Problem 1 (10 points): Give the correct definition or statement.

(1) Define differentiability of a function \( f : \mathbb{R}^n \to \mathbb{R}^n \).

(2) State the contraction mapping principle.

(3) State the inverse function theorem.

(4) State the rank theorem.

(5) Define a partition of unity.

(6) Define a differential form of order \( k \) on \( \mathbb{R}^n \).

(7) State Stokes theorem.

(8) Define measurable set.

(9) State the monotone convergence theorem.

(10) State the Lebesgue dominated convergence theorem.

Problem 2 (10 points): Give an example of each, or explain why it can’t exist:

(1) A sequence of functions on \([0, 1]\) that converges pointwise, but not uniformly.

(2) A sequence \( \{f_n\} \) on \([0, 1]\) so that \( f_n(x) \to 0 \) for every \( x \) but \( \int_0^1 f_n \, dx \not\to 0 \).

(3) A function \( f(x, y) \) so that the \( x \) and \( y \) partials exist, but \( f \) is not differentiable at \((0, 0)\).

(4) A subset \( E \subset \mathbb{R}^2 \) and a strict contraction \( f : E \to E \) that has no fixed point.

(5) A function that is \( C^1 \) on \( \mathbb{R} \), but is not \( C^2 \).

(6) A measurable function on \( \mathbb{R} \) that is nowhere continuous.

(7) A sequence of functions on \([0, 1]\) so that \( \int_0^1 |f_n| \, dx \to 0 \) but \( f_n(x) \) does not converge to zero for any \( x \).

(8) An integrable function \( f \) on \([0, 1]\) so that \( f^2 \) is not integrable.

(9) An example of strict inequality in Fatou’s theorem.

(10) An uncountable set of Lebesgue measure zero.
**Problem 3 (10 points):** Give a complete and correct proof of two of the following statements (your choice). You may use results from the text if they are correctly quoted.

1. If $f \geq 0$ on $E$ and $\int_E f \,dx = 0$, then $f = 0$ almost everywhere on $E$.
2. If $f_n$ is a sequence of measurable functions, then the set where $f_n$ converges is measurable.
3. Prove that there is a non-measurable set in $[0, 1]$.
4. Prove that the continuous functions are dense in the integrable functions ($L^1$ norm).