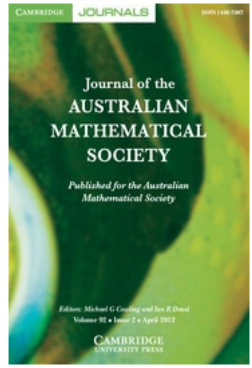


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# AN ENTIRE FUNCTION WHICH HAS WANDERING DOMAINS

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**Abstract**

Let $f(z)$ denote a rational or entire function of the complex variable $z$ and $f_n(z)$, $n = 1, 2, \cdots$, the $n$-th iterate of $f$. Provided $f$ is not rational of order 0 or 1, the set $\mathfrak{C}$ of those points where $\{f_n(z)\}$ forms a normal family is a proper subset of the plane and is invariant under the map $z \to f(z)$. A component $G$ of $\mathfrak{C}$ is a wandering domain of $f$ if $f_k(G) \cap f_n(G) = \varnothing$ for all $k \geqq 1$, $n \geqq 1$, $k \neq n$. The paper contains the construction of a transcendental entire function which has wandering domains.

The theory of the iteration of a rational or entire function $f(z)$ of the complex variable $z$ deals with the sequence of natural iterates $f_n(z)$ defined by

$$f(z) = z, \quad f_1(z) = f(z), \quad f_{n+1}(z) = f_1(f_n(z)), \qquad n = 0, 1, 2, \cdots.$$

In the theory developed by Fatou (1919, 1926) and Julia (1918) an important part is played by the set $\mathfrak{F} = \mathfrak{F}(f)$ of these points of the complex plane where $\{f_n(z)\}$ is not a normal family. Unless $f(z)$ is a rational function of order 0 or 1, (which we henceforth exclude) the set $\mathfrak{F}(f)$ is a non-empty perfect set, whose complement $\mathfrak{C} = \mathfrak{C}(f)$ consists of an at most countably infinite collection of (open) components $G_i$, each of which is a maximal domain of normality of $\{f_n\}$.

It is shown by Fatou (1919, 1926) that $\mathfrak{F}(f)$ is completely invariant under the mapping $z \to f(z)$, i.e. if $\alpha$ belongs to $\mathfrak{F}(f)$ then so do $f(\alpha)$ and every solution $\beta$ of $f(\beta) = \alpha$. It follows that $\mathfrak{C}(f)$ is also completely invariant and, in particular, for each component $G_i$ of $\mathfrak{C}(f)$ there is just one component $G_j$ such that $f(G_i) \subset G_j$. By definition, the component $G_0$ of $\mathfrak{C}(f)$ is a *wandering domain* of $f$ if

$$f_k(G_0) \cap f_n(G_0) = \varnothing \qquad \text{for all} \qquad 1 \leqq k, n < \infty, k \neq n.$$

No examples of wandering domains for either entire or rational functions seem to be known and indeed Jacobson (1969) raises the question whether they can occur at all for rational $f$. Pelles also discusses the notion.

In Baker (1963) an entire function $g(z)$ was constructed as follows:

Let $C = (4e)^{-1}$ and $\gamma_1 > 4e$. Then define inductively

(1) $$\gamma_{n+1} = C\gamma_n^2\left(1+\frac{\gamma_n}{\gamma_1}\right)\left(1+\frac{\gamma_n}{\gamma_2}\right)\cdots\left(1+\frac{\gamma_n}{\gamma_n}\right), \qquad n = 1, 2, \cdots.$$

Then $1 < \gamma_1 < \gamma_2 < \cdots$ and [c.f. Baker (1963): lemmas 1 and 2]

(2) $$g(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\gamma_n}\right)$$

is an entire function which satisfies

(3) $$|g(e^{i\theta})| < \tfrac{1}{4}, \qquad 0 \leqq \theta \leqq 2\pi,$$

(4) $$\gamma_{n+1} < g(\gamma_n) < 2\gamma_{n+1}, \quad n = 1, 2, \cdots,$$

(5) $$g(\gamma_n^{1/2}) < \gamma_{n+1}^{1/2}, \qquad n = 1, 2, \cdots,$$

and

(6) $$g(\gamma_n^2) > \gamma_{n+1}^2, \qquad n = 1, 2, \cdots.$$

Moreover, if $A_n$ denotes the annulus

(7) $$A_n : \gamma_n^2 < |z| < \gamma_{n+1}^{1/2},$$

then by Baker (1963) Theorem 1, there is an integer $N$ such that for all $n > N$ the mapping $z \to g(z)$ maps $A_n$ into $A_{n+1}$, so that $g_k(z) \to \infty$ uniformly in $A_n$ as $k \to \infty$. Since by (3) $g_k(z) \to 0$ uniformly for $|z| \leqq 1$, it is clear that each $A_n$, $n > N$, belongs to a multiply connected component $C_n$ of $\mathfrak{C}(g)$ and that $C_n$ does not meet $\{z : |z| \leqq 1\}$, which belongs to a component of $\mathfrak{C}(g)$ which we designate $C_0$. It is natural to ask whether the $C_n$, $n > N$, are all different, but this question was left unanswered in Baker (1963). The solution is given by the

THEOREM. For $n > N$ *the components* $C_n$ *of* $\mathfrak{C}(g)$ *described above are all different and each is a wandering domain of* $g$.

PROOF. Suppose that there are two values of $n > N$ for which $A_n$ belong to the same component of $\mathfrak{C}(g)$. Suppose $n = m > N$ and $n = m + l$, $l > 0$, are such values. Then there is a path $\Gamma$ in $\mathfrak{C}(g)$ which joins a point of $A_m$ to a point of $A_{m+l}$. The path $\Gamma$ must meet $A_{m+1}$, which therefore belongs to the same component of $\mathfrak{C}(g)$ as $A_m$. So we may take $l = 1$. By the complete invariance of $\mathfrak{C}(g)$ the path $g_k(\Gamma)$ lies in $\mathfrak{C}(g)$ and it joins $A_{m+k}$ to $A_{m+k+1}$, $k = 1, 2, \cdots$. Thus all $A_n$, $n > m$, belong to the same component of $\mathfrak{C}(g)$, which is therefore multiply-connected and unbounded.

It suffices to show that for all sufficiently large $n$ the annuli $A_n$ and $A_{n+2}$ cannot be joined in $\mathfrak{C}(g)$. Now for all sufficiently large $n(> N_0)$ we have, since $\gamma_n \to \infty$ in (1) that,

$$(8) \qquad 4\gamma_n^2 < \gamma_{n+1}^{1/2}.$$

Take any $n > \text{Max}(N, N_0)$ and assume that $A_n$, $A_{n+2}$ can be joined in $\mathfrak{C}(g)$. Then $z_1 = 2\gamma_n^2 \in A_n$ and $z_2 = \frac{1}{2}\gamma_{n+3}^{1/2} \in A_{n+2}$. There is then a simple polygon joining $z_1$ and $z_2$ in $\mathfrak{C}(g)$ and so $z_1$, $z_2$ belong to a simply-connected subdomain, say $H$, of $\mathfrak{C}(g)$. $H$ may be mapped conformally by $z = \psi(t)$ onto $|t| < 1$ so that $\psi(0) = z_1$ and $\psi(u) = z_2$ where $u$ is some value for which $|u| < 1$.

Since $g_k(z) \to \infty$ locally uniformly, as $k \to \infty$ for $z \in A_n$, the same is true locally uniformly in the component $G$ of $\mathfrak{C}(g)$ to which $A_n$ belongs. Thus for each integer $p > 0$, $g_p(G)$ is a domain in which $G_k(z) \to \infty$ locally uniformly, so $g_p(G)$ does not meet the component $G_0$ of $\mathfrak{C}(g)$ which includes the disc $\{z : |z| \leqq 1\}$, as $g_k(z) \to 0$ in $G_0$. Thus in $G$, and in particular in $H$, $g(z)$ omits the values $0, 1$. Similarly the functions $F_p(t) = g_p\{\psi(t)\}$ omit the values $0, 1$ in $|t| < 1$. By Schottky's theorem there is a constant $B$, independent of $p$, such that

$$(9) \qquad \begin{aligned} |g_p(z_2)| &= |F_p(u)| \\ &\leqq \exp\left[\left(\frac{1}{1-|u|}\right)\left\{(1+|u|)\log\max(1, |F_p(0)|) + 2B\right\}\right] \end{aligned}$$

Now $g_p(z_1)$ is positive and $\to\infty$ as $p \to \infty$. so for all sufficiently large $p$ (9) gives, noting $F_p(0) = g_p(z_1)$,

$$|g_p(z_2)| \leqq k\,|g_p(z_1)|^L,$$

where $L$, $K$ are constants which depend on $z_1$, $z_2$ but not on $p$. Thus for all sufficiently large $p$ we have

$$(10) \qquad 0 < g_p(\tfrac{1}{2}\gamma_{n+3}^{1/2}) \leqq K\{g_p(2\gamma_n^2)\}^L.$$

By (8), however, we have

$$2\gamma_n^2 < \gamma_{n+1}^{1/2} < \gamma_{n+1},$$

and every iterate $g_k$ is positive and increasing on the positive real axis, so for $k \leqq 1$

$$\begin{aligned} g_k(2\gamma_n^2) &< g_k(\gamma_{n+1}) = g_{k-1}\{g(\gamma_{n+1})\} \\ &< g_{k-1}(2\gamma_{n+2}) < g_{k-1}(\tfrac{1}{2}\gamma_{n+3}^{1/2}, \end{aligned}$$

using (4) and (8). For all sufficiently large $x$ one has $g(x) > Kx^L$ and so for all sufficiently large $k$

$$g_k(\tfrac{1}{2}\gamma_{n+3}^{1/2}) = g\{g_{k-1}(\tfrac{1}{2}\gamma_{n+3}^{1/2})\} > g\{g_k(2\gamma_n^2)\} > K\{g_k(2\gamma_n^2)^L\},$$

which contradicts (10). Thus the first assertion of the theorem is established: for $n > N$ the components $C_n$ of $\mathfrak{C}(g)$ which contain $A_n$ are all different, and each is a bounded domain.

It follows at once that each $C_n$ is a wandering domain for $g$. If this is not the case, then there exist integers $n > N$, $k > 0$, $l > 0$ such that $g_k(C_n)$ meets $g_{k+l}(C_n)$, i.e. since $g_k(C_n) \subset C_{n+k}$, $g_l(G') \subset G'$, where $G' = C_{n+k}$. The sequence $\{g_{ln}(z)\}$, $n = 1, 2, \cdots$ is bounded in $G'$, taking values only in $G'$. But this contradicts the fact that the whole sequence $\{g_k\}$, $k = 1, 2, \cdots$, tends locally uniformly to $\infty$ in $G'$, as in every $C_n$, $n > N$.

The theorem is now established and clears up the problem of the existence of wandering domains, at least in the case of entire functions. It adds a little to the discussion of Baker (1963) where it was shown that, if for entire $g$ the set $\mathfrak{C}(g)$ has a multiply-connected component, $G$, then there are just two alternatives, namely:

I. $G$ is unbounded and completely invariant and every other component of $\mathfrak{C}(f)$ is simply-connected, or

II. All components of $\mathfrak{C}(f)$ are bounded and infinitely many of them are multiply-connected.

It was conjectured in Baker (1963) that alternative II occurred in the case of the $g$ of our theorem and this is now established. It is interesting to note [c.f Baker (1963)] that truncating the infinite product in (2) gives a polynomial

$$P(z) = Cz^2 \prod_{n=1}^{k} \left(1 + \frac{z}{\gamma_n}\right)$$

such that alternative I applies to $\mathfrak{C}(P)$ which has an unbounded and multiply-connected component.

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