\xi) = 0$.*

Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then there exist a sequence (f_k) in $\mathcal{F}$, a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ such that $z_k \to z_0$, $\rho_k \to 0$ and

$$\rho_k^\alpha f_k(z_k + \rho_k z) \to f(z)$$

locally uniformly in $\mathbb{C}$. Moreover, $f^\#(z) \le f^\#(0) = mK + 1$ for all $z \in \mathbb{C}$.

We omit the proof, but mention only that its spirit is similar to the proof of the original Zalcman Lemma, but the technical details are more convoluted.

We note that $\{1/f : f \in \mathcal{F}\}$ is normal if and only if $\mathcal{F}$ is normal. Thus we obtain an analogous result for $-1 < \alpha \le \ell$, if the poles of the functions in $\mathcal{F}$ have multiplicity at least ℓ, with an additional hypotheses if $\alpha = \ell$. Note that no hypothesis on the zeros or poles is required when $-1 < \alpha < 1$.

The Zalcman Lemma 1.4.1 is of course the case $\alpha = 0$. As mentioned, the idea to introduce the exponent α seems to be due to X. Pang [83, 84], who proved that one can always take $-1 < \alpha < 1$. It was shown by X. Pang and G. Xue [117] that $\alpha < 0$ is admissible if the functions in $\mathcal{F}$ have no zeros. Then H. Chen and X. Gu [31, Thm. 2] proved that one can take $-m < \alpha \le 0$ if the zeros of the functions in $\mathcal{F}$ have multiplicity at least m. Finally, the case $\alpha = -m$ is due to X. Pang and L. Zalcman [88, Lem. 2]. The special case $\alpha = -m = -1$ had been treated before by X. Pang [85].

The Zalcman-Pang Lemma shows that the Zalcman Principle 1.5.1 may be modified by replacing the condition (ii) by

(ii′) *there exists $\alpha \in (-1, 1)$ such that if $\langle f, D\rangle \in P$ and $\varphi(z) = \rho z + c$ where $\rho, c \in \mathbb{C}$, $\rho \ne 0$, then $\langle \rho^\alpha (f \circ \varphi), \varphi^{-1}(D)\rangle \in P$;*

or

(ii″) *there exists $m \in \mathbb{N}$ and $\alpha \in (-m, 1)$ such that if $\langle f, D\rangle \in P$, then all zeros of f have multiplicity at least m, and if $\varphi(z) = \rho z + c$ where $\rho, c \in \mathbb{C}$, $\rho \ne 0$, then $\langle \rho^\alpha (f \circ \varphi), \varphi^{-1}(D)\rangle \in P$.*

We leave it to the reader to formulate a condition for the case that $\alpha = -m$, or the case that the functions with property P have only multiple poles of multiplicity at least ℓ.

If $\langle f, D\rangle \in P$ implies that f has no zeros, then we can take any m and α in (ii″). The choice $\alpha = -k$ implies that the property occurring in the Theorems of Pólya-Saxer-Ullrich, Miranda, Hayman and Gu is indeed a Bloch property. In particular, Gu's Theorem can then be deduced from Hayman's, and Miranda's from that of Pólya-Saxer-Ullrich. However, as with the Theorems of Picard and Montel, the Zalcman-Pang Lemma not only shows that the Theorems of Miranda and Pólya-Saxer-Ullrich are *equivalent*, it can also be used to *prove* them; see [53, 87].

Proof of Miranda's Theorem (following [87])**.** We write $\langle f, D\rangle \in P$ if f is holomorphic in D and if f and $f^{(k)} - 1$ have no zeros in D. As seen above, it follows from the Zalcman Principle, with (ii) replaced by (ii$''$), that P is a Bloch property. The argument given at the end of Section 1.5 shows that the Theorems of Pólya-Saxer-Ullrich and Miranda are equivalent to the condition (c) given there.

Let thus $\langle f, \mathbb{C}\rangle \in P$ where f has bounded spherical derivative. By the Clunie-Hayman Theorem 1.5.2, f has exponential type. Since f has no zeros this implies that f has the form $f(z) = e^{az+b}$ with $a, b \in \mathbb{C}$. Since $f^{(k)}(z) - 1 = a^k e^{az+b} - 1$ has no zeros we deduce that $a = 0$ and thus f is constant. ∎

We mention, however, that we have not been able to find a proof of Hayman's or Gu's Theorem based on these ideas.

2.3. Multiple values of derivatives. Nevanlinna's Theorem 1.7.1 can be seen as a generalization of the Theorems of Picard and Montel, where the hypothesis that f does not take a value a_j is replaced by the hypothesis that the a_j-points of f have high multiplicity. One may ask whether the Theorems of Hayman and Gu (or of Pólya-Saxer-Ullrich and Miranda) admit similar generalizations. This question was considered in [15, 32, 39, 116, 119, 120]. The results below are taken mostly from [15].

Let k be a positive integer and let $0 < M \leq \infty$, $0 < N \leq \infty$. For a function f meromorphic in a domain D we say that f has the property $P(k, M, N)$, written again as $\langle f, D\rangle \in P(k, M, N)$, if all zeros of f in D have multiplicity at least M, while all zeros of $f^{(k)} - 1$ in D have multiplicity at least N. Here $M = \infty$ or $N = \infty$ should be interpreted as meaning that there are no corresponding zeros in D.

2.3.1. Theorem. *Let $k, M, N \in \mathbb{N}$. Then $P(k, M, N)$ is a Bloch property.*

Proof. Suppose first that $M \leq k$. Define $f_n(z) = 2n(z - a)^k$ with $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Then f_n satisfies $P(k, M, N)$ for any N. Moreover, f_n is non-constant and entire, and there is no neighbourhood of a on which the f_n form a normal family. So both statements (a) and (b) occurring in the definition of Bloch property in Section 1.5 are false.

Suppose now that $M > k$. Then $P(k, M, N)$ satisfies condition (ii$''$) in Section 2.2 with $m := M$ and $\alpha := -k$. It is obvious that $P(k, M, N)$ also satisfies condition (i) of Zalcman's Principle 1.5.1. An application of Hurwitz's Theorem shows that condition (iii) of 1.5.1 is also satisfied. The conclusion thus follows from the Zalcman Principle, as generalized in Section 2.2. ∎

Theorem 2.3.1 says that whether a family $\{f : \langle f, D\rangle \in P(k, M, N)\}$ is normal is equivalent to whether $\langle f, \mathbb{C}\rangle \in P(k, M, N)$ contains only constant functions f. It does not say whether these statements are true or false. For example, for $(k, M, N) = (1, 3, 3)$ both statements are false, as shown by the example $f := 1/\wp'$, where $\wp$ is the Weierstraß elliptic function satisfying the equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4\wp^3 + 3\wp - 1$. Then f has only triple zeros and so has $f' - 1 = -4(\wp + 1/2)^3/(\wp')^2$. Thus $\langle f, \mathbb{C}\rangle \in P(1, 3, 3)$, and f is non-constant. But I do not know whether property $P(k, M, N)$ forces a meromorphic function in the plane to be constant — and a family of meromorphic functions to be normal — if, for example, $(k, M, N) = (1, 3, 4)$ or $(k, M, N) = (1, 4, 3)$.

So while the precise conditions on M and N yielding this are not known, some partial answers are available.

2.3.2. Theorem. *Let k be a positive integer and let $0 < M \leq \infty$, $0 < N \leq \infty$ with*

$$\frac{2k + 3 + \frac{2}{k}}{M} + \frac{2k + 4 + \frac{2}{k}}{N} < 1.$$

Then $P(k, M, N)$ is a Picard-Montel property.

We omit the proof, which — among other things — is based on Nevanlinna theory. The interested reader is referred to [15].

The following result is due to Y. Wang and M. Fang [116, Thm. 7].

2.3.3. Wang-Fang Theorem. *Let k be a positive integer. Then $P(k, k+2, \infty)$ is a Picard-Montel property.*

One ingredient in the proof of this theorem is the following result proved in [13, Cor. 3]. Here a complex number w is called a critical value of f if there exists ζ such that $f'(\zeta) = 0$ and $f(\zeta) = w$.

2.3.4. Theorem. *If a meromorphic function of finite order ρ has only finitely many critical values, then it has at most 2ρ asymptotic values.*

Sketch of proof of the Wang-Fang Theorem. First we note that — as remarked at the end of Section 1.5 — it suffices to prove condition (c) for the property $P = P(k, k + 2, \infty)$. So suppose that f is a function meromorphic in the plane which has bounded spherical derivative such that all zeros of f have multiplicity $k + 2$ at least and such that $f^{(k)} - 1$ has no zeros. As remarked in Section 1.5, f has finite order.

We consider the auxiliary function $g(z) = z - f^{(k-1)}(z)$. Then g' has no zeros. Moreover, g has finite order. By Theorem 2.3.4, g has only finitely many asymptotic values.

Suppose now that f has a zero ζ. Since this zero has multiplicity $k + 2$ at least, we find that $g(\zeta) = \zeta$ and $g'(\zeta) = 1$. In the terminology of complex dynamics (see, for example, [5, 8, 68, 108]) the point ζ is thus a parabolic fixed point of g. By a classical result from complex dynamics, sometimes called the Leau-Fatou Flower Theorem, there exists a domain U with $\zeta \in \partial U$ where the iterates g^n of g tend to ζ as $n \to \infty$. Moreover, a maximal domain U with this property contains a critical or asymptotic value of g. Since g has no critical and only finitely many asymptotic values, this implies that f has only finitely many zeros. Hayman's Theorem 2.1.4, or more precisely the remark made after it, implies that f is rational. A discussion of this case then completes the proof. ■

The examples

$$f(z) = \frac{(z+a)^{k+1}}{k!(z+b)} \tag{2.3.5}$$

with $a, b \in \mathbb{C}, a \neq b$ show that $P(k, k+1, \infty)$ is not a Picard-Montel property. However, Wang and Fang [116, Lem. 8] have shown that every non-constant rational function f satisfying $\langle f, \mathbb{C}\rangle \in P(k, k+1, \infty)$ has this form. The argument used in the proof of the Wang-Fang Theorem shows that there is no transcendental function f meromorphic in the plane and of finite order such that $\langle f, \mathbb{C}\rangle \in P(k, k+1, \infty)$. However, this restriction on the order turns out not to be necessary. In fact, Wang and Fang [116, Thm. 3] proved that there is no transcendental function f meromorphic in the plane satisfying $\langle f, \mathbb{C}\rangle \in P(k, 3, \infty)$ for some $k \in \mathbb{N}$. And Nevo, Pang and Zalcman [82] have recently shown that there is no transcendental function f meromorphic in the plane such that $\langle f, \mathbb{C}\rangle \in P(1, 2, \infty)$.

The examples (2.3.5) also show that the family $\{f : \langle f, D\rangle \in P(k, k+1, \infty)\}$ is not normal for any domain D and $k \in \mathbb{N}$. However, Nevo, Pang and Zalcman [82] have recently shown that $\{f : \langle f, D\rangle \in P(1, 2, \infty)\}$ is quasinormal of order 1, and, as Larry Zalcman has kindly informed me, their method can be extended to yield that $\{f : \langle f, D\rangle \in P(k, k+1, \infty)\}$ is quasinormal of order 1 for every $k \in \mathbb{N}$. For further results in this direction we refer to [81, 86].

The following result strengthens the Wang-Fang Theorem.

2.3.6. Theorem. *Let k be a positive integer. Then there exists a positive integer T_k such that $P(k, k+2, T_k)$ is a Picard-Montel property.*

Proof. Again it suffices that to show that there exists T_k such that (c) holds for the property $P(k, k+2, T_k)$. Suppose that this is not the case. Then for each $n \in \mathbb{N}$ there exists a non-constant entire function f_n such that $\langle f_n, \mathbb{C}\rangle \in P(k, k+2, n)$, and we may assume that $f_n^{\#}(z) \leq f_n^{\#}(0) = 1$ for all $z \in \mathbb{C}$. Thus the f_n form a normal family so that $f_{n_j} \to f$ for some subsequence (f_{n_j}) of (f_n). Hurwitz's Theorem now implies that $\langle f, \mathbb{C}\rangle \in P(k, k+2, \infty)$, contradicting the Wang-Fang Theorem. ■

2.4. Exceptional values of differential polynomials. Let $n \in \mathbb{N}$ and let $a, b \in \mathbb{C}$, $a \neq 0$. We say that a meromorphic function $f : D \to \mathbb{C}$ satisfies property $P(n, a)$ if $f(z)^n f'(z) \neq a$ for all $z \in D$. And we say that f has property $Q(a, b, n)$ if $f'(z) + af(z)^n \neq b$ for all $z \in D$. Note that $\langle f, D\rangle \in P(n, a)$ if and only if $\langle 1/f, D\rangle \in Q(n+2, a, 0)$.

It follows from the Zalcman-Pang Lemma with $\alpha := -1/(n+1)$ that $P(n,a)$ is a Bloch property. This was a key ingredient in the proof of the following result [13, 30, 122].

2.4.1. Theorem. *$P(n,a)$ is a Picard-Montel property for all $n \geq 1$ and $a \neq 0$.*

That functions meromorphic in the plane which satisfy $P(n,a)$ are constant was proved by Hayman [55, Cor. to Thm. 9] for $n \geq 3$ and by E. Mues [74, Satz 3] for $n = 2$. The case that f is entire is due to W. K. Hayman [55, Thm. 10] if $n \geq 2$ and to J. Clunie [37] if $n = 1$. By the remarks made above, Theorem 2.4.1 is equivalent to the following result.

2.4.2. Theorem. *$Q(n,a,0)$ is a Picard-Montel property for all $n \geq 3$ and $a \neq 0$.*

Hayman [55, Thm. 9] also proved the following result.

2.4.3. Theorem. *If $\langle f, \mathbb{C}\rangle \in Q(n,a,b)$ where $a, b \in \mathbb{C}$, $a \neq 0$, and $n \geq 5$, then f is constant.*

The conclusion of this theorem is not true for $n = 3$ and $n = 4$, as shown by examples due to Mues [74]. For $n = 4$ such an example is given by $f(z) := \tan z$. Then $f'(z) = 1 + f(z)^2 \neq 0$ so that

$$f' + \frac{1}{2}f^4 - \frac{1}{2} = \frac{1}{2}(1+f^2)^2$$

has no zeros. Thus $\langle \tan, \mathbb{C}\rangle \in Q\left(4, 1/2, 1/2\right)$. The examples for $n = 3$, or for different values of a and b, are similar.

However, an argument due to Pang [83, 84] shows that $Q(n,a,b)$ does imply normality for $n \geq 3$.

2.4.4. Pang's Theorem. *If $n \geq 3$ and $a \neq 0$, then, for each domain $D \subset \mathbb{C}$, the family $\{f : \langle f, D\rangle \in Q(n,a,b)\}$ is normal on D*

Sketch of proof. The idea is to deduce Theorem 2.4.4 from Theorem 2.4.2. We note that $Q(n,a,b)$ does not satisfy the condition (ii′) stated after the Zalcman-Pang Lemma in Section 2.2. However, with $\alpha := 1/(n-1) \in (-1,1)$ and $\varphi(z) := \rho z + c$ where $\rho, c \in \mathbb{C}$, $\rho \neq 0$ we see that $\langle f, D\rangle \in Q(n,a,b)$ is equivalent to

$$\langle \rho^{\alpha}(f \circ \varphi), \varphi^{-1}(D)\rangle \in Q(n, a, \rho^{n/(n-1)}b).$$

Since in the Zalcman-Pang Lemma one considers a sequence of ρ-values tending to 0 we see that the limit function f occurring in this lemma satisfies $\langle f, \mathbb{C}\rangle \in Q(n,a,0)$, contradicting Theorem 2.4.2. ■

When Pang wrote his papers, the conclusion of Theorem 2.4.2 was known only for $n \geq 4$. Therefore he could prove his result only for $n \geq 4$. But his argument also extended to the case $n = 3$, once Theorem 2.4.2 was known; see also [102, p. 143] for further discussion.

For $n \in \{3,4\}$ and $a, b \neq 0$ the property $Q(n,a,b)$ is thus a counterexample to the Bloch Principle as stated in Section 1.5. However, L. Zalcman and, independently, the present writer have in recent years suggested a variant of Bloch's Principle which allows to deal with properties such as $Q(n,a,b)$. Instead of the condition

(a) *if* $\langle f, \mathbb{C}\rangle \in P$, *then* f *is constant;*

introduced in Section 1.5 we consider the following condition:

(a′) *the family* $\{f : \langle f, \mathbb{C}\rangle \in P\}$ *is normal on* $\mathbb{C}$.

Note that (a′) is satisfied in particular if $\{f : \langle f, \mathbb{C}\rangle \in P\}$ consists only of constant functions. In other words, the condition (a) implies (a′). Recall the condition (b) in the formulation of the original Bloch Principle:

(b) *the family* $\{f : \langle f, D\rangle \in P\}$ *is normal on* D *for each domain* $D \subset \mathbb{C}$.

The variant of Bloch's Principle mentioned now says that (a′) should be equivalent to (b). Note that (b) trivially implies (a′), so what this modification of Bloch's Principle is really asking for is that (a′) implies (b).

We have seen that for $n \in \{3,4\}$ and $a, b \neq 0$ the property $P := Q(n,a,b)$ is an example where the original Bloch Principle "(a)⇔(b)" fails, but where the variant "(a′)⇔(b)" holds.

Some further cases where this is true will be discussed in Sections 2.5, 3.1 and 4.1.

2.5. Meromorphic functions with derivatives omitting zero. The following result was proved by W. K. Hayman [55, Thm. 5] for $k = 2$ and by J. Clunie [36] for $k \geq 3$.

2.5.1. Hayman-Clunie Theorem. *Let* f *be entire and let* $k \geq 2$. *Suppose that* f *and* $f^{(k)}$ *have no zeros. Then* f *has the form* $f(z) = e^{az+b}$ *where* $a, b \in \mathbb{C}$, $a \neq 0$.

Hayman obtained the case $k = 2$ as a corollary of his Theorem 2.1.4. In fact, if the functions f and f'' have no zeros, then $F := f/f'$ satisfies $F(z) \neq 0$ and $F'(z) - 1 = -f(z)f''(z)/f'(z)^2 \neq 0$ for all $z \in \mathbb{C}$. Thus F is constant by Theorem 2.1.4, and this implies that f has the form stated.

So we see that the conclusion of the above theorem is equivalent to the statement that f'/f is constant. Bloch's Principle thus suggests the following normal families analogue proved by W. Schwick [103, Thm. 5.1].

2.5.2. Schwick's Theorem. *Let* $k \geq 2$ *and let* $\mathcal{F}$ *be a family of functions holomorphic in a domain* D. *Suppose that* f *and* $f^{(k)}$ *have no zeros in* D, *for all* $f \in \mathcal{F}$. *Then* $\{f'/f : f \in \mathcal{F}\}$ *is normal.*

Theorem 2.5.1 was extended to meromorphic functions by G. Frank [47] for $k \geq 3$ and by J. K. Langley [61] for $k = 2$.

2.5.3. Frank-Langley Theorem. *Let f be meromorphic in $\mathbb{C}$ and let $k \geq 2$. Suppose that f and $f^{(k)}$ have no zeros. Then f has the form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-n}$, where $a, b \in \mathbb{C}$, $a \neq 0$, and $n \in \mathbb{N}$.*

We note that if f has the form $f(z) = (az+b)^{-n}$, then $f'(z)/f(z) = -n/(z+b/a)$ is not constant. So the original Bloch Principle does not suggest that there is a normal family analogue. However, the family of all functions f'/f of the above form is normal, and thus the following extension of Schwick's Theorem proved in [14] is in accordance with the variant of Bloch's Principle discussed in Section 2.4.

2.5.4. Theorem. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic in a domain D. Suppose that f and $f^{(k)}$ have no zeros in D, for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is normal.*

For $k = 2$ the result had been obtained already earlier in [11]. We omit the proofs of the above results and refer to the papers mentioned.

Instead of considering the condition that $f^{(k)}$ has no zeros one may, more generally, consider the condition that

$$L(f) := f^{(k)} + a_{k-1}f^{(k-1)} + \dots a_1 f' + a_0 f$$

has no zeros, for certain constants or functions $a_0, a_1, \dots, a_{k-1}$. For functions meromorphic in the plane this has been addressed in [22, 61, 62, 48, 107], and results about normality appear in [35].

Similarly one may replace the exceptional values of $f^{(k)}$ by exceptional values of $L(f)$ in many of the results discussed in Sections 2.1–2.3. In fact, already in 1940 it was proved by C. T. Chuang [33] that in Miranda's Theorem 2.1.3 one may replace the condition $f^{(k)} \neq 1$ by $L(f) \neq 1$ if the a_j are holomorphic. We note that this result can be deduced from the Pólya-Saxer-Ullrich Theorem using the Zalcman-Pang Lemma in the same way Miranda's Theorem was proved in Section 2.2. There are a large number of papers concerning exceptional values of $L(f)$. Here we only refer to [39, 102].

3. Fixed points and periodic points

3.1. Introduction. Let X, Y be sets, let $f \colon X \to Y$ be a function, and define the iterates $f^n \colon X_n \to Y$ by $X_1 := X, f^1 := f$ and $X_n := f^{-1}(X_{n-1} \cap Y)$, $f^n := f^{n-1} \circ f$ for $n \in \mathbb{N}$, $n \geq 2$. Note that $X_2 = f^{-1}(X_1 \cap Y) \subset X = X_1$ and thus $X_{n+1} \subset X_n \subset X$ for all $n \in \mathbb{N}$.

A point $\xi \in X$ is called a *periodic point of period p* of f if $\xi \in X_p$ and $f^p(\xi) = \xi$, but $f^m(\xi) \neq \xi$ for $1 \leq m \leq p-1$. A periodic point of period 1 is called a *fixed point*. The periodic points of period p are thus the fixed points of f^p which are not fixed points of f^m for any m less than p. The periodic points play an important role in complex dynamics.

By the Fundamental Theorem of Algebra, every non-constant polynomial f which is not of the form $f(z) = z + c$, with $c \in \mathbb{C} \setminus \{0\}$, has a fixed point, and so does every iterate of f. Transcendental entire functions need not have fixed points, as shown by the example $f(z) = z + e^z$. P. Fatou [45, p. 345] proved that the second iterate of a transcendental entire function has a fixed point, and this result was crucial for his proof that the Julia set of such a function is non-empty. P. C. Rosenbloom [96] proved that in fact any iterate of a transcendental entire function has infinitely fixed points. Summarizing these results we obtain the following theorem.

3.1.1. Fatou-Rosenbloom Theorem. *Let f be a entire function and $p \in \mathbb{N}$, $p \geq 2$. If f^p has no fixed point, then f has the form $f(z) = z+c$ where $c \in \mathbb{C}\setminus\{0\}$.*

We note that the family of all functions f of the form $f(z) = z + c$, with $c \in \mathbb{C} \setminus \{0\}$, is normal in $\mathbb{C}$. Thus the variant of Bloch's Principle discussed in Section 2.4 suggests a normal family analogue not incorporated in the original Bloch Principle. This normal family analogue was proved by M. Essén and S. Wu [43], thereby answering a question of L. Yang [118, Prob. 8].

3.1.2. Essén-Wu Theorem. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be the family of all holomorphic functions $f\colon D \to \mathbb{C}$ for which there exists $p = p(f) > 1$ such that f^p has no fixed point. Then $\mathcal{F}$ is normal.*

We shall sketch the proof of the Essén-Wu Theorem after Theorem 3.2.4.

3.2. Periodic points and quasinormality. In Section 3.1 we discussed the property P defined by $\langle f, D\rangle \in P$ if f is holomorphic in D and if there exists $p = p(f) > 1$ such that f^p has no fixed point in D. The Essén-Wu Theorem 3.1.2 says that P implies normality.

We shall now be concerned with the weaker property Q defined by $\langle f, D\rangle \in Q$ if f is holomorphic in D and if there exists $p = p(f) > 1$ such that f has no periodic point of period p in D. We note that for $n \in \mathbb{N}$ the function $f_n(z) := nz$ has no periodic points of period greater than 1 so that $\langle f_n, \mathbb{C}\rangle \in Q$ for all n, but the f_n do not form a normal family.

We first discuss what entire functions have property Q. For polynomials we have the following result due to I. N. Baker [3].

3.2.1. Baker's Theorem. *Let f be a polynomial of degree $d \geq 2$ and let $p \in \mathbb{N}$, $p \geq 2$. Suppose that f has no periodic point of period p. Then $d = p = 2$. Moreover, there exists a linear transformation L such that $f(z) = L^{-1}(g(L(z)))$, with $g(z) = -z + z^2$.*

Note that for $g(z) = -z+z^2$ we have $g(z)-z = z(z-2)$ and $g^2(z)-z = z^3(z-2)$ so that there are no periodic points of period 2.

The case of transcendental functions is covered by the following generalization of the Fatou-Rosenbloom Theorem 3.1.1 which was conjectured in [57, Prob. 2.20] and proved in [6, Thm. 1] and [7, §1.6, Satz 2].

3.2.2. Theorem. *Let f be a transcendental entire function and let $p \in \mathbb{N}$, $p \geq 2$. Then f has infinitely many periodic points of period p.*

These results may be summarized as follows.

3.2.3. Theorem. *Let $\langle f, \mathbb{C}\rangle \in Q$. Then the function f is a polynomial of degree at most 2. If f has degree 2, then there exists a linear transformation L such that $f(z) = L^{-1}(g(L(z)))$, with $g(z) = -z + z^2$.*

As mentioned above, property Q does not imply normality. However, we have the following result [4].

3.2.4. Theorem. *For every domain $D \subset \mathbb{C}$ the family $\{f : \langle f, D\rangle \in Q\}$ is quasinormal of order 1 in D.*

The proofs of Theorem 3.2.4 and the Essén-Wu Theorem 3.1.2 are based on similar arguments.

Sketch of Proof. For simplicity we only prove that $\mathcal{F}$ is normal if f^2 has no fixed point for all $f \in \mathcal{F}$, and that $\mathcal{F}$ is quasinormal of order 1 if f has no periodic point of period 2 for all $f \in \mathcal{F}$. The general case is proved along the same lines; we refer to the papers cited for the details.

First we prove that a family $\mathcal{F}$ of functions holomorphic in a domain D is quasinormal of order 3 if f has no periodic point of period 2 for all $f \in \mathcal{F}$. Suppose that $\mathcal{F}$ is not quasinormal of order 3. Then there exists a sequence (f_n) in $\mathcal{F}$ and four points $a_1, a_2, a_3, a_4 \in D$ such that no subsequence of (f_n) is normal at any of the points a_j.

Applying Ahlfors's Theorem 1.7.5 with a domain D_3 containing ∞ we see that if D_1, D_2 are Jordan domains in $\mathbb{C}$ with disjoint closures, if Ω is a neighborhood of one of the points a_j, and if n is sufficiently large, then f_n has an island U contained in Ω over one of the domains D_1 or D_2. We choose $\varepsilon > 0$ such that the closures of the disks of radius ε around the a_j are pairwise disjoint. We see that if n is sufficiently large and $j, k_1, k_2 \in \{1, 2, 3, 4\}$ with $k_1 \neq k_2$, then f_n has an island U in $D(a_j, \varepsilon)$ over $D(a_{k_1}, \varepsilon)$ or $D(a_{k_2}, \varepsilon)$. Thus f_n has an island in $D(a_j, \varepsilon)$ over $D(a_k, \varepsilon)$ for at least three values of k. This implies that there exists $j, k \in \{1, 2, 3, 4\}$, $j \neq k$, such that f_n has an island U in $D(a_j, \varepsilon)$ over $D(a_k, \varepsilon)$ and an island V in $D(a_k, \varepsilon)$ over $D(a_j, \varepsilon)$. We now consider a component W of $U \cap f_n^{-1}(V)$ and see that $f_n^2|_W \colon W \to D(a_j, \varepsilon)$ is a proper map. In particular, f_n^2 takes the value a_j in W.

For $z \in \partial W$ we have

$$|(f_n^2(z) - a_j) - \big(f_n^2(z) - z\big)| = |z - a_j| < \varepsilon = |f_n^2(z) - a_j|.$$

Rouché's Theorem implies that $f_n^2(z) - z$ has a zero in W, say $f_n^2(\xi) = \xi$ where $\xi \in W$. Since $f_n(\xi) \in D(a_k, \varepsilon)$ and $W \cap D(a_k, \varepsilon) = \emptyset$ we see that ξ is a periodic point of period 2. This contradicts our assumption. Thus $\mathcal{F}$ is quasinormal of order 3 if f has no periodic point of period 2 for all $f \in \mathcal{F}$ — and thus in particular if f^2 has no fixed point for all $f \in \mathcal{F}$.

To complete the proof that $\mathcal{F}$ is normal if f^2 has no fixed point for all $f \in \mathcal{F}$, suppose that $\mathcal{F}$ is not normal. Then there exists a sequence (f_n) in $\mathcal{F}$ and a point $a_1 \in D$ such that no subsequence of (f_n) is normal at a_1. Since $\mathcal{F}$ is quasinormal of order 3 we may, passing to a subsequence if necessary, assume that f_n converges in $D \setminus \{a_1, a_2, a_3\}$ where $a_2, a_3 \in D$. The Maximum Principle implies that $f_n \to \infty$ in $D \setminus \{a_1, a_2, a_3\}$. We find that if $\varepsilon > 0$ is such that the closure of the disk $D(a_1, \varepsilon)$ is contained in $D \setminus \{a_2, a_3\}$ and if n is large enough, then f_n has an island U in $D(a_1, \varepsilon)$ over $D(a_1, \varepsilon)$. As above Rouché's Theorem implies that f_n has a fixed point in U. This fixed point is also a fixed point of f_n^2, contradicting the assumption.

The proof that $\mathcal{F}$ is quasinormal of order 1 if f has no periodic point of period 2 for all $f \in \mathcal{F}$ is completed in a similar fashion. Assuming that this is not the case we find a sequence (f_n) in $\mathcal{F}$ and two points $a_1, a_2 \in D$ such that no subsequence of (f_n) is normal at a_1 or a_2. Passing to a subsequence we may again assume that f_n converges in $D \setminus \{a_1, a_2, a_3\}$ for some $a_3 \in D$, and hence $f_n \to \infty$ in $D \setminus \{a_1, a_2, a_3\}$ by the Maximum Principle. For suitable $\varepsilon > 0$ and sufficiently large n we find that f_n has an island U in $D(a_1, \varepsilon)$ over $D(a_2, \varepsilon)$ and an island V in $D(a_2, \varepsilon)$ over $D(a_1, \varepsilon)$. Again we consider a component W of $U \cap f_n^{-1}(V)$ and see that $f_n^2|_W \colon W \to D(a_1, \varepsilon)$ is a proper map. As above, we see that W contains a periodic point ξ of period 2 of f_n. ■

A fixed point ξ of a holomorphic function f is called *repelling* if $|f'(\xi)| > 1$. Repelling periodic points are defined accordingly. They play an important role in complex dynamics. Many of the results mentioned above have generalizations where instead of fixed points and periodic points only repelling fixed points and periodic points are considered. For example, the Theorems 3.1.2 and 3.2.2 hold literally with the word "repelling" added. But the results about polynomials are somewhat different; see [6, 7, 12, 44] for more details.

The condition that f has no (repelling) periodic points of some period — or that some iterate does not have (repelling) fixed points — has also been considered for meromorphic functions. We refer to [29, 43, 105, 106, 115] for results when this implies (quasi)normality, to [3, 60] for results concerning rational functions, and to [8, §3] for the case of transcendental functions meromorphic in the plane.

4. Further topics

4.1. Functions sharing values. Two meromorphic functions f and g are said to share a value $a \in \widehat{\mathbb{C}}$ if they have the same a-points; that is, $f(z) = a$ if $g(z) = a$

and vice versa. A famous result of Nevanlinna [78] says that if two functions meromorphic in the plane share five values, then they are equal. E. Mues and N. Steinmetz [75] proved that if f is meromorphic in the plane and if f and f' share three values, then $f' = f$ so that $f(z) = ce^z$ for some $c \in \mathbb{C}$. Now the family $\{ce^z : c \in \mathbb{C}\}$ is normal and thus the variant of Bloch's Principle discussed in Section 2.4 suggests that the family of functions f meromorphic in a domain and sharing three fixed values with their derivative is normal. W. Schwick [104] proved that this is in fact the case.

On the other hand, G. Frank and W. Schwick [49, 50] showed that for a function f meromorphic in the plane the condition that f and $f^{(k)}$ share three values for some $k \geq 2$ still implies that $f = f^{(k)}$, but the family of all functions f which are meromorphic in some domain and satisfy this condition is not normal. This is in accordance with both the original Bloch Principle and its variant introduced in Section 2.4, since the functions f which are meromorphic in the plane and satisfy $f = f^{(k)}$ do not form a normal family for $k \geq 2$.

There is an enormous amount of literature on functions meromorphic in the plane that share values, and in recent years many papers on corresponding normality results have appeared. Here we only refer to [51, 85, 88, 89] and the literature cited there. We note that some of these results generalize the results about exceptional values of derivatives described in Section 2.1, since if two functions omit the same value, then they of course also share this value.

4.2. Gap series. A classical result of L. Fejér [46, p. 412] says that an entire function f of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \qquad \text{where } \sum_{k=0}^{\infty} \frac{1}{n_k} = \infty \tag{4.2.1}$$

has at least one zero. S. Ruscheweyh and K.-J. Wirths [99] have shown that the family of all functions f of the form (4.2.1) which are holomorphic in the unit disk and do not vanish there form a normal family.

There are a number of further results, as well as open questions, on exceptional values of entire functions with gap series; see [76] for a survey. Here we only mention a question of G. Pólya [90, p. 639] whether Fejér's condition $\sum_{k=0}^{\infty} 1/n_k = \infty$ in (4.2.1) can be replaced by Fabry's condition $\lim_{k\to\infty} n_k/k = \infty$.

In accordance with Bloch's Principle, Ruscheweyh and Wirths [99] have made the following conjecture.

4.2.2. Conjecture. For $\Lambda \subset \mathbb{N} \cup \{0\}$ and a function f holomorphic in a domain D containing 0 define $\langle f, D\rangle \in P_\Lambda$ if f has a power series expansion

$$f(z) = \sum_{\lambda\in\Lambda} a_\lambda z^\lambda$$

and if $f(z) \neq 0$ for all $z \in D$. Then $\{f : \langle f, \mathbb{D}\rangle \in P_\Lambda\}$ is normal in $\mathbb{D}$ if and only if $\langle f, \mathbb{C}\rangle \in P_\Lambda$ implies that f is constant.

We note that one direction in this conjecture is obvious: Normality of the family $\{f : \langle f, \mathbb{D}\rangle \in P_\Lambda\}$ implies that $\langle f, \mathbb{C}\rangle \in P_\Lambda$ only for constant functions f. In fact, if a non-constant entire function f has property P_Λ, then so does every function in the family $\{f(nz) : n \in \mathbb{N}\}$, and this family is not normal at 0.

We mention that Zalcman's Principle 1.5.1 cannot apply to properties concerning gap series since condition (ii) is not satisfied. However, there are some further results in addition to [99] which support the above conjecture. W. K. Hayman [58] has considered entire functions with gaps in arithmetic progressions. Normal family analogues of some of the results have been obtained by S. Ruscheweyh and L. Salinas [98] and by J. Grahl [52].

4.3. Holomorphic curves. It is easily seen that Picard's Theorem is equivalent to the statement that if f_1, f_2, f_3 are non-vanishing entire functions and if c_1, c_2, c_3 are non-zero complex numbers such that $\sum_{j=1}^{3} c_j f_j = 0$, then each quotient f_j/f_k is constant. In fact, writing $F := -c_1 f_1/(c_3 f_3)$ we see that F is an entire function without zeros. Moreover, since $F = 1 + c_2 f_2/(c_3 f_3)$ we see that F also omits the value 1. Thus F is constant by Picard's Theorem. This implies that not only f_1/f_3 but also the other quotients f_1/f_2 and f_2/f_3 are constant.

Another way to phrase this result is that the hypothesis that the three functions f_1, f_2, f_3 are linearly dependent already implies that two of them are linearly dependent.

A generalization of this statement was proved by É. Borel [21].

4.3.1. Borel's Theorem. *Let $p \in \mathbb{N}$, let $f_1, \dots, f_p$ be entire functions without zeros and let $c_1, \dots, c_p \in \mathbb{C} \setminus \{0\}$. Suppose that $\sum_{j=1}^{p} c_j f_j = 0$. Then $\{f_1, \dots, f_p\}$ contains a linearly dependent subset of less than p elements.*

Repeated application of this result shows that under the hypotheses of Borel's Theorem the set $\{1, \dots, p\}$ can be written as the union of disjoint subsets I_μ, each of which has at least two elements, and such that if j, k are in the same set I_μ, then f_j/f_k is constant.

One may ask whether this Picard type theorem also has an analogue in the context of normal families. However, as noted already by Bloch [18, p. 311] himself, it is not clear at first sight what such an analogue could look like. This problem was then addressed by H. Cartan [27] who proved such an analogue in the case that $p = 4$ and made a conjecture for the general case. Cartan's conjecture was disproved by A. Eremenko [40]. However, Eremenko [41] also showed that a weakened form of the conjecture is true for $p = 5$. For a connection to gap series we refer to a paper by J. Grahl [52].

4.4. Quasiregular maps. Many of the concepts used in this survey are also applicable for quasiregular maps in higher dimensions; see [93] for the definition and basic properties of quasiregular maps. One of the most important results in this theory is the analogue of Picard's Theorem which was obtained by S. Rickman [92]. He proved that there exists $q = q(d, K) \in \mathbb{N}$ with the property that every K-quasiregular map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ which omits q points is constant. The corresponding normality result was proved by R. Miniowitz [70], using an extension of the Zalcman Lemma to quasiregular maps. A. Eremenko [42] has used Miniowitz's result to extend the classical Covering Theorem of Bloch [19] to quasiregular maps.

Recall that the limit function f occurring in Zalcman's Lemma has bounded spherical derivative. The corresponding conclusion in the context of quasiregular maps is that the limit function is uniformly continuous. This plays an important role in the work of M. Bonk and J. Heinonen [20] on closed, connected and oriented Riemannian d-manifolds N for which there exists a non-constant quasiregular map $f\colon \mathbb{R}^d \to N$. They use Miniowitz's extension of Zalcman's Lemma to show that if there is such a mapping, then there is also a uniformly continuous one.

Many of the results of Sections 3.1 and 3.2 concerning fixed points have been extended to quasiregular mappings by H. Siebert [105, 106]. For example, Theorems 3.1.2, 3.2.2 and 3.2.4 hold literally for quasiregular maps. We omit a detailed discussion of the results about quasiregular mappings here and refer to the papers cited.

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