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AN ENTIRE FUNCTION WHICH HAS WANDERING DOMAINS

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Abstract

Let $f(z)$ denote a rational or entire function of the complex variable $z$ and $f_n(z)$, $n = 1, 2, \ldots$, the $n$-th iterate of $f$. Provided $f$ is not rational of order 0 or 1, the set $\mathcal{C}$ of those points where $\{f_n(z)\}$ forms a normal family is a proper subset of the plane and is invariant under the map $z \rightarrow f(z)$. A component $G$ of $\mathcal{C}$ is a wandering domain of $f$ if $f_k(G) \cap f_n(G) = \emptyset$ for all $k \geq 1$, $n \geq 1$, $k \neq n$. The paper contains the construction of a transcendental entire function which has wandering domains.

The theory of the iteration of a rational or entire function $f(z)$ of the complex variable $z$ deals with the sequence of natural iterates $f_n(z)$ defined by

$$f(z) = z, \quad f_1(z) = f(z), \quad f_{n+1}(z) = f(f_n(z)), \quad n = 0, 1, 2, \ldots$$

In the theory developed by Fatou (1919, 1926) and Julia (1918) an important part is played by the set $\hat{\mathcal{N}} = \hat{\mathcal{N}}(f)$ of those points of the complex plane where $\{f_n(z)\}$ is not a normal family. Unless $f(z)$ is a rational function of order 0 or 1, (which we henceforth exclude) the set $\hat{\mathcal{N}}(f)$ is a non-empty perfect set, whose complement $\mathcal{G} = \mathcal{G}(f)$ consists of an at most countably infinite collection of (open) components $G_i$, each of which is a maximal domain of normality of $\{f_n\}$.

It is shown by Fatou (1919, 1926) that $\hat{\mathcal{N}}(f)$ is completely invariant under the mapping $z \rightarrow f(z)$, i.e. if $\alpha$ belongs to $\hat{\mathcal{N}}(f)$ then so do $f(\alpha)$ and every solution $\beta$ of $f(\beta) = \alpha$. It follows that $\mathcal{G}(f)$ is also completely invariant and, in particular, for each component $G_i$ of $\mathcal{G}(f)$ there is just one component $G_i$ such that $f(G_i) \subset G_i$. By definition, the component $G_0$ of $\mathcal{G}(f)$ is a wandering domain of $f$ if

$$f_k(G_0) \cap f_n(G_0) = \emptyset \quad \text{for all} \quad 1 \leq k, n < \infty, k \neq n.$$ 

No examples of wandering domains for either entire or rational functions seem to be known and indeed Jacobson (1969) raises the question whether they can occur at all for rational $f$. Pelles also discusses the notion.

In Baker (1963) an entire function $g(z)$ was constructed as follows:
Let $C = (4e)^{-1}$ and $a > 4e$. Then define inductively

\[ y_{n+1} = C \gamma_n^2 \left( 1 + \frac{\gamma_n}{\gamma_1} \right) \left( 1 + \frac{\gamma_n}{\gamma_2} \right) \cdots \left( 1 + \frac{\gamma_n}{\gamma_n} \right), \quad n = 1, 2, \cdots. \]

Then $1 < y_1 < y_2 < \cdots$ and [c.f. Baker (1963): lemmas 1 and 2]

\[ g(z) = Cz^2 \prod_{n=1}^{\infty} \left( 1 + \frac{z}{\gamma_n} \right) \]

is an entire function which satisfies

\[ |g(e^{i\theta})| < \frac{1}{2}, \quad 0 \leq \theta \leq 2\pi, \]

\[ y_{n+1} < g(y_n) < 2y_{n+1}, \quad n = 1, 2, \cdots, \]

\[ g(y_n^{1/2}) < \gamma_{n+1}^{1/2}, \quad n = 1, 2, \cdots, \]

and

\[ g(\gamma_n^2) > \gamma_{n+1}^2, \quad n = 1, 2, \cdots. \]

Moreover, if $A_n$ denotes the annulus

\[ A_n : \gamma_n^2 < |z| < \gamma_{n+1}^{1/2}, \]

then by Baker (1963) Theorem 1, there is an integer $N$ such that for all $n > N$ the mapping $z \to g(z)$ maps $A_n$ into $A_{n+1}$, so that $g_k(z) \to \infty$ uniformly in $A_n$ as $k \to \infty$. Since by (3) $g_k(z) \to 0$ uniformly for $|z| \leq 1$, it is clear that each $A_n$, $n > N$, belongs to a multiply connected component $C_n$ of $G(g)$ and that $C_n$ does not meet $\{z : |z| = 1\}$, which belongs to a component of $G(g)$ which we designate $C_0$. It is natural to ask whether the $C_n$, $n > N$, are all different, but this question was left unanswered in Baker (1963). The solution is given by the

**Theorem.** For $n > N$ the components $C_n$ of $G(g)$ described above are all different and each is a wandering domain of $g$.

**Proof.** Suppose that there are two values of $n > N$ for which $A_n$ belong to the same component of $G(g)$. Suppose $n = m > N$ and $n = m + l$, $l > 0$, are such values. Then there is a path $\Gamma$ in $G(g)$ which joins a point of $A_m$ to a point of $A_{m+l}$. The path $\Gamma$ must meet $A_{m+1}$, which therefore belongs to the same component of $G(g)$ as $A_m$. So we may take $l = 1$. By the complete invariance of $G(g)$ the path $g_{\infty}(\Gamma)$ lies in $G(g)$ and it joins $A_{m+k}$ to $A_{m+k+1}$, $k = 1, 2, \cdots$. Thus all $A_n$, $n > m$, belong to the same component of $G(g)$, which is therefore multiply-connected and unbounded.

It suffices to show that for all sufficiently large $n$ the annuli $A_n$ and $A_{n+2}$ cannot be joined in $G(g)$. Now for all sufficiently large $n (> N_0)$ we have, since $\gamma_n \to \infty$ in (1) that,
(8) \[ 4\gamma_n^2 < \gamma_{n+1}^{1/2}. \]

Take any \( n > \text{Max}(N, N_0) \) and assume that \( A_n, A_{n+2} \) can be joined in \( \mathcal{C}(g) \). Then 
\[ z_1 = 2\gamma_n^2 \in A_n \quad \text{and} \quad z_2 = \frac{1}{2} \gamma_{n+3}^{1/2} \in A_{n+2}. \]
There is then a simple polygon joining \( z_1 \) and \( z_2 \) in \( \mathcal{C}(g) \) and so \( z_1, z_2 \) belong to a simply-connected subdomain, say \( H \), of \( \mathcal{C}(g) \). \( H \) may be mapped conformally by \( z = \psi(t) \) onto \( |t| < 1 \) so that \( \psi(0) = z_1 \) and \( \psi(u) = z_2 \) where \( u \) is some value for which \( |u| < 1 \).

Since \( g_k(z) \to \infty \) locally uniformly, as \( k \to \infty \) for \( z \in A_n \), the same is true locally uniformly in the component \( G \) of \( \mathcal{C}(g) \) to which \( A_n \) belongs. Thus for each integer \( p > 0 \), \( g_p(G) \) is a domain in which \( G_k(z) \to \infty \) locally uniformly, so \( g_p(G) \) does not meet the component \( G_0 \) of \( \mathcal{C}(g) \) which includes the disc \( \{z : |z| \leq 1\} \), as \( g_k(z) \to 0 \) in \( G_0 \). Thus in \( G \), and in particular in \( H \), \( g(z) \) omits the values 0, 1. Similarly the functions \( F_p(t) = g_p(\psi(t)) \) omit the values 0, 1 in \( |t| < 1 \).

By Schottky's theorem there is a constant \( B \), independent of \( p \), such that
\[
|g_p(z)| = |F_p(u)| \leq \exp\left(\frac{1}{1 - |u|}\left\{(1 + |u|)\log\max(1,|F_p(0)|) + 2B\right\}\right)
\]

Now \( g_p(z_1) \) is positive and \( \to \infty \) as \( p \to \infty \). so for all sufficiently large \( p \) (9) gives, noting \( F_p(0) = g_p(z_1) \),
\[
|g_p(z)| \leq k |g_p(z_1)|^L,
\]
where \( L, K \) are constants which depend on \( z_1, z_2 \) but not on \( p \). Thus for all sufficiently large \( p \) we have
\[
0 < g_p(\frac{1}{2} \gamma_{n+3}^{1/2}) \leq K(g_p(2\gamma_n^2))^L.
\]

By (8), however, we have
\[
2\gamma_n^2 < \gamma_{n+1}^{1/2} < \gamma_{n+1},
\]
and every iterate \( g_k \) is positive and increasing on the positive real axis, so for \( k \leq 1 \)
\[
g_k(2\gamma_n^2) < g_k(\gamma_{n+1}) = g_{k-1}(g(\gamma_{n+1}))
\]
\[
< g_{k-1}(2\gamma_{n+2}) < g_{k-1}(\frac{1}{2} \gamma_{n+3}^{1/2}),
\]
using (4) and (8). For all sufficiently large \( x \) one has \( g(x) > Kx^L \) and so for all sufficiently large \( k \)
\[
g_k(\frac{1}{2} \gamma_{n+3}^{1/2}) = g(g_{k-1}(\frac{1}{2} \gamma_{n+3}^{1/2})) > g(g_k(2\gamma_n^2)) > K\{g_k(2\gamma_n^2)^L\},
\]
which contradicts (10). Thus the first assertion of the theorem is established: for \( n > N \) the components \( C_n \) of \( \mathcal{C}(g) \) which contain \( A_n \) are all different, and each is a bounded domain.
It follows at once that each $C_n$ is a wandering domain for $g$. If this is not the case, then there exist integers $n > N$, $k > 0$, $l > 0$ such that $g_k(C_n)$ meets $g_{k+l}(C_n)$, i.e. since $g_k(C_n) \subset C_{n+k}$, $g_l(G') \subset G'$, where $G' = C_{n+k}$. The sequence \{g_n(z)\}, $n = 1, 2, \cdots$ is bounded in $G'$, taking values only in $G'$. But this contradicts the fact that the whole sequence \{g_k\}, $k = 1, 2, \cdots$, tends locally uniformly to $\infty$ in $G'$, as in every $C_n$, $n > N$.

The theorem is now established and clears up the problem of the existence of wandering domains, at least in the case of entire functions. It adds a little to the discussion of Baker (1963) where it was shown that, if for entire $g$ the set $\mathcal{C}(g)$ has a multiply-connected component, $G$, then there are just two alternatives, namely:
I. $G$ is unbounded and completely invariant and every other component of $\mathcal{C}(f)$ is simply-connected, or
II. All components of $\mathcal{C}(f)$ are bounded and infinitely many of them are multiply-connected.

It was conjectured in Baker (1963) that alternative II occurred in the case of the $g$ of our theorem and this is now established. It is interesting to note [c.f. Baker (1963)] that truncating the infinite product in (2) gives a polynomial

$$P(z) = Cz^2 \prod_{n=1}^{k} \left( 1 + \frac{z}{\gamma_n} \right)$$

such that alternative I applies to $\mathcal{C}(P)$ which has an unbounded and multiply-connected component.

References
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