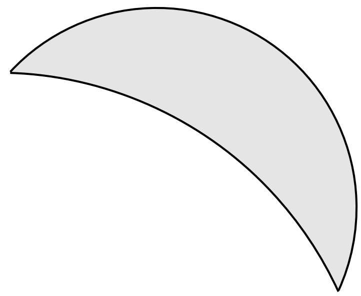
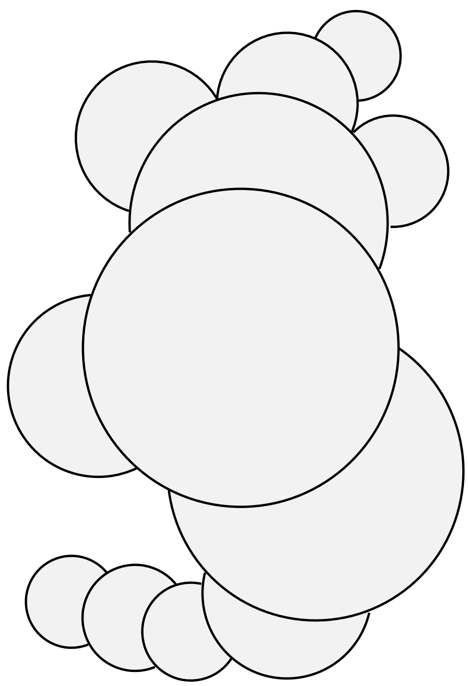
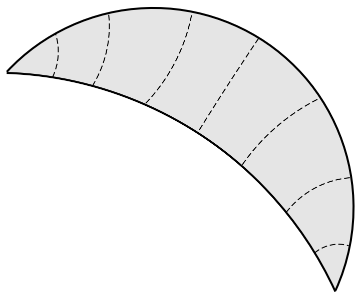
A crescent is a domain bounded by two circular arcs.

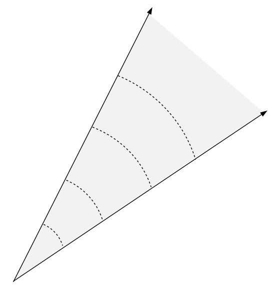


A cresent domain is the union of the unit disk and finitely many crescents, as below. (Every finite union of disks has this form.)

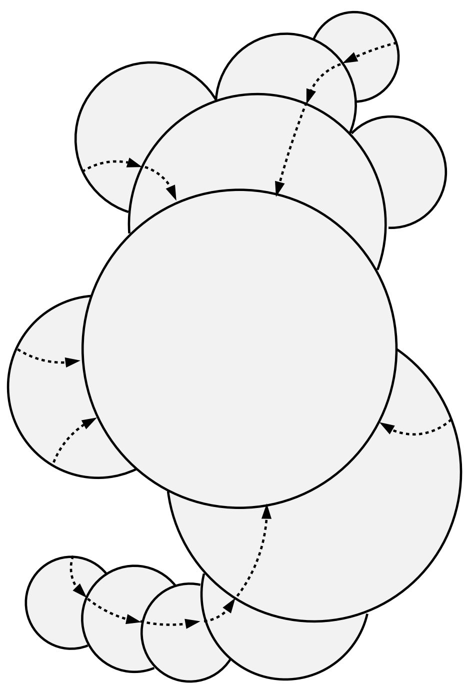


A crescent can be foliated by circular arcs orthogonal to both boundary arcs.





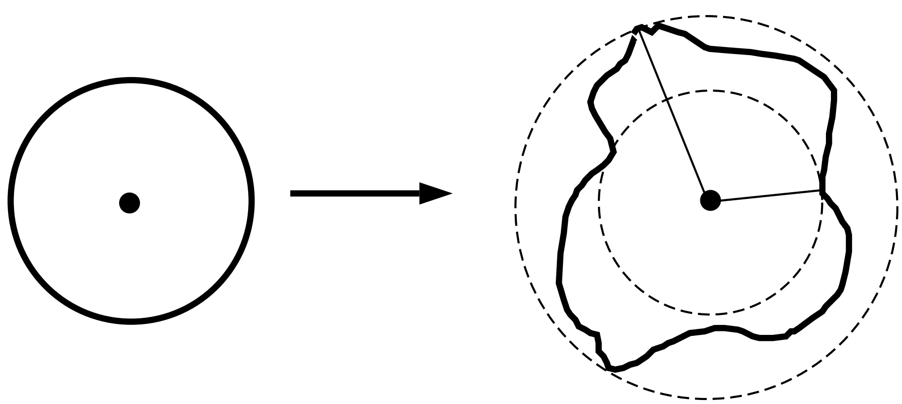
Following arcs in the foliation gives a map $\iota : \partial\Omega \to \partial D$.



Theorem: ι extends to a K-quasiconformal, map $F : \Omega \to \mathbb{D}$. Moreover, $F' \in L^\infty$ and K is independent of Ω.

A map f is K-quasiconformal if

$$\sup_x \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \leq K < \infty.$$



If $K = 1$, then f is conformal.

Corollary: If Ω is simply connected and quasiconvex (i.e., interior path metric comparable to Euclidean metric) then there is Lipschitz homeomorphism of Ω to a disk.

Corollary: Any quasicircle can be mapped onto a circle by a Lipschitz map of the plane.

Corollary: [The factorization theorem] Any conformal map $f : D \to \Omega$ can be written as $f = g \circ h$ where h is a K-quasiconformal self-map of D and $|g'|$ is bounded away from zero.

Indeed $|g'(tz)| \leq C|g'(z)|$ for any $0 \leq t \leq 1$

If f is conformal on the disk then f' is never 0, but it can tend to 0 near the boundary. Many results limit how close it can be to 0

Hayman-Wu theorem If $f : \Omega \to \mathbb{D}$ is conformal and L is a line (or circle) then $f(L)$ has finite length (independent of f or Ω).

Makarov's theorem: If $E \subset \partial D$ has psotive length then $\dim(f(E)) \geq 1$. In general, if $\dim(E) = \alpha$ then $\dim(f(E)) \geq \varphi(\alpha)$.

Brennan's conjecture: If $f : \Omega \to \mathbb{D}$ is conformal then $f' \in L^p(\Omega, dxdy)$ for all $p < 4$.

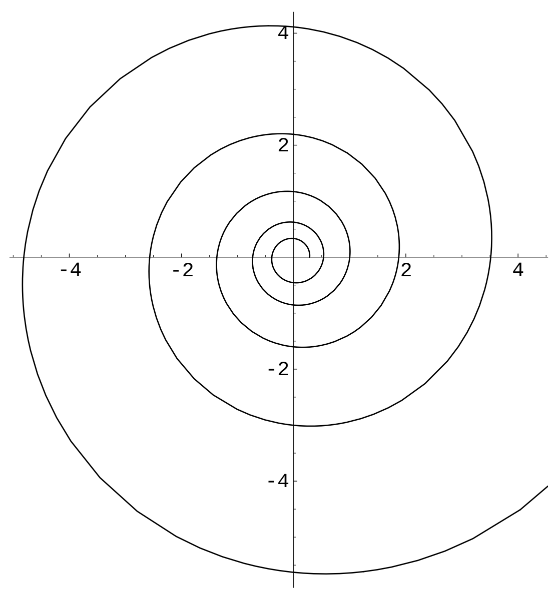
What is best K? Thurston conjectured $K = 2$. Epstein-Marden proved ≈ 80. I can prove ≈ 8.

Important result of Kari Astala says K-QC map has deriviative in weak L^p for $p = 2K/(K-1)$. If $K = 2$ then $p = 4$.

Then any conformal map $f : \Omega \to D$ can be written $f = h \circ g$ where $g' \in L^\infty$ and h is 2-QC, so f' is weak L^4. Thus "$K = 2$" implies Brennan's conjecture.

Best result for Brennan so far is $p = 3.442$ by Bertillson. This corresponds to $K = 2.4064$.

Epstein-Markovic have announced counterexample to $K = 2$. They claim that complement of logrithmic spiral has $K \approx 2.1 > 2$.



So Thurston's $K = 2$ conjecture seems to be false. Might the factorization theorem for conformal maps still be true for $K = 2$?

The hyperbolic metric in the ball is

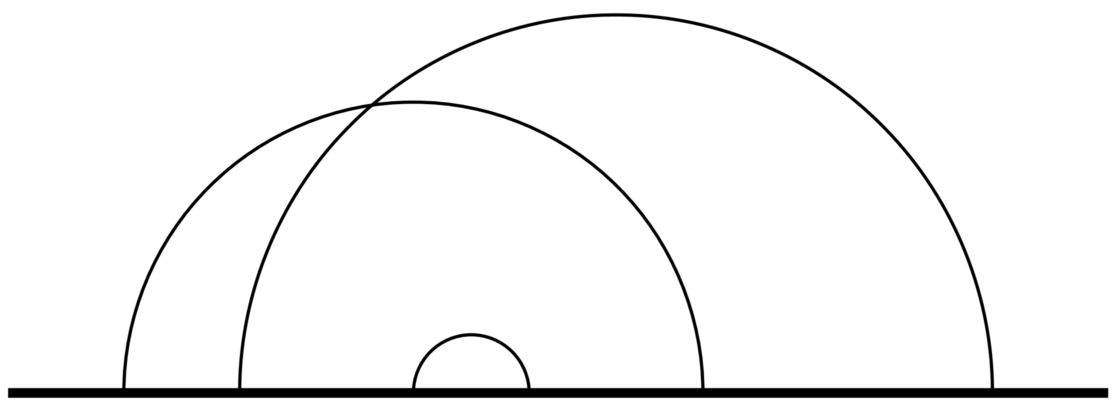
$$|d\rho| = \frac{|dz|}{1 - |z^2|}$$

and in the upper halfspace is

$$|d\rho| = \frac{|dz|}{y}$$

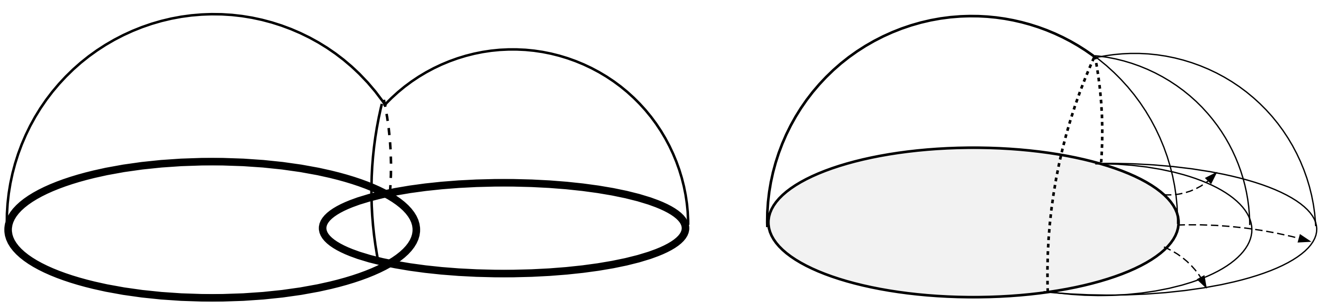
Geodesics are circle orthogonal to boundary. Also will use in ball,

$$\exp(-\rho(0, z)) \simeq 1 - |g(0)|.$$



Given $\Omega \subset \mathbb{R}^2$ take union of all hemispheres with base in Ω to get region in $\mathbb{R}^3_+$. The "upper" boundary is called S. It is also a boundary component of the hyperbolic convex hull of $\partial\Omega$.

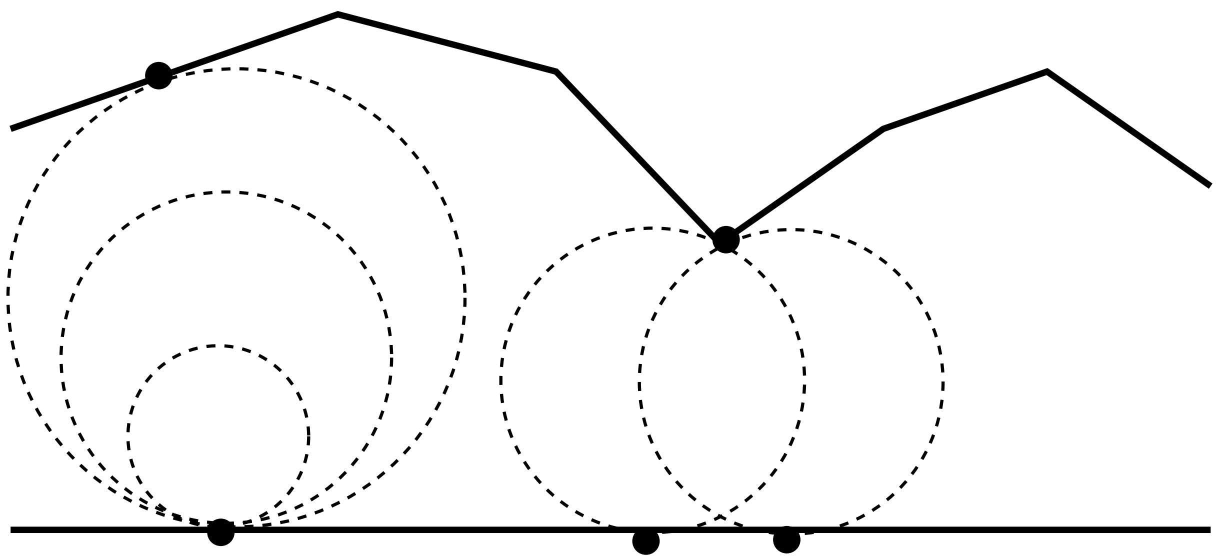
Give S the intrinsic hyperbolic path metric. Thurston observed there is an isometry ι from S to the hyperbolic disk. If Ω is a crescent domain, S is a finite union of geodesic polygons, and this ι agrees with the previous one on the boundary.



Theorem: [Sullivan, Epstein-Marden] There is a K-quasiconformal map $\sigma : \Omega \to S$ which is the identity on the boundary.

Thus $\iota \circ \sigma$ is the K-QC extension of ι.

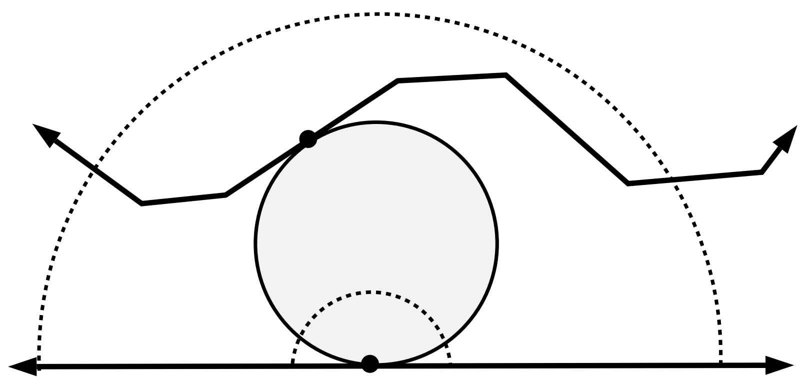
Define nearest point retraction $R : \Omega \to S$ by expanding horoball tangent at $z \in \Omega$ until it first hits S at $R(z)$.



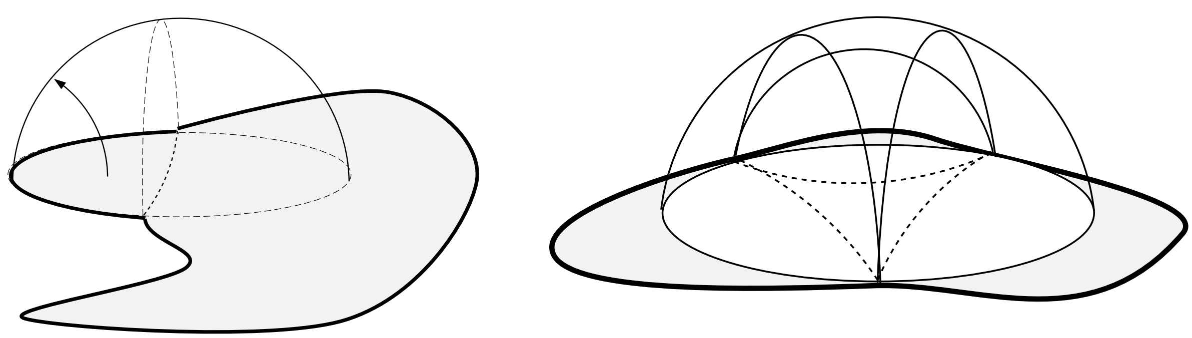
In general, the retraction is **not** one to one.

Several facts are easy to check.

Fact 1: $\mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2)$.

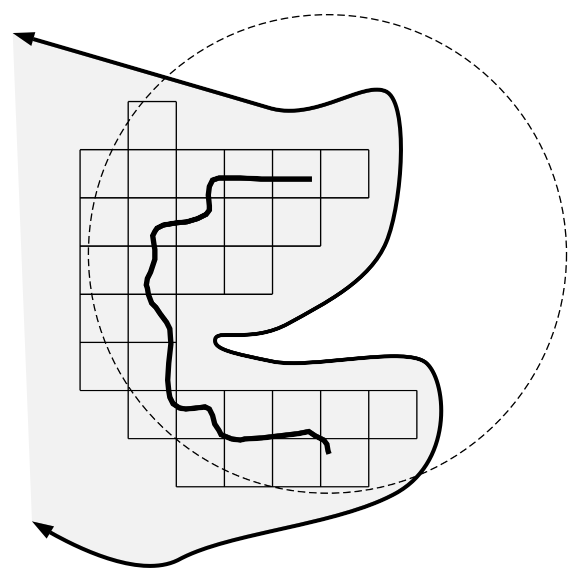


Fact 2: R is Lipschitz.



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

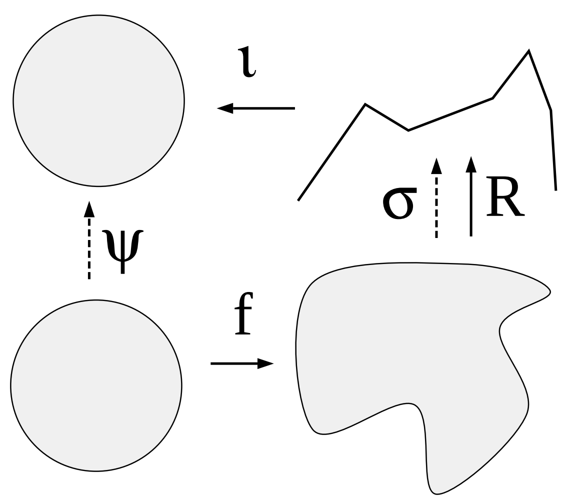
Suppose $R(z)$ is at height r from boundary. Consider a path of length 1 on S between $R(z)$ and $R(w)$. Every point has height $\simeq r$ and every preimage in Ω is distance $\simeq r$ from $\partial\Omega$. Thus preimage covered by bounded number of squares, each diameter $\simeq r$ and distance $\simeq r$ from $\partial\Omega$, i.e., uniform hyperbolic diameter.



Facts 2 and 3 imply R is a rough isometry,

$$\frac{1}{A}\rho(z,w) - B \le \rho(R(z), R(w)) \le A\rho(z,w) + B.$$

Thus $\iota \circ R \circ f$ is rough isometry of D to itself. Standard results say it extends to homeomorphism of the boundary and there is a QC self-map ψ of D with same boundary values. Then $\sigma = \iota^{-1} \circ \psi \circ f^{-1}$ is Sullivan's map.



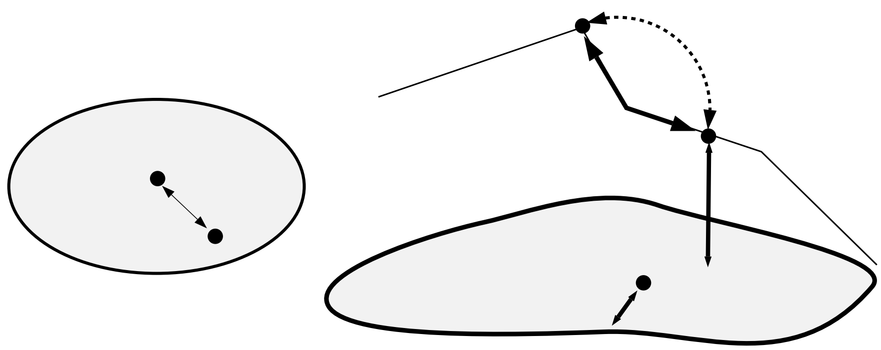
Moreover, $\rho(R(z), \sigma(z)) < C$ uniformly.

Why is $g = \iota \circ \sigma$ Lipschitz? Standard estimates show

$$|g'(z)| \simeq \frac{\mathrm{dist}(\mathrm{g}(\mathrm{z}), \partial\mathbb{D})}{\mathrm{dist}(\mathrm{z}, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\mathrm{dist}(\mathrm{z}, \partial\Omega) &\simeq \mathrm{dist}(\sigma(\mathrm{z}), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \mathrm{dist}(\mathrm{g}(\mathrm{z}), \partial\mathrm{D})
\end{aligned}$$



Suppose Ω is simply connected and invariant under a Kleinian group G. If $f : \mathbb{D} \to \Omega$ is conformal then $G_1 = f^{-1} \circ G \circ f$ is Fuchsian.

Bowen's theorem: If $R = \Omega/G$ is compact then $\partial\Omega$ is either circle or has Hausdorff dimension > 1.

The α-content of E is

$$H^{\varphi}(E) = \inf\{\sum (r_j)^{\alpha} : E \subset \cup D(x_j, r_j)\}.$$

The Hausdorff dimension is

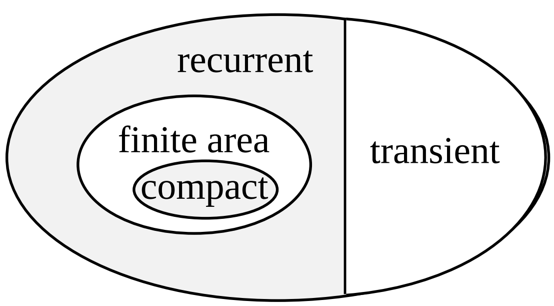
$$\dim(E) = \inf\{\alpha : H^{\alpha}(E) = 0\}.$$

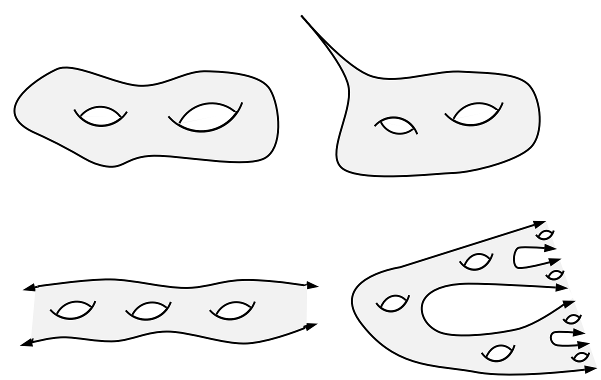
Roughly speaking a covering by ϵ-balls must have $\epsilon^{-\dim(E)}$ elements.

Mass distribution principle + Frostman's Lemma: E has positive α-content iff E supports a measure such that $\mu(D(x,r)) \leq r^{\alpha}$.

Also true for finite area (Sullivan). Astala and Zinsmeister showed it was false for some infinite area cases, i.e.,

Theorem: [Astala-Zinsmeister] If R has a Green's function (i.e., is transient), then there is a simply connected Ω and Kleinian group G, so that $R = \Omega/G$ and $\partial\Omega$ is a non-circular, rectifiable curve.



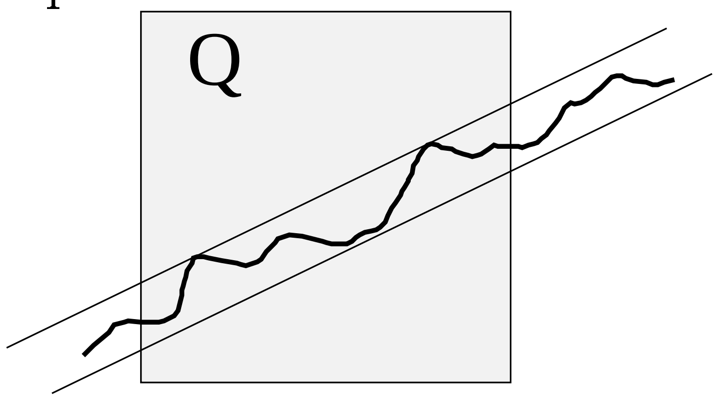


Theorem: Suppose $R = \Omega/G$ is recurrent. Then either $\partial\Omega$ is a circle or $\dim(\partial\Omega) > 1$.

Given a set E and a dyadic square Q, we define

$$\beta_E(Q) = \inf_L \sup_{z \in E \cap Q} \mathrm{dist(z, L)},$$

where the sup is over all lines interseting Q.



Jones' Traveling Salesman Theorem: The minmum length curve containing E has length comparable to

$$\mathrm{diam(E)} + \sum_{\mathrm{Q}} \beta_{\mathrm{E}}(\mathrm{Q})^2 \ell(\mathrm{Q}).$$

The minimum length of a curve coming within ϵ of every point of E is comparable to

$$\mathrm{diam(E)} + \sum_{\mathrm{Q}:\ell(\mathrm{Q}) \geq \epsilon} \beta_{\mathrm{E}}(\mathrm{Q})^2 \ell(\mathrm{Q}).$$

A set is uniformly wiggly if there is a $\beta_0 > 0$ so that $\beta_E(Q) \geq \beta_0$ for all Q such that $\frac{1}{3}Q \cap E \neq \emptyset$.

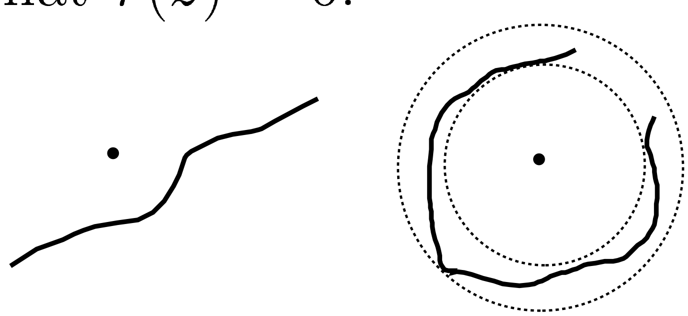
Theorem: If E is connected and uniformly wiggly, then $\dim(E) \geq 1 + C\beta_0^2$.

Theorem: If $R = \Omega/G$ is compact, then $\partial\Omega$ is either a circle or is uniformly wiggly.

For $z \notin E$, define

$$\eta_E(z) = \inf_\tau \log \frac{\max_{x \in E} |\tau(x)|}{\min_{y \in E} |\tau(y)|},$$

where the infimum is over all Möbius transformations such that $\tau(z) = 0$.

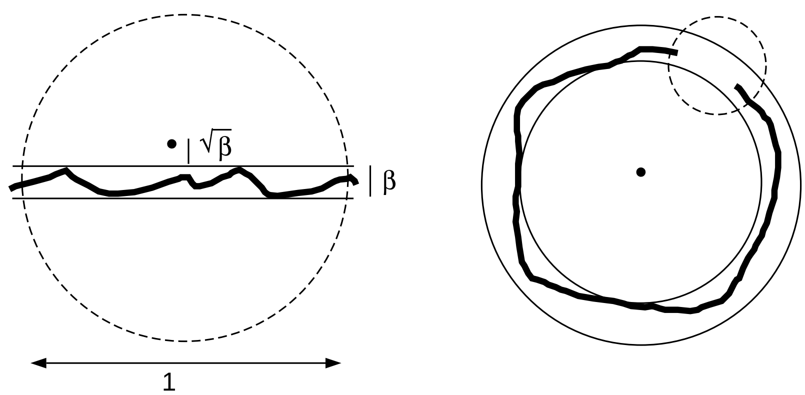


FACT 1: $\eta_E(z)$ is continuous in z.

FACT 2: η is invariant under Möbius transmormations, i.e., $\eta_E(z) = \eta_{\sigma(E)}(\sigma(z))$.

FACT 3: $\eta_E(z) = 0$ for any z implies E is a subset of a circle (or line).

FACT 4: If η is strictly positive, so is β.

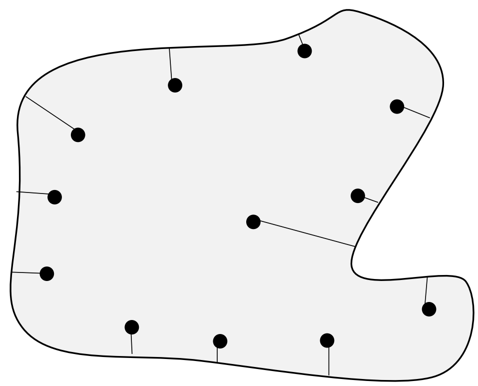


Thus if $E = \partial\Omega$ is invariant under a Möbius group and Ω/G is compact, then either $\eta(z) \equiv 0$ or η takes a positive minimum. Thus $\partial\Omega$ is circle or is uniformly wiggly.

The critical exponent δ is infimum of all s so that

$$\sum_{g \in G} \mathrm{dist}(\mathrm{g}(\mathrm{z}), \Lambda)^{\mathrm{s}} < \infty.$$

For all non-elementary groups $\delta \leq \dim(\Lambda)$ with equality in many cases.



Theorem: If $R = \Omega/G$ is recurrent then either $\partial\Omega$ is a circle or $\delta > 1$.

Easier version: If $R = \Omega/G$ is recurrent then

$$\sum_{g \in G} \mathrm{dist}(\mathrm{g}(\mathrm{z}), \Lambda) = \infty,$$

i.e., the Poincaré series diverges at $s = 1$.

A Fuchsian group is called convergence type if

$$\sum_{g \in G} 1 - |g(0)| < \infty$$

and otherwise is called divergence type.

Fact 1: Divergence type is a QC invariant (Pfluger, 1949), i.e., if $G = f \circ G \circ f^{-1}$ are both Fuchsian groups and f is QC then G is divergence type iff H is.

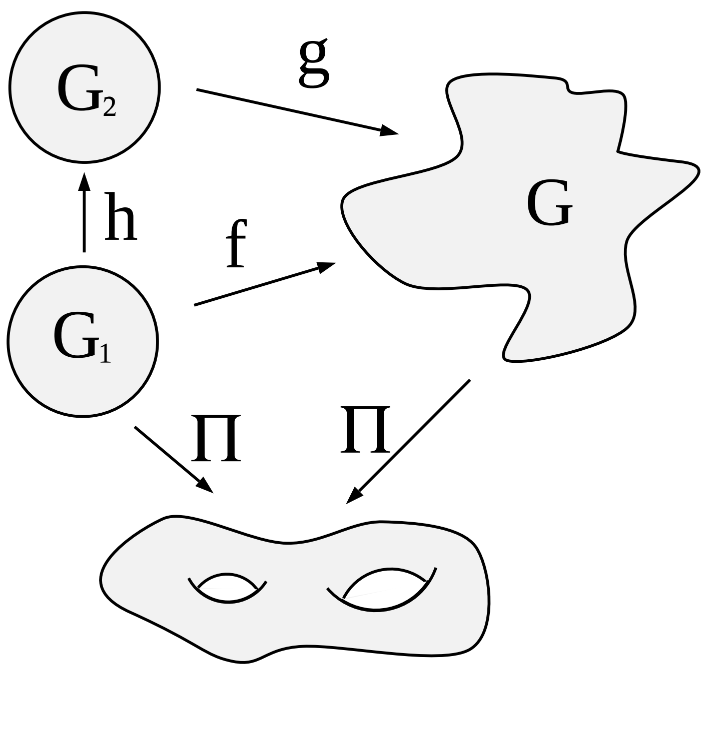
Fact 2: G is divergence type iff orbits are non-tangentially dense almost everywhere on circle.

Fact 3: G is divergence type iff $R = \mathbb{D}/G$ is recurrent (has no Green's function). This is easy by explicit formulas.

Suppose $f = g \circ h$ as in Factorization theorem. Let $G_1 = f^{-1} \circ G \circ f$ and $G_2 = g^{-1} \circ G \circ g$. We are assuming G_1 is divergence type. Since

$$\mathrm{dist}(\gamma(\mathrm{z}_0), \Lambda) \gtrsim \mathrm{dist}(\mathrm{g}^{-1}(\gamma(\mathrm{z}_0)), \partial\mathbb{D}),$$

it suffices to show G_2 is also divergence type. But G_2 is QC conjugate to G_1 (via h) so it is divergence type by Pfluger's theorem.

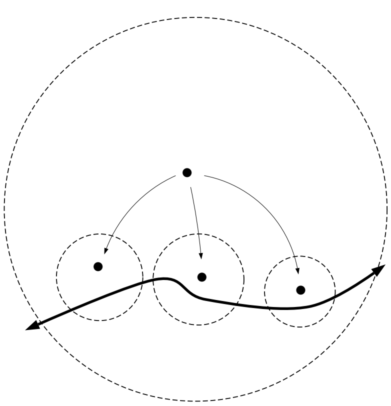


Next prove the series diverges for some $s > 1$. Given $w \in G(z_0)$ we define a finite collection of orbit points $\mathcal{C}(w)$ so that for $z \in \mathcal{C}(w)$,

$$|z - w| \leq 2\mathrm{dist}(\mathrm{w}, \Lambda), \tag{1}$$

$$z_1, z_2 \in \mathcal{C}(w) \Rightarrow |z_1 - z_2| \geq 2\mathrm{dist}(\mathrm{z}_1, \Lambda), \tag{2}$$

$$\sum_{z \in \mathcal{C}(w)} \mathrm{dist}(\mathrm{z}, \Lambda)^{\mathrm{s}} \geq \mathrm{dist}(\mathrm{w}, \Lambda)^{\mathrm{s}} \tag{3}$$



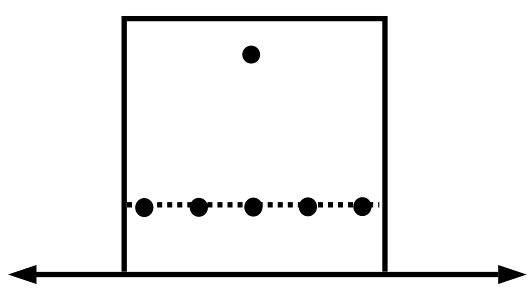
If we use iteration to define sucessive generations of points the first two conditions imply all chosen points are distinct and the third implies the sum over these points diverges (since the sum over any generation is larger than the sum over the previous generation). (3) follows from

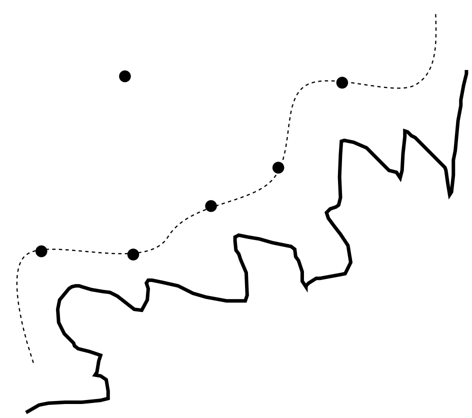
$$\sum_{z \in \mathcal{C}(w)} \mathrm{dist}(\mathrm{z}, \Lambda) \geq 2\mathrm{dist}(\mathrm{w}, \Lambda) \qquad (3')$$

$$\mathrm{dist}(\mathrm{z}, \Lambda) \geq \epsilon \mathrm{dist}(\mathrm{w}, \Lambda), \qquad (3'')$$

for some $s = s(\epsilon)$.

STEP 1: If Λ is not a circle, then every sub-arc has infinite length. Given w consider level lines so close to boundary that their length is $\gg \text{dist}(w, \Lambda)$. Choose points on this curve which satisfy conditions (1), (2) and (3'). These points need not be orbit points.





STEP 2: For each z chosen in step down, choose orbit points x_n which satisfy (1), (2) and

$$\sum \mathrm{dist}(\mathrm{x_n}, \Lambda) \geq \mathrm{Cdist}(\mathrm{z}, \Lambda).$$

This can be done using the almost monotone growth part of the factorization theorem to compare to G_2, where it holds because G_2 is divergence type and hence orbits are non-tangentially dense.