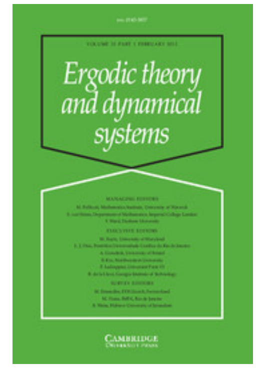

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Simple proofs of some fundamental properties of the Julia set

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Abstract. Let f be a holomorphic self-map of $\mathbb{C}\backslash\{0\}$, $\mathbb{C}$, or the extended complex plane $\overline{\mathbb{C}}$ that is neither injective nor constant. This paper gives new and elementary proofs of the well-known fact that the Julia set of f is a non-empty perfect set and coincides with the closure of the set of repelling cycles of f. The proofs use Montel–Caratheodory's theorem but do not use results from Nevanlinna theory.

1. *Introduction*

Let D be either $\mathbb{C}\backslash\{0\}, \mathbb{C}$, or the extended complex plane $\overline{\mathbb{C}}$. Let $f : D \to D$ be a holomorphic function that is neither injective nor constant. Denote the Julia set of f, i.e. the set of all $z \in D$ at which the iterates $\{f^n : n \in \mathbb{N}\}$ of f do not form a normal family, by $\mathcal{J}$. Recall that a periodic point $p \in D$ of f with period $n \in \mathbb{N}$ is called repelling if the eigenvalue $(f^n)'(p)$ of f^n at p has absolute value bigger than one.

The aim of this paper is to give new and elementary proofs of the following fundamental properties of the Julia set.

THEOREM 1. *$\mathcal{J}$ is a perfect set, i.e. $\mathcal{J}$ is a closed and non-empty subset of D and does not contain isolated points.*

THEOREM 2. *$\mathcal{J}$ is the closure of the set of all repelling periodic points of f.*

The first result is easy to prove for the case when f is rational (due to Fatou [**5**] and Julia [**7**], see also [**13**, p. 28], and [**2**, p. 159], for a different proof) or when f is a transcendental holomorphic self-map of the punctured plane (due to Radstrœm [**8**]). For the case when f is a transcendental entire function, Theorem 1 is also due to Fatou [**6**] but is more difficult to prove. Fatou himself used estimates on the growth of composite functions to show that the Julia set of a transcendental entire function is a perfect set. More recent proofs of this statement are often based on Nevanlinna theory (see for instance [**2**]).

We shall give an elementary proof of Theorem 1 which does not use results from Nevanlinna theory and which is also different from Fatou's proof. The proof is based on a theorem of Bohr [**4**], which is an easy application of Montel–Caratheodory's theorem.

Theorem 2 is due to Fatou [5] and Julia [7] for the case when $D = \overline{\mathbb{C}}$. They gave different proofs, neither of which applies to the case when f is a transcendental entire function. In this case the theorem was first proved by Baker [1] using a method based on Ahlfors's theory of covering surfaces. For the case when $D = \mathbb{C}\backslash\{0\}$, Theorem 2 was first proved by Bhattacharyya [3]. Recently Schwick [12] has given a new proof of Theorem 2 which applies to all cases and is much more elementary. He makes use of a normality criterion provided by Zalcman [14]. Moreover, his proof is based on the fact that a transcendental meromorphic function g has at most four perfectly branched values, i.e. at most four values $a \in \overline{\mathbb{C}}$ for which only finitely many a-points of g are simple. This is a consequence of Nevanlinna's second fundamental theorem.

In our proof of Theorem 2, we shall modify Schwick's method so that no results from Nevanlinna theory are necessary. Instead of this we shall only use Picard's theorem and the fact that the recurrent but not periodic points are dense in $\mathcal{J}$. The latter is an easy consequence of Baire's theorem and Theorem 1. More precisely, one does not need the whole statement of Theorem 1 but only the fact that $\mathcal{J}$ does not contain isolated points. As we shall see in §2, this is easier to prove than the fact that $\mathcal{J} \neq \emptyset$. On the other hand Theorem 2 would be useless if $\mathcal{J} = \emptyset$.

2. *Proof of Theorem 1*

Since there are already simple proofs for the case when $D = \overline{\mathbb{C}}$ or $D = \mathbb{C}\backslash\{0\}$, we confine ourselves to the case when $D = \mathbb{C}$ and f is transcendental.

It follows immediately from the definition of $\mathcal{J}$ that $\mathcal{J}$ is closed in D. The first non-trivial part of the proof is to show the following.

Step 1. $\mathcal{J}$ does not contain isolated points.

Proof. We may assume that $\mathcal{J} \neq \emptyset$. Assume that $\mathcal{J}$ is finite. Since $f^{-1}(x) \subset \mathcal{J}$, for each $x \in \mathcal{J}$, it follows from Picard's theorem that $\mathcal{J}$ consists of one single point. By conjugating if necessary, we may assume that $\mathcal{J} = \{0\}$. Now $f|_{\mathbb{C}\backslash\{0\}}$ is a transcendental holomorphic self-map of the punctured plane and thus, has a non-empty Julia set. This implies that $\mathcal{J} \setminus \{0\} \neq \emptyset$, a contradiction.

Hence, $\mathcal{J}$ is an infinite set. Once this is known it is easy to show that $\mathcal{J}$ does not contain isolated points (see for instance [2, p. 159]). □

The essential part of the proof is to show the following.

Step 2. $\mathcal{J} \neq \emptyset$.

We shall make use of two lemmata, which will be proved first.

LEMMA 1. *Let g be an entire function such that $g \neq \mathrm{id}_{\mathbb{C}}$ and $(g^2 - \mathrm{id}_{\mathbb{C}})/(g - \mathrm{id}_{\mathbb{C}})$ is constant. Then g is a polynomial of degree ≤ 1.*

Lemma 1 was stated by Fatou [6] in a similar form. The proof given here is due to Rosenbloom [9].

Proof. We may assume that g is not constant. Let $c \in \mathbb{C}$ be such that

$$(g^2 - \mathrm{id}_{\mathbb{C}}) = c(g - \mathrm{id}_{\mathbb{C}}).$$

If $c = 0$ then $g^2 = \mathrm{id}_{\mathbb{C}}$, which implies that g is injective and hence, g is a polynomial of degree ≤ 1. If $c = 1$ then $g^2 = g$, which implies that g is constant or $g = \mathrm{id}_{\mathbb{C}}$. Thus, we may assume that $c \notin \{0, 1\}$. By differentiation we obtain

$$g' \circ g g' - 1 = c(g' - 1)$$

and hence

$$g'(g' \circ g - c) = 1 - c.$$

Since $c \neq 1$, we conclude that g' omits zero and $g' \circ g$ omits $c \neq 0$. Since g is not constant, it follows from Picard's theorem that g' is constant. Hence, g is a polynomial of degree 1. □

LEMMA 2. *Let g be an entire function such that $g(0) = 0$, $|g'(0)| < 1$ and $\lim_{n\to\infty} g^n = 0$ locally uniformly in $\mathbb{C}$. Then $g(z) = g'(0)z$, for each $z \in D$.*

Lemma 2 is based on a theorem of Bohr [**4**], which is an easy application of Montel–Caratheodory's theorem. To state Bohr's theorem, we need a some more notation. For each $z \in D$ and $r > 0$, denote the open disk centered at z with radius r by $D(z; r)$ and its boundary by $\partial D(z; r)$. As usual, for an entire function h and $r \geq 0$, denote the supremum of the set $\{|h(z)| : z \in \partial D(0; r)\}$ by $M(r; h)$.

THEOREM 3. (Bohr) *Define $\mathcal{H}$ to be the set of all holomorphic functions $h : D(0; 1) \to \mathbb{C}$ which satisfy $h(0) = 0$ and $M(\frac{1}{2}; h) \geq 1$. For each $h \in \mathcal{H}$, let $c(h) := \sup\{r > 0 : \partial D(0; r) \subset h(D(0; 1))\}$. Then $\inf\{c(h) : h \in \mathcal{H}\} > 0$.*

Proof. Assume that $\inf\{c(h) : h \in \mathcal{H}\} = 0$. Then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} c(h_n) = 0$. Then, for all large $n \in \mathbb{N}$, the circles $\partial D(0; 1)$ and $\partial D(0; 2)$ are not contained in $h_n(D(0; 1))$, which by the extended Montel–Caratheodory theorem implies that $\{h_n : n \in \mathbb{N}\}$ forms a normal family. By passing over to a subsequence if necessary, we may assume that $(h_n)_{n\in\mathbb{N}}$ converges to a holomorphic function $h : D(0; 1) \to \mathbb{C}$. By Hurwitz's theorem we conclude that $h \in \mathcal{H}$. Hence, h is not constant and there exists $r > 0$ such that $\partial D(0; r) \subset h(D(0; \frac{1}{2}))$. Applying Hurwitz's theorem again we see that, for all large $n \in \mathbb{N}$, $c(h_n) \geq r$ and hence we obtain a contradiction. □

Proof of Lemma 2. We may assume that g is not constant. Since $|g'(0)| < 1$, there exists an open disk $K \subset \mathbb{C}$ centered at zero such that $g(K) \subset K$. For each $n \in \mathbb{N}$, let

$$K_n := g^{-n}(K) \quad \text{and} \quad r_n := \sup\{r > 0 : \partial D(0; r) \subset K_n\}.$$

Since $K_n \subset K_{n+1}$, for each $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{C}$, we conclude that $(r_n)_{n\in\mathbb{N}}$ is increasing and $\lim_{n\to\infty} r_n = \infty$. Now, for each $n \in \mathbb{N}$ and $x \in D(0; r_n)$, there exists $s \in (|x|, r_n)$ such that $\partial D(0; s) \subset K_n$. This implies that $f^n(\partial D(0; s)) \subset K$, and from the maximum principle we conclude that $f^n(x) \in K$ and hence, $x \in K_n$. Thus, we have proved that, for each $n \in \mathbb{N}$,

$$r_n = \max\{s > 0 : D(0; s) \subset K_n\}.$$

Now, for each $n \in \mathbb{N}$, let $s_n := r_n/2$ and

$$h_n : D(0; 1) \to \mathbb{C}, \quad z \mapsto M(s_n; g)^{-1} g(2 s_n z).$$

By Bohr's theorem there exists $c > 0$ such that, for each $n \in \mathbb{N}$, there exists $t_n \geq c$ satisfying $\partial D(0; t_n) \subset h_n(D(0; 1))$. Hence, for each $n \in \mathbb{N}$,

$$\partial D(0; t_n M(s_n, g)) \subset g(D(0; r_n)) \subset g(K_n) \subset K_{n-1} \subset K_n,$$

which by definition of r_n implies that

$$c M(s_n; g) \leq t_n M(s_n; g) \leq r_n = 2 s_n.$$

Hence, $(M(s_n; g) s_n^{-1})_{n \in \mathbb{N}}$ is a bounded sequence. This implies that the meromorphic function $\phi := (g - g'(0)\,\mathrm{id}_{\mathbb{C}})/\,\mathrm{id}_{\mathbb{C}}$ satisfies $\lim_{n\to\infty} M(s_n; \phi) = 0$. Since zero is a removable singularity of ϕ, it follows from the maximum principle and Liouville's theorem that $\phi = 0$. □

Proof of Step 2. Since f is transcendental, Lemma 1 yields that $g := (f^2 - \mathrm{id}_{\mathbb{C}})/(f - \mathrm{id}_{\mathbb{C}})$ is a non-constant meromorphic function. By Picard's theorem we conclude that $g^{-1}(\{0, 1, \infty\}) \neq \emptyset$. This implies that f has a periodic point $p \in \mathbb{C}$. We may assume that $p = 0$ and $f(0) = 0$. If $0 \in \mathcal{J}$ then we have finished. Thus, we may assume that zero belongs to the Fatou set of f, which especially implies that $|f'(0)| \leq 1$. Let E be the component of the Fatou set of f which contains zero.

If $|f'(0)| = 1$ then by Weierstrass' theorem each limit function of the sequence $(f^n|_E)_{n \in \mathbb{N}}$ is not constant. From this, one easily concludes that $f|_E$ is injective (see, for example, [**2**, p. 160]). Since f is transcendental, we conclude that $E \neq \mathbb{C}$. Hence, $\mathcal{J} \neq \emptyset$.

If $|f'(0)| < 1$ then it follows from Lemma 2 that $E \neq \mathbb{C}$ and hence $\mathcal{J} \neq \emptyset$. □

3. *Proof of Theorem 2*

Let $p \in D$ be a repelling periodic point of f with period $n \in \mathbb{N}$. The chain rule yields that $\lim_{k\to\infty} |(f^{kn})'(p)| = \infty$, which by Weierstrass' theorem implies that no subsequence of $(f^{kn})_{k \in \mathbb{N}}$ is uniformly convergent in a neighbourhood of p. Hence, p belongs to the Julia set of f^n, which coincides with the Julia set of f. Since $\mathcal{J}$ is closed in D, we see that the closure (in D) of the set of repelling periodic points of f is also contained in $\mathcal{J}$.

Verification of the other inclusion takes place in two steps. We define $\mathcal{M}$ to be the set of recurrent but not periodic points of $\mathcal{J}$, i.e. the set of all $z \in \mathcal{J}$ which belong to the closure of the set $\{f^n(z) : n \in \mathbb{N}\} \setminus \{z\}$. The first step is to prove the following.

Step 1. $\mathcal{M}$ is contained in the closure of the set of repelling periodic points of f.

To prove Step 1, we use Picard's little theorem and the following result, which is a local adaptation of Zalcman's lemma [**14**] and was also used by Schwick [**12**].

LEMMA 3. (Zalcman) *Let F be a family of meromorphic functions in a domain $U \subset \mathbb{C}$. Let $z_0 \in U$ be such that F is not normal at z_0. Then there exist sequences $(f_n)_{n\in\mathbb{N}}$ in F, $(z_n)_{n\in\mathbb{N}}$ in U and $(\rho_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ and a non-constant meromorphic function $g : \mathbb{C} \to \overline{\mathbb{C}}$ such that $\lim_{n\to\infty} z_n = z_0$, $\lim_{n\to\infty} \rho_n = 0$ and the sequence $(f_n \circ (\rho_n \operatorname{id}_{\mathbb{C}} + z_n))_{n\in\mathbb{N}}$ converges to g locally uniformly in $\mathbb{C}$.*

Proof. See [**11**]. □

Proof of Step 1. Let $z_0 \in \mathcal{M}$. By conjugating if necessary, we may assume that $z_0 \in \mathbb{C}$. Let $U \subset \mathbb{C}$ be an open neighbourhood of z_0. It follows from Zalcman's lemma that there exists a strictly increasing sequence α in $\mathbb{N}$, sequences $(z_n)_{n\in\mathbb{N}}$ in U and $(\rho_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ and a non-constant meromorphic function $g : \mathbb{C} \to \overline{\mathbb{C}}$ such that

$$\lim_{n\to\infty} z_n = z_0, \quad \lim_{n\to\infty} \rho_n = 0 \quad \text{and} \quad \lim_{n\to\infty} f^{\alpha(n)} \circ (\rho_n \operatorname{id}_{\mathbb{C}} + z_n) = g.$$

Since $z_0 \in \mathcal{M}$ and g is not constant, it follows from Picard's little theorem that there exists $j \in \mathbb{N}_0$ such that $f^j(z_0) \in U \cap g(\mathbb{C})$. Let $x \in g^{-1}(f^j(z_0))$. Then there exists an open neighbourhood V of x in $\mathbb{C}$ such that $g(V) \subset U$ and $g'(v) \neq 0$ for each $v \in V \setminus \{x\}$. Since $z_0 \in \mathcal{M}$, we have that $f^j(z_0) \in \mathcal{M}$ too. Hence, there exist $i \in \mathbb{N}$ and $v \in V \setminus \{x\}$ such that $g(v) = f^i(z_0)$. Then v is an isolated zero of the function

$$h := g - f^i(z_0) = \lim_{n\to\infty} (f^{\alpha(n)} \circ (\rho_n \operatorname{id}_{\mathbb{C}} + z_n) - f^i \circ (\rho_n \operatorname{id}_{\mathbb{C}} + z_n)).$$

Thus, Hurwitz's theorem implies that there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\lim_{n\to\infty} v_n = v$ and

$$f^{\alpha(n)}(\rho_n v_n + z_n) - f^i(\rho_n v_n + z_n) = 0$$

for all large $n \in \mathbb{N}$. Thus, $p_n := f^i(\rho_n v_n + z_n)$ is a fixed point of $f^{\alpha(n)-i}$ for all large $n \in \mathbb{N}$. Using Weierstrass' theorem and the chain rule we obtain that

$$g'(v) = \lim_{n\to\infty} (f^{\alpha(n)} \circ (\rho_n \operatorname{id}_{\mathbb{C}} + z_n))'(v_n) = \lim_{n\to\infty} ((f^{\alpha(n)-i})'(p_n)(f^i)'(\rho_n v_n + z_n)\rho_n).$$

Since $v \in V \setminus \{x\}$, we have that $g'(v) \neq 0$. Since we also know that

$$\lim_{n\to\infty} (f^i)'(\rho_n v_n + z_n)\rho_n = (f^i)'(z_0) \lim_{n\to\infty} \rho_n = 0,$$

we conclude that $\lim_{n\to\infty} |(f^{\alpha(n)-i})'(p_n)| = \infty$, so that all but at most finitely many periodic points p_n, $n \in \mathbb{N}$, are repelling. Finally, since

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} f^i(\rho_n v_n + z_n) = f^i(z_0) \in U,$$

we see that U contains a repelling periodic point of f. □

Our desired result follows immediately by combining Step 1 with the following.

Step 2. $\mathcal{M}$ is dense in $\mathcal{J}$.

Since $\mathcal{J}$ does not contain isolated points, this is an easy consequence of the following theorem which in Steinmetz's book [**13**, p. 38] is shown to be true for the case when f is a rational function.

THEOREM 4. *$\{z \in \mathcal{J} : \{f^n(z) : n \in \mathbb{N}\}$ is dense in $\mathcal{J}\}$ is dense in $\mathcal{J}$.*

Steinmetz's proof of the rational case applies to the general case without further difficulties. For the sake of completeness we shall give the proof. The main tool is the following well-known theorem.

THEOREM 5. (Baire) *Let (X, d) be a complete metric space. Let $\mathcal{Q}$ be a countable set of open and dense subsets of X. Then $\bigcap\{Q; Q \in \mathcal{Q}\}$ is also dense in X.*

Proof. See for instance [**10**]. □

Proof of Theorem 4. We may assume that $\mathcal{J} \neq \emptyset$, for otherwise there is nothing to prove. Let d be the chordal metric for the case when $D = \overline{\mathbb{C}}$, the euclidian metric for the case when $D = \mathbb{C}$, and defined to be

$$\mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0\} \to \mathbb{R}, (z, w) \mapsto |z - w| + \left|\frac{1}{z} - \frac{1}{w}\right|$$

for the case when $D = \mathbb{C}\backslash\{0\}$. In either case $(\mathcal{J}, d_{\mathcal{J}\times\mathcal{J}})$ forms a separable complete metric space. Now, for each $n \in \mathbb{N}$, there is a sequence $(B_{n,k})_{k\in\mathbb{N}}$ of open d-balls with radius $1/n$ such that $\mathcal{J} \subset \bigcup_{k\in\mathbb{N}} B_{n,k}$ and $B_{n,k} \cap \mathcal{J} \neq \emptyset$, for each $k \in \mathbb{N}$. Since $\mathcal{J}$ does not contain isolated points, we conclude that, for each $n, k \in \mathbb{N}$, the set $B_{n,k} \cap \mathcal{J}$ is an infinite set, which by Montel–Caratheodory's theorem implies that

$$Q_{n,k} := \mathcal{J} \cap \bigcup_{j\in\mathbb{N}} f^{-j}(B_{n,k})$$

is open and dense in $\mathcal{J}$. By Baire's theorem we conclude that

$$Q := \bigcap_{n,k\in\mathbb{N}} Q_{n,k}$$

is also dense in $\mathcal{J}$. Now let $q \in Q$. Then $\{f^j(q) : j \in \mathbb{N}\} \cap B_{n,k} \neq \emptyset$, for each $n, k \in \mathbb{N}$. Hence, $\{f^j(q) : j \in \mathbb{N}\}$ is dense in $\mathcal{J}$. □

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