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REAL ANALYSIS II

FOLLAND'S REAL ANALYSIS: CHAPTER 8
ELEMENTS OF FOURIER ANALYSIS

Christopher Bishop

Chapter 8: Elements of Fourier analysis

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Chapter 8.1: Preliminaries

In this chapter we work in $\mathbb{R}^n$, $n =$ dimension.

$m =$ Lebesgue measure on $\mathbb{R}^n$, $L^p(E) = L^p(E, dm)$.

If $U \subset \mathbb{R}^n$ is open, $C^k(U) =$ functions with k continuous partial derivatives.

$C^\infty(U) = \cap_{k=1}^\infty C^k(U)$.

$C_c(E) =$ continuous functions with compact support contained in E.

$L^p = L^p(\mathbb{R}^n)$, $C^k = C^k(\mathbb{R})$, $C_c^\infty = C_c^\infty(\mathbb{R})$, ...

$x \cdot y = \sum_{k=1}^n x_k y_k$, $|x| = \sqrt{x \cdot x}$.

$\partial_j = \frac{\partial}{\partial x_j}$.

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$,

$$|\alpha| = \sum_1^n \alpha_k$$

$$\alpha! = \prod_1^n \alpha_k!$$

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

$$x^\alpha = \prod_{k=1}^n x_k^{\alpha_k}.$$

Taylor series:

$$f(x)\ \sum_{|\alpha|\leq k}(\partial^{\alpha}f(x_0)\frac{(a-x_0)^{\alpha}}{\alpha!}+R_k(x),\quad \lim_{x\to x_0}\frac{|R(_k(x)|}{|x-x_0|^k}=0.$$

Product Rule:

$$\partial^{\alpha}(fg)=\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!}\gamma!)\partial^{\beta}f)(\partial^{\gamma}g).$$

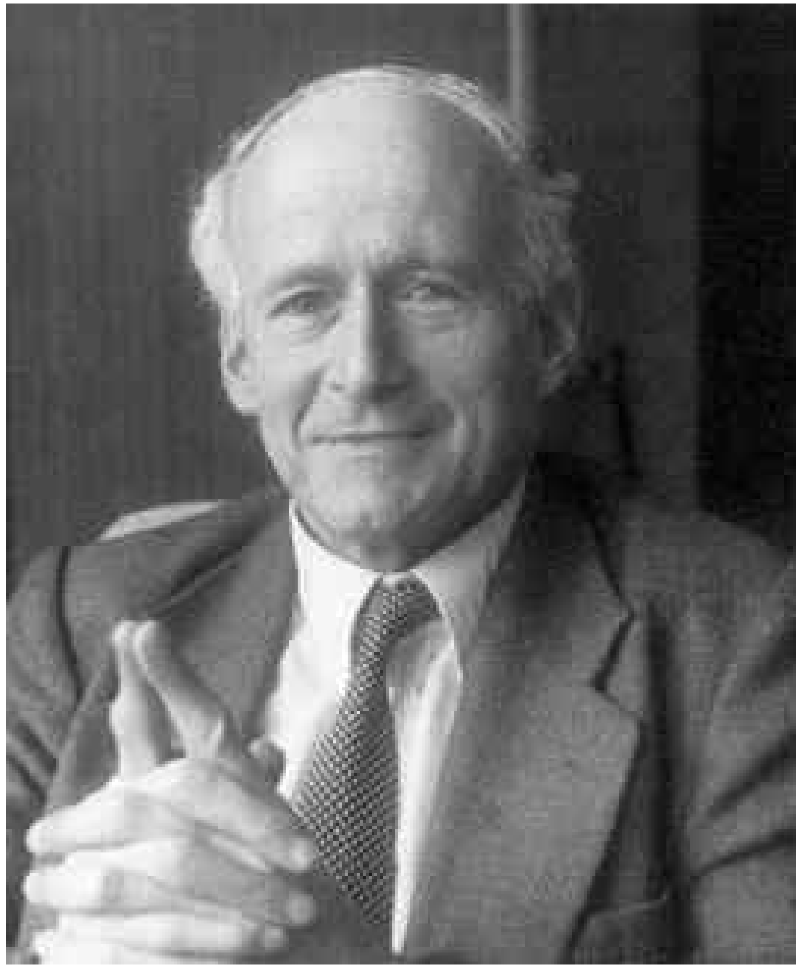
C^∞ functions of compact support:

$$\psi(x) = \begin{cases} \exp((|x|^2-1)^{-1}) & if |x| < 1 \\ 0 & if |x| \geq 1 \end{cases}$$

Schwarz space, $\mathcal{S}$: C^∞ functions which, together with there derivatives, tend to zero at ∞ faster than any power. A TVS given by the finiteness of the seminorms:

$$\|f\|_{N,\alpha} = \sup_{x\in\mathbb{R}^n} (1+|x|^N)|\partial^\alpha f(x)| < \infty.$$

Example: $\exp(-|x|^2)x^\alpha$,



Laurent Moise Schwartz (1915–2002)

Prop 8.2: $\mathcal{S}$ is a Fréchet space with the topology defined by the norms $\|\cdot\|_{(N,\alpha)}$.

Proof. Only nontrivial part is completeness. If $\{f_k\}$ is Cauchy, then $\|f_j - f_k\|_{(M,\alpha)} \to 0$ for every norm. Thus every $\partial^\alpha f$ converges to some continuous function g_α. Let e_j be the jth unit coordinate vector. Then

$$f_k(x + te_j) - f_k(x) = \int_0^t \partial_j f_k(x + se_j)ds$$

As $k \to \infty$ we have $f_k \to g_0$ and $(\partial^\alpha f_k) \to g_\alpha$, so by uniform convergence,

$$g_0(x + te_j) - g_0(x) = \int_0^t g_{e_j}(x + se_j)ds$$

By the Fundamental theorem of Calculus we must have $g_{e_j} = \partial_j g_0$. Then use induction of $|\alpha|$ to prove $g_\alpha = \partial^\alpha g_0$ for all α. Then

$$\begin{aligned}
\|f_k - g_0\|_{(N,\alpha)} &= \sup(1+|x|^N)|\partial^\alpha(f_k - g_0)| \\
&= \sup(1+|x|^N)|\partial^\alpha f_k - g_\alpha| \\
&= \sup(1+|x|^N)|\partial^\alpha f_k - \lim_j \partial^\alpha f_j| \\
&\leq \sup_{j\geq k}\sup(1+|x|^N)|\partial^\alpha f_k - \partial^\alpha f_j| \\
&\leq \sup_{j\geq k}\|f_k - f_j\|_{(N,\alpha)}
\end{aligned}$$

and the last line tends to 0 as $k \nearrow \infty$ because $\{f_k\}$ was Cauchy. $\square$

Prop. 8.3: If $f \in C^\infty$ then $f \in \mathcal{S}$ iff $x^\beta \partial^\alpha f$ is bounded for all α, β iff $\partial^\alpha (x^\beta f)$ is bounded for all α, β.

Proof. Let $N = |\beta|$. Then

$$\sup |x^\beta \partial^\alpha f| \leq (1 + x^N) |\partial^\alpha f| \|f\|_{(N,\alpha)},$$

so $f \in \mathcal{S}$ then $|x^\alpha \partial^\alpha f|$ is bounded.

Conversely, $|x|^N$ and $\sum_{k=1}^M |x_k|^N$ are both homogeneous of order N ($f(\lambda x) = \lambda^N f(x)$) and both are bounded and bounded below on the (compact) unit sphere in $\mathbb{R}^N$ so

$$|x|^N \leq C \sum_{k=1}^{M} |x_k|^N$$

for some $C < \infty$. Thus (using $x^0 = 1$)

$$\begin{aligned} \|f\|_{(N,\alpha)} &= \sup(1 + |x|^N)|\partial^\alpha f| \\ &\leq \sup(1 + \sum_{k=1}^{N} |x_k|^N)|\partial^\alpha f| \\ &\leq \sup \sum_{|\beta| \leq N} |x_k|^\beta |\partial^\alpha f|. \end{aligned}$$

Second equivalence follows using product rule: $\partial^\alpha(x^\beta f)$ is a linear combination of terms of form $x^\gamma \partial^\delta f$, each of which is bounded iff $f \in \mathcal{S}$. $\square$

Defn: if f is a function on $\mathbb{R}^n$ and $y \in \mathbb{R}^n$, the **translation operator**

$$\tau_y f(x) = f(x-y).$$

τ_y is an isometry onL^p and with respect to the uniform norm.

Defn: a function f is called uniformly continuous if $\|\tau_y f - f\|_u \to 0$ as $y \to 0$.

Lemma 8.4: If $f \in C_c$ then then f is uniformly continuous.

Proof. Given $\epsilon > 0$, for each $x \in \text{supp}(f)$ there exists $\delta_x > 0$ such that

$$|y| < \delta)x \quad \Rightarrow \quad |f(x-y) - f(x)| < \frac{\epsilon}{2}.$$

Since $\text{supp}(f)$ is compact, it is covered by a finite collection of balls $B_j = B(x_j, \delta_{x_j})$. In fact, $\{z : \text{dist}(z, \text{supp}(f)) \leq 1\}$ is also compact, so assume that is covered by the balls B_j.

Set $\delta = \inf \delta_{x_j}$. Then if $|y| < \delta$, and $x \in B_j$,

$$|f(x-y) - f(x)| \;\leq\; |f(x-y) - f(x_j)| + |f(x_j) - f(x)|$$

Since $x \in B_j$, we have $|x - x_j| \leq \delta_j$ so setting $y = x - x_j$ we have $x = x_j - y$ with $|y| < \delta_{x_j}$ and hence

$$|f(x_j) - f(x)| = |f(x_j) - f(x_j - y)| < \epsilon.$$

To bound $|f(x-y)-f(x_j)|$ we use the fact that $x \in B_j$ implies $|x-x_j| \leq \delta_{x_j}/2$ and $|y| < \delta_{x_j}$ to write

$$x - y = x_j - (x_j - x + y) = x_j - z$$

where $|z| \leq \delta_{x_j}$, so

$$|f(x_j) - f(x-y)| \leq |f(x_j) - f(x_j - z)| \leq \epsilon.$$

This gives the estimate if $x \in B_j$ for some j. Otherwise, x is outside the support of f and $f(x) = 0$. If $\operatorname{dist}(x, \operatorname{supp}(f)) > 1$, and $|y| < 1$, then

$$|f(x) - f(x-y)| = |0 - 0| = 0,$$

so we are done. □

Prop 8.5: If $1 \le p < \infty$, translation is continuous in the L^p norm.

Proof. We want to show that if $f \in L^p$

$$\lim_{y \to z} \|\tau_y f - \tau_z f\|_p \to 0.$$

Since $\tau_{y+z} = \tau_y \tau_z$, by replacing f by $\tau_z f$ it suffices to assume that $z = 0$.

If $g \in C_c$ and $|y| \le 1$, $\text{supp}(\tau_y g) \subset K$, for some compact K. By Lemma 8.4,

$$\int |\tau_y g - g|^p dm \le \|\tau_y g - g\|_u^p m(K) \to 0$$

If $f \in L^p$, by Proposition 7.9 there exists $g \in C_c$ with $\|g - f\|_p < \epsilon/3$, so

$$\|\tau_y f - f\|_p \le \|\tau_y(f - g)\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \le \frac{2}{3}\epsilon + \|\tau_y g - g\|_p$$

and we already showed the last term is small if $|y|$ is small enough. $\square$

Proposition 8.5 is false for $p = \infty$: $f = \chi_{[0,\infty)}$.

Defn: a function f on $\mathbb{R}^n$ is periodic if $f(x+k) = f(x)$ for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$.

Every periodic function is thus completely determined by its values on the unit cube $Q = [-\frac{1}{2}, \frac{1}{2})^n$.

Periodic functions may be regarded as functions on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Chapter 8.2: Convolutions

Defn: The **convolution** of f and g is

$$f * g(x) = \int f(x-y)f(y)dy,$$

for all x for which the integral exists

Fact: if f is measurable on $\mathbb{R}^n$ then $K(x,y) = f(x-y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ (Exercise 5).

Prop. 8.6: Assuming all the integrals exist,

a. $f * g = g * f$,

b. $(f * g) * h = f * (g * h)$,

c. For $z \in \mathbb{R}^n$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$,

d. If $A =$ closure of $\{x + y : x \in \text{supp}(f), y \in \text{supp}(y)\}$ then $\text{supp}(f * g) \subset A$.

Proof. **Proof of a:** Use Fubini's theorem and set $z = x - y$

$$f * g(x) = \int f(x-y)g(y)dy = \int f(z)gf(x-z)dy = g * f(x).$$

Proof of b: Use Fubini again,

$$\begin{aligned}(f * g) * h(x) &= \int\int f(y)g(x-z-y)h(z)dydz \\ &= \int\int f(y)g(x-y-z)h(z)dzdy \\ &= f * (g * h)(x).\end{aligned}$$

Proof of c:

$$\begin{aligned}\tau_z(f*g)(x) &= \int f(x-z-y)g(y)dy \\ &= \int \tau_z f(x-y)g(y)dy \\ &= \int \tau_z f(x-y)g(y)dy \\ &= (\tau_z f*)g(x).\end{aligned}$$

and by (a)

$$\tau_z(f*g)(x) = \tau_z(g*f)(x) = (\tau_z g)*f(x) = f*(\tau_z g)(x).$$

Proof of d: if $x \notin A$ and $y \in \text{supp}(g)$ then $x - y \notin \text{supp}(f)$. So for all y one of the two terms in $f(x-y)g(y)$ is zero. Thus $f*g(x) = 0$. $\square$

Young's inequality: If $f \in L^1$ and $g \in L^p$, $1 \leq p \leq \infty$, then $f * g$ exists for a.e. x, $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p$.

Proof. One can use Minkowski's inequality for integrals:

$$\begin{aligned} \|f * g\|p &= \|\int f(y)g(\cdot - y)dy\|_p \\ &\leq \int |f(y)| \|\tau_y g\|_p \\ &= \|f\|_1 \cdot \|\tau_y g\|_p. \end{aligned}$$

□

Also can think of this as a Banach space valued integral.

William Henry Young (1863–1942)

Grace Chisholm Young (1868–1944)

Prop. 8.8: If p and q are conjugate exponents, $f \in L^p$, and $g \in L^q$, then

a. $f * g(x\ )$ exists for every x,

b. $\|f * g\|_u \leq \|f\|_p \cdot \|g\|_q$,

c. $f * g$ is uniformly continuous,

d. If $1 < p, q < \infty$ then $f * g \in C_0(\mathbb{R}^n)$.

Proof. **Proof of a and b:** Holder's inequality.

Proof of c: If $1 \leq p < \infty$,

$$\|\tau_y(f * g) - f * g\|_u = \|\tau_y f - f) * g\|_\infty = \|\tau_y f - f\|_p \|g\|_q,$$

which tends to zero as $y \to 0$.

Proof of d:

Choose sequences $\{f_n\}$, $\{g_n\}$ of compact support converging to f, g in L^p, L^q respectively. Then $f_n * g_n \in C_c$ by Prop 8.6.d and

$$\|f * g - f_n * g_n\|_u \leq \|f - f_n\|_p \cdot \|g_n\|_q \|f\|_p \cdot \|g_n - g\|_q \to 0.$$

This proves d since C_0 is closure of C_c in uniformly topology. □

Defn: weak-L^p is space of functions so that

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p m(\{x : |f(x)| > \alpha\}) \right)^{1/p}.$$

For the sake of completeness we state also the following.

Prop 8.9: Suppose $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

a. (Young's Inequality, General Form) If $f \in L^p$, $g \in L^q$ then $f * g \in L^r$ and and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

b. Suppose also that $p > 1, q > 1$, and $r < \infty$. If $f \in L^p$ and $g \in$ weak-L^q, then $f * g \in L^r$ and $\|f * g\|_r \leq C_{pq} \|f\|_p [g]_q$. where C_{pq} is independent of f, g.

c. Suppose that $p = 1$ and $r = q > 1$. If $f \in L^1$ and $g \in$ weak-L^q, then $f * g$ is in weak-L^q and $[f * g]_q \leq C \|f\|_1$.

Proof. See Folland.

$f * g$ is at least as smooth as f or g is.

$$\partial^\alpha (f * g) = \partial^\alpha \int f(x-y)g(y)dy = \int \partial^\alpha f(x-y)g(y)dy = (\partial^\alpha f) * g$$

How do we justify interchanging differentiation and integration?

Prop. 8.10: If $f \in L^1$, and $g \in C_k$, and $\partial^\alpha g$ is bounded for $|\alpha| \leq k$ then $f * g \in C^k$ and $\partial^\alpha (f * g) = f * (\partial^\alpha g)$.

Proof: Immediate from Theorem 2.27.

Prop 8.11: If $f, g \in \mathcal{S}$ then $f * g \in \mathcal{S}$.

Proof. First, $f * g \in C_0$ by Proposition 8.10. Since

$$1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|)$$

we have

$$\begin{aligned}
(1 + |x|)^N |\partial^\alpha (f * g)(x)| &\leq \int (1 + |x - y|)^N |\partial^\alpha f(x - y)| (1 + |y|)^N |g(y) dy \\
&\leq \|f\|_{(N,\alpha)} \cdot \|g\|_{(N+n+1,\alpha)} \cdot \int (1 + |y|)^{-n-1} dy \\
&< \infty. \quad \square
\end{aligned}$$

Convolutions of functions on the torus $\mathbb{T}^n$ are defined just as for functions on $\mathbb{R}^n$.

All of the preceding results remain valid, with the same proofs.

Notation:

if ϕ is any function on $\mathbb{R}^n$ and $t > 0$, we set

$$\phi_t(x) = t^{-n}\phi(x/t).$$

This has same L^1 norm as ϕ but is more "concentrated" around zero if t is small; more "dispersed" if t is large.

Theorem 8.14: Suppose $\pi \in L^1$ and $\int \phi dx = a$.

a. If $f \in L^p$, $1 \leq p < \infty$, then $f * \phi \to af$ in the L^p norm as $t \to 0$.

b. If f is bounded and uniformly continuous, then $f * \phi \to af$ uniformly.

c. If $f \in L^\infty$ and f is continuous on an open set, then $f * \phi_t \to af$ uniformly on compact subsets of U.

Proof of a: Setting $y = tz$, we have

$$\begin{aligned} f * \phi(x) - af(x) &= \int [f(x-y) - f(x)]\phi_t(y)dy \\ &= \int [f(x-tz) - f(x)]\phi(z)dz = \int [\tau_{tz}f(x) - f(x)]\phi(z)dz \end{aligned}$$

so by Minkowski's inequality for integrals

$$\|f\phi_t - af\|_p \leq \int \|\tau_{tz}f - f\|_p |\phi(z)|dz.$$

Now $\|\tau_{tz}f - f\|_p$ is bounded by $2\|f\|_p$ and tends to zero as $t \to \infty$ for each z by Prop. 8.5. Thus (a) follows by the dominated convergence theorem.

Proof of b: The proof of (b) is exactly the same, with L^p norm replaced by the uniform norm. The estimate for $\|\tau_{tz}f - af\|_u$ is obvious, and $\|\tau_{tz}f - f\|_u \to 0$ by uniform continuity.

Proof of c: Given $\epsilon > 0$ let us choose a compact $E \subset \mathbb{R}^n$ such that $\int_{E^c} |\phi| dm < \epsilon$. Also, let K be a compact subset of U. If t is sufficiently small, then, we will have $x - tz \in U$ for all $x \in Ki$ and $z \in E$, so from the compactness of K it follows as in Lemma 8.4 that

$$\sup_{x \in K, z \in E} |f(x - tz) - f(x)| < \epsilon,$$

for small t. But then

$$\begin{aligned} \sup_{x \in K} |f * \phi_t(x) - af(x)| &\leq \sup_{x \in K} \int_E + \int_{E^c} [f(x - tz) - f(x)] \\ phi(z)|dz \\ &\leq \epsilon \int |\phi| + 2\|f\|_\infty \epsilon \end{aligned}$$

If we impose slightly stronger conditions on ϕ, we can also show that $f * \phi_t$ converges to af pointwise almost everywhere for $f \in L^p$.

The device in the following proof of breaking up an integral into pieces corresponding to the dyadic intervals $[2^k, 2^{k+1}]$ and estimating each piece separately is a standard trick of the trade in Fourier analysis.

Defn: Lebesgue set of f is

$$L_f = \{x : \lim_{t \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0\}.$$

Theorem 8.15: Suppose $\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for some $C < \infty$, $\epsilon > 0$ (this implies $\phi \in L^1$) and $\int \phi = a$. If $f \in L^p$, $1 \leq p \leq \infty$ then $f * \phi_t(x) \to af(x)$ for every x in the Lebesgue set of f; in particular at almost every x, and everywhere f is continuous.

Proof. If $x \in L_f$, then for any $\delta > 0$ there is a $\eta > 0$ so that

$$\int_{|y|<r} |f(x-y) - f(x)| dy \leq \delta r^n$$

for $r \leq \eta$. We claim

$$I_1 = \int_{|y|<\eta} |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \leq A\delta,$$

and

$$I_2 = \int_{|y|\geq\eta} |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \to 0$$

If we can prove these claims, then

$$\limsup_{t\to 0} |f * \phi_t(x) - af(x)| \leq A\delta,$$

for any $\delta > 0$, so

$$\limsup_{t\to 0} |f * \phi_t(x) - af(x)| = 0.$$

Proof of Claim 1: Let K be the integer so that $2^K \leq \eta/t < 2^{K+1}$, if $\eta/t \geq 1$ and $K = 0$ if $\eta/t < 1$.

Cut the ball $\{|y| < \eta\}$ into annuli

$$A_k = \{2^{-k}\eta \leq |y| \leq 2^{1-k}\eta\}$$

and a ball

$$B = \{2|y| \leq 2^{-K}\eta\}.$$

On A_k we have

$$\phi_t(y)| \leq Ct^{-n}|\frac{y}{t}|^{-n-\epsilon} \leq CT^{-n}(\frac{2^{-k}\eta}{t})^{-n-\epsilon},$$

and on the ball B we have

$$|\phi_t(y)| \leq Ct^{-n}.$$

Then

$$\begin{aligned} I_1 \;\le\; & \sum_{k=1}^{K} Ct^{-n}[2^{-K}\eta/t]^{-n-\epsilon}\int_{A_k}|f(x-y)-f(y)|dy \\ & +Ct^{-n}\int_B |f(x-y)-f(y)|dy. \end{aligned}$$

Thus

$$\begin{aligned} I_1 \;\le\; & C\delta\sum_{k=1}^{K}(2^{1-k}\eta)^n t^{-n}(2^{-k}\eta/t]^{-n-\epsilon} + C\delta t^{-n}(2^{-K}\eta)^n \\ \le\; & 2^n C\delta(\eta/t)^{-\epsilon}\sum_{k=1}^{K}2^{k\epsilon} + C\delta(2^{-K}\eta/t)^n \\ \le\; & 2^n C\delta(\eta/t)^{-\epsilon}\frac{2^{(K+1)\epsilon}-2^{\epsilon}}{2^{\epsilon}-1} + C\delta(2^{-K}\eta/t)^n \\ \le\; & 2^n C\delta C_{\epsilon} + C\delta. \end{aligned}$$

Proof of Claim 2: Let p' be the conjugate exponent to p and let χ be the characteristic function of $\{|y| \geq \eta\}$. By Hölder's inequality

$$\begin{aligned} I_2 &\leq \int_{|y|\geq\eta} |f(x-y)| + |f(x)| \cdot |\phi_t(y)| dy \\ &\leq \|f\|_p \|\chi\phi_t\|_{p'} + |f(x)| \|\chi\phi_t\|_1, \end{aligned}$$

so it suffices to show for $q = 1$ and $q = p'$ that

$$|\chi\phi_t\|_q \to 0,$$

as $t \to 0$

If $q = \infty$ this is easy

$$\|\chi\phi_t\|_\infty \leq Ct^{-n}(1 + (\eta/t)|^{-n-\epsilon} = Ct^{\epsilon}|t+\eta|^{-n-\epsilon} \leq C\eta^{-n-\epsilon}t^{\epsilon}.$$

If $q < \infty$,

$$\begin{aligned}
\|\chi\phi_t\|_q^q &= \int_{|y|\geq\eta} t^{-nq}|\phi(y/t)|^q dy \\
&= t^{n(1-q)} \int_{|y|\geq\eta/t} |\phi(z)|^q dz \\
&\leq t^{n(1-q)} \int_{\eta/t}^{\infty} r^{n-1-(n+\epsilon)q} dr \\
&\leq C_2 t^{n(1-q)} [\eta/t]^{n-(n+\epsilon)q} \\
&\leq C_3 t^{\epsilon q} \to 0 \quad \square
\end{aligned}$$

If $a = 1$ in the previous result, we call $]phi_t$ an approximate identity, as it furnishes an approximation to the identity operator on L^p by convolution operators.

This construction is useful for approximating L^p functions by functions having specified regularity properties. For example,

Prop. 8.17: C_c^∞ (and hence also S) is dense in L^p $1 \leq p < \infty$ and in C_0.

Proof. Given $f \in L^p$ and$\epsilon > 0$, there exists $g \in C_c$ with $\|f - g\|_p < \epsilon$ by Proposition 7.9. Let ϕ be a function in C_c^∞ such that $\int \phi dx = 1$. Then $g * \phi_t \in C_c^\infty$ and $\|g - g * \phi_t\|_p$ is small.

The same argument works for the uniform norm. $\square$

The C^∞ Urysohn Lemma: If $K \subset \mathbb{R}^n$ is compact and U is an open set containing K, there exists $f \in C_c^\infty$ such that $0 \leq f \leq 1$, $f = 1$ on K , and $\text{supp}(f) \subset U$.

Proof. Let δ be the distance from K to U^c, which is positive since K is compact. Let $V = \{x : \rho(x, K) < \delta/3\}$. Choose a nonnegative $\phi \in C_c^\infty$ so that $\int \phi dx = 1$ and $\phi(x) = 0$ if $|x| > \delta/3$. Set $f = \chi_V * \phi$. Then $f \in C_c^\infty$, $0 \leq f \leq 1$, $f = 1$ on K and $\text{supp}(f) \subset \{x : \rho(x, V) \leq 2\delta/3\} \subset U$. □

Chapter 8.3: The Fourier transform

Defn: a **unitary character** of a group G is a homomorphism $G \to \mathbb{T}$. For $G = \mathbb{R}$ a character satisfies

$$f(x + y) = f(x) \cdot f(y).$$

In harmonic analysis on groups we often try to write general functions as sums of characters.

Theorem 8.19: If $\phi\mathbb{R}^n\mathbb{T}$ is a measurable function and $\phi(x + y) = \phi(x)\phi(y)$, then $\phi(x) = \exp(ix\dot{\zeta})$ for some $\zeta\mathbb{R}^n$. A similar result holds for functions on $\mathbb{T}^n$.

Proof. First consider $n = 1$. Let $a \in \mathbb{R}$ be such that $\int_0^a \phi(t)dt \neq 0$; there is such an a or Lebesgue differentiation implies $\phi = 0$.

Set $A = (\int_0^a \phi(t)dt)^-1$. Then

$$\phi(x) = A\int_0^a \phi(x)\phi(t)dt = A\int_0^a \phi(x+t)dt = A\int_x^{x+a} \phi(x+t)dt.$$

Thus ϕ is continuous (even Lipschitz since its derivative is bounded). Moreover

$$\phi(x) = A(\phi(x+a) - \phi(x)) = A(\phi(a) - \phi(1))\phi(x) = B\phi(x).$$

Thus $(e^{-Bx}\phi(x))' = 0$ so this function is constant C, so$\phi(x) = Ce^{Bx}$. Since ϕ takes values in $\mathbb{T}$ and $\phi(0) = 1$, we have $C = 1$ and B is imaginary.

If $\{e_j\}$ is the standard basis for $\mathbb{R}^n$ then the argument above shows

$$\psi_t(t) = \phi(te_j)\exp(x \cdot 2\pi i\xi_j)$$

so

$$\phi(x) = \phi(\sum x_j e_j) = \prod \phi(x_j e_j) = \prod \exp(x_j \cdot 2\pi i\xi_j) = \exp(x \cdot 2\pi\xi).$$

$\square$

For $\mathbb{T}^n$ we must also have $\xi \in \mathbb{Z}^n$.

The idea now is to decompose more or less arbitrary functions terms of the exponentials. In the case of $\mathbb{T}^n$ this works out very simply for L^2 functions.

Theorem 8.20: Let $E_\kappa(x) = \exp(2\pi i\kappa \cdot x)$. Then $\{E_\kappa\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$.

Proof. Verification of orthonormality is an easy exercise in calculus; by Fubini's theorem it boils down to the fact that $\int_0^1 \exp(2\pi i\kappa t)dt$ equals 1 if $\kappa = 0$ and equals 0 otherwise.

Next, $E_\kappa \cdot E_\lambda = E_{\kappa+\lambda}$ so the set of finite linear combinations is an algebra. It separates points so contains all continuous functions by Stone-Weierstrass, hence is dense in L^2. Hence the linear span is complete and so these functions form a basis. $\square$

Defn: for $f \in L^2(\mathbb{T})$ its **Fourier Transform** is

$$\hat{f}(\kappa = \langle f, E_\kappa \rangle = \int_{\mathbb{T}^n} f(x) \exp(-2\pi i \kappa \cdot x) dx$$

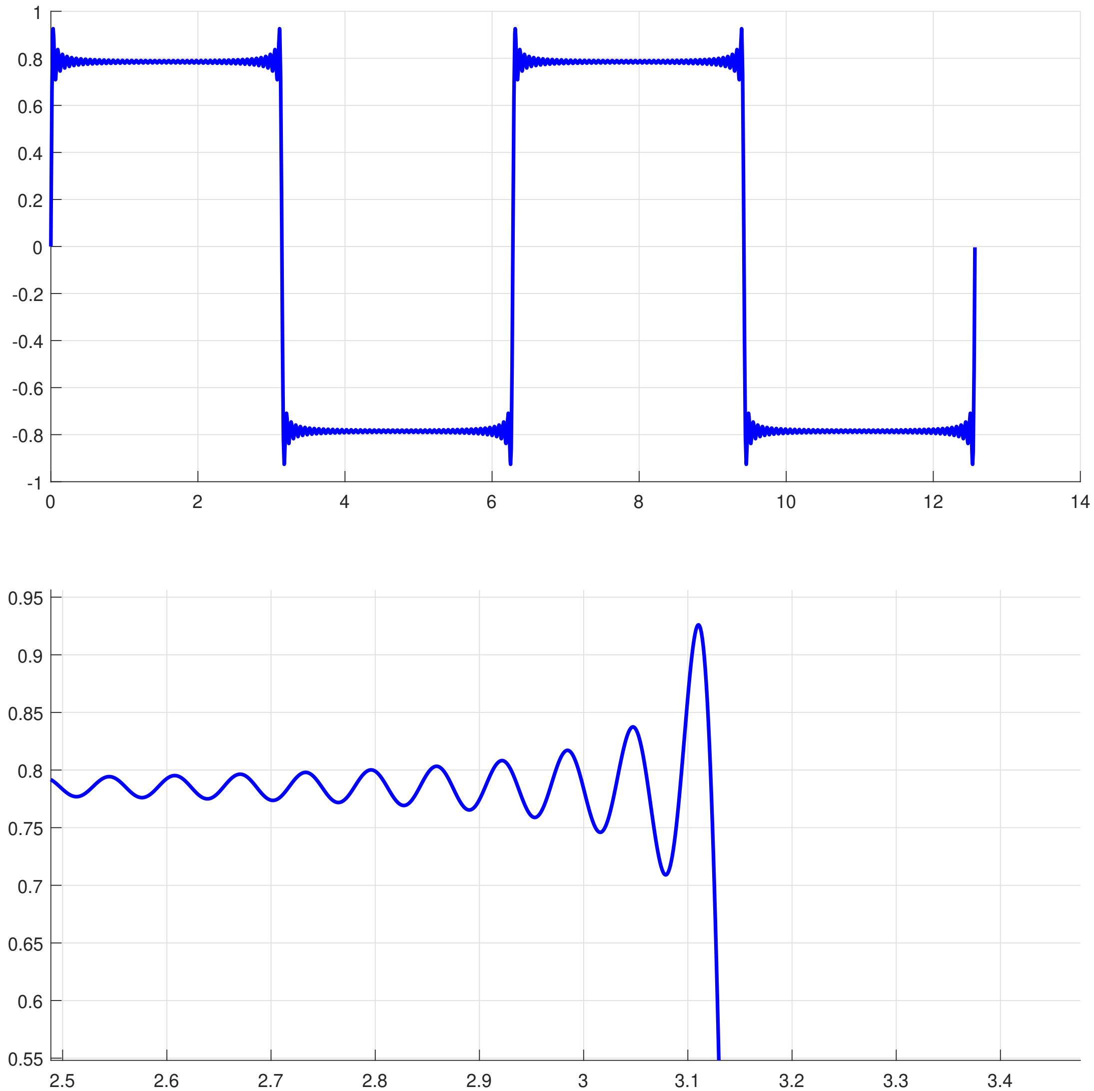
and

$$\sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa) E_\kappa,$$

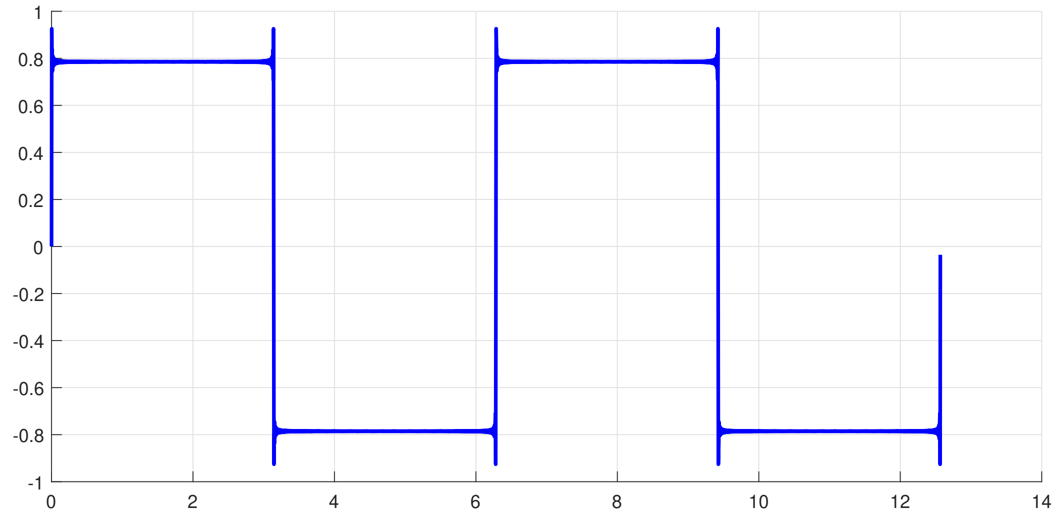
is the **Fourier Series** of f.

The Fourier transform is an isometry from $L^2(\mathbb{T}^n)$ to $\ell^2(\mathbb{Z}^n)$ and the Fourier series converges in the L^2 norm.

We mentioned earlier in the course that the Fourier series of a continuous function need not converge pointwise to the function.



Gibbs phenomena


1
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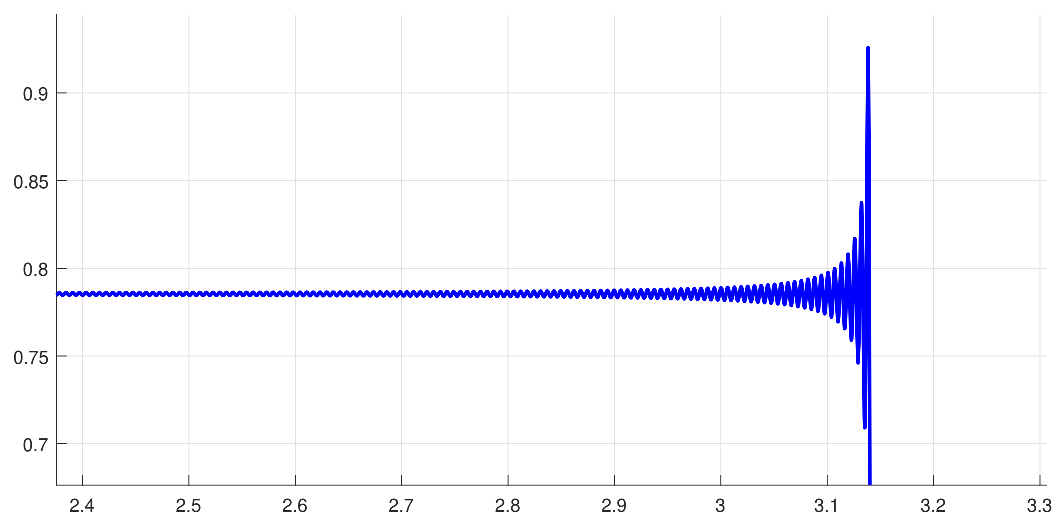
0.9
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3.1
3.2
3.3

Jean Baptiste Joseph Fourier (1768–1830)

The Hausdorff-Young Inequality: Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent to p. If $f \in L^p(\mathbb{T}^n)$, then $\hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \leq \|f\|_p$.

Proof. Since $\|\hat{f}\|_\infty \leq \|f\|_1$ $\|\hat{f}\|_2 \leq \|f\|_2$ this follows from the Riesz-Thorin interpolation theorem. □

M. Riesz-Thorin interpolation theorem: If T is a linear map from $L^{p_1} + L^{p_2}$ to $L^{q_1} + L^{q_2}$ so that

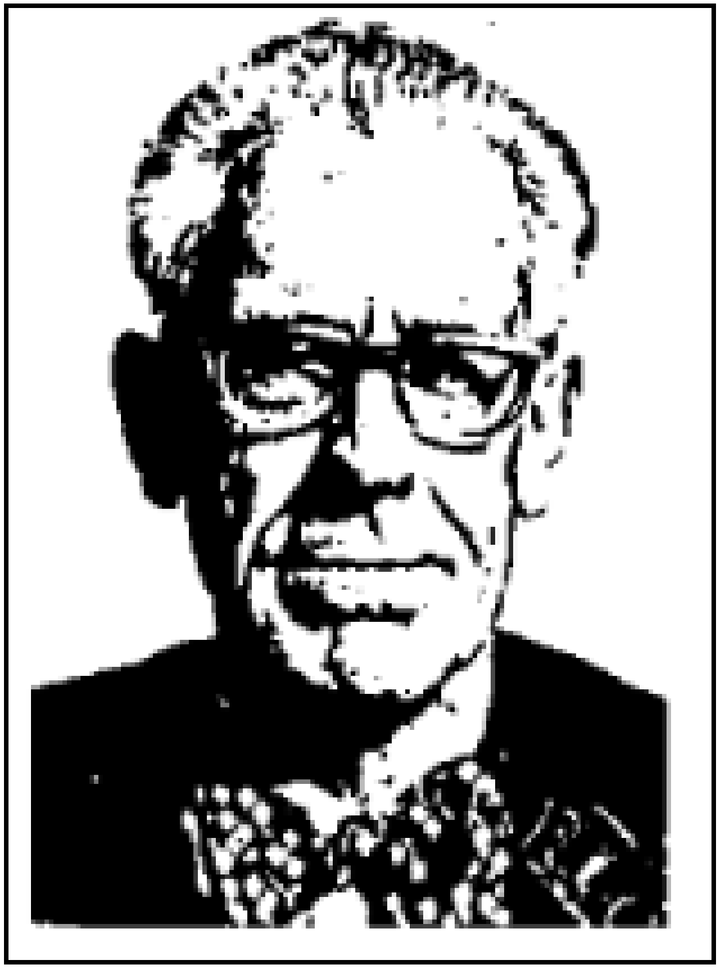
$$\|Tf\|_{q_1} \leq M_1 \|f\|_{p_2}, \qquad \|Tf\|_{q_2} \leq M_2 \|f\|_{p_2},$$

Then T is bounded from L^{p_t} to L^{q_t} where

$$\frac{1}{p_t} = \frac{1-t}{p_1} + \frac{t}{p_2}, \frac{1}{q_t} = \frac{1-t}{q_1} + \frac{t}{q_2}, 0 \leq t \leq t$$

with norm $M \leq M_1^{1-t} M_2^t$.

Thus if T is bounded $L^2 \to L^2$ and $L^1 \to L^\infty$, then it is bounded $L^p \to L^q$ for $1 \leq p \leq 2$ and $q =$ conjugate of p.



Olof Thorin (1912–2004)

Situation in $\mathbb{R}^n$ is more intricate since $f \in L^2$ does not imply

$$\int f(x) \exp(ix \cdot \xi) dx$$

exists. Need to restrict to $f \in L^1$.

Defn: for $f \in L^1$ define the **Fourier Transform** of f as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}}^{n} f(x) \exp(-2\pi i \xi \cdot x) dx.$$

Clearly $\|\hat{f}\|_u \leq q\|f\|_1$.

Theorem. 8.22: Suppose $f, g \in L^1(\mathbb{R}^n)$.

a. $\widehat{(\tau_y f)}(\xi) = \exp(-2\pi i \xi \cdot y)\hat{f}(\xi)$ and $\tau_\eta(\hat{f}) = \hat{h}$ where $h = \exp(2\pi i \eta \cdot x) f(x)$.

b. If T is an invertible linear transformation of $\mathbb{R}^n$ and $S = (T*)^{-1}$ is its inverse transpose, then $\widehat{(f \circ T)} = |\det T|^{-1} \hat{f} \circ S$. In particular, if T is a rotation, then $\widehat{(f \circ T)} = \hat{f} \circ T$. If $Tx = x/t$ then then $\widehat{(f \circ T)}(\xi) = t^n \hat{f}(t\xi)$. and $\widehat{f_t}(\xi) = \hat{f}(t\xi)$.

c. $\widehat{f * g.} = \hat{f}\,\hat{g}$.

d. If $x^\alpha f \in L^1$ for $|\alpha| \le k$ then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = [(-2\pi i x)^\alpha f]^\wedge$.

e. If $f \in C^k, \partial^\alpha f \in L^1$ for $|\alpha| \le k$ and $\partial^\alpha \in C_0$ for $|\alpha| \le k-1$, then $(\partial^\alpha f)^\wedge(\xi) = (2\pi\xi)^\alpha \hat{f}(\xi)$.

f. (**The Riemann-Lebesgue Lemma**) $\mathcal{F}(L^1) \subset C_0$.

Proof of a:

$$
\begin{aligned}
(\tau_y f)^\wedge(\xi) &= \int f(x-y)\exp(-2\pi i\xi\cdot x)dx \\
&= \int f(x)\exp(-2\pi i\xi\cdot(x+y))dx \\
&= \exp(-2\pi i\xi\cdot y\hat{f}(\xi).
\end{aligned}
$$

Proof of b:

$$
\begin{aligned}
(f\circ T)^\wedge(\xi) &= \int f(Tx)\exp(-2\pi i\xi\cdot x)dx \\
&= |\det T|^{-1}\int f(x)\exp(-2\pi i\xi\cdot T^{-1}x)dx \\
&= |\det T|^{-1}\int f(x)\exp(-2\pi iS\xi\cdot x)dx \\
&= |\det T|^{-1}\hat{f}(D\xi).
\end{aligned}
$$

Proof of c:

$$\begin{aligned}
(f*g)^\wedge(\xi) &= \int\int f(x-y)g(y)\exp(-2\pi i\xi\cdot x)dydx \\
&= \int\int f(x-y)\exp(-2\pi i\xi\cdot(x-y))g(y)\exp(-2\pi i\xi\cdot y)dydx \\
&= \hat{f}(\xi)\int g(y)\exp(-2\pi i\xi\cdot y)dy \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

Proof of d: By induction on $|\alpha|$,

$$\begin{aligned}
\partial^\alpha\hat{f}(\xi) &= \partial_\xi^\alpha\int f(x)\exp(-2\pi i\xi\cdot x)dx \\
&= \int f(x)(-2\pi x)^\alpha\exp(-2\pi i\xi\cdot x)dx \\
&= (-2\pi x)^\alpha\hat{f}(\xi).
\end{aligned}$$

Proof of e: Since $f \in C_0$ we can use integration by parts:

$$\int f'(x)\exp(-2\pi i\xi\cdot x)dx \;=\; f(x)\exp(-2\pi i\xi\cdot x)|_{-\infty}^{\infty}$$

$$-\int f(x)(-2\pi\xi)^{\alpha}\exp(-2\pi i\xi\cdot x)dx$$

$$= \;= 2\pi i\xi\hat{f}(\xi).$$

General case follows from induction on $|\alpha|$.

Proof of f: If $f \in C^1 \cap C_c$ then $|\xi|\hat{f}(\xi 0$ is bounded, so $\hat{f} \in C_0$. This set is dense in L^1 and C_0 is closed in the uniform norm so $\mathcal{F}L^1 \subset C_0$.

Corollary 8.23: $\mathcal{F}$ is a continuous map of $\mathcal{S}$ to itself.

Proof. $f \in \mathcal{S}$ implies $x^\alpha \partial^\beta f \in L^1 \cap C_0$ for all α, β, so

$$(x^\alpha \partial^\beta f)^\wedge = (-1)^{\alpha|}(2\pi i)^{|\beta|-|\alpha|}\partial^\alpha)\xi^\beta f\|_u,$$

so $\partial^\alpha x^\beta \hat{f}$ is bounded for all α, β. Thus $\hat{f} \in \mathcal{S}$.

$$\|(x^\alpha \partial^\beta f)^\wedge\|_u \leq \|x^\alpha \partial^\beta f\|_1 \leq C\|(1+|x|)^{n+1} x^\alpha \partial^\beta f\|_u$$

so $\mathcal{F}$ is continuous as a map between TVSs. $\square$

Prop. 8.24: If $f(x) = e^{-\pi a|\mathbf{x}|^2}$, $a > 0$ them $\hat{f} = a^{-n/1} e^{-\pi|x|^2/a}$.

Proof. First we consider dimension $n = 1$. Since

$$(e^{-\pi a x^2})' = -2\pi a x e^{-\pi z x^2},$$

we have

$$\begin{aligned}
(\hat{f})'(\xi) &= (-2\pi i x f)^\wedge \\
&= (\frac{i}{a} f')^\wedge \\
&= \frac{i}{a}(2\pi i \xi \hat{f}(\xi) \\
&= -\frac{2\pi}{a} \xi \hat{f}(\xi)
\end{aligned}$$

If follows that

$$\hat{f}(\xi) = C e^{2\pi\xi^2/a},$$

for some constant C. To find C, set $\xi = 0$

$$\hat{f}(0) = \int e^{-\pi a x^2} dx = \frac{1}{\sqrt{a}}.$$

This formula is Prop 2.53 and is proven with a famous trick using polar coordinates

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-ax^2} dx\right)^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-a(x^2+y^2)^2} dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} re^{-ar^2} drd\theta \\
&= \int_0^{2\pi}\int_0^{\infty} re^{-ar^2} drd\theta \\
&= 2\pi \cdot \frac{1}{2a} e^{-ar^2}|_0^{\infty} \\
&= \frac{\pi}{a}
\end{aligned}$$

The $n > 1$ dimensional case follows using Fubini's theorem

$$\begin{aligned}
\hat{f}(\xi) &= \prod_{k=1}^{n} \int \exp(-\pi a x_k^2 - 2\pi \xi_k x_k) dx_k \\
&= \prod_{k=1}^{n} \left[a^{-1/2} \exp(-\pi \xi_k^2/a)\right] \\
&= a^{-n/2} \exp(-\pi|\xi|_k^2/a).
\end{aligned}$$

Lemma 8.25: If $f, g \in L^1$, then $\int \hat{f} \cdot g dx = \int f \cdot \hat{g} dx$.

Proof. Both integrals equal

$$\int\int f(x)g(\xi)e^{-2\pi i\xi\cdot x}dxdxi.$$

□

The Fourier Inversion Theorem: If $f \in L^1$ and $\hat{f} \in L^1$ then f agrees almost everywhere with a continuous function f_0 and $(\hat{f})^\vee = (f^\vee)^\wedge = f_0$.

Proof. Given $t > 0$ and $x \in \mathbb{R}^n$, set

$$\phi(\xi) = \exp(2\pi i \xi cdot x - \pi t^2 |\xi|^2)).$$

By Theorem 8.22a and Proposition 8.24,

$$\hat{\phi}(y) = t^{-n} \exp(-\pi |x - y|^2 / t^2) = g_t(x - y),$$

where $g(x) = \exp(-\pi |x|^2)$, and the subscript t has the meaning in (8.13), dilating in L^1.

By Lemma 8.25,

$$\int e^{-\pi t^2 xi|^2} e^{2\pi i \xi \cdot x} \hat{f}(\xi) d\xi = \int f \hat{\phi} = \int \hat{f} \phi = f * g_t(x).$$

Since $\int g dx = 1$, by Theorem 8.14 we have $f * g_t f$ in the L^1 norm as $t \to 0$. On the other hand, since $\hat{f} \in l^1$, the dominated convergence theorem yields

$$\lim_{t \to 0} \int e^{-\pi t^2 xi|^2} e^{2\pi i \xi \cdot x} \hat{f}(\xi) d\xi = \int e^{2\pi i \xi \cdot x} \hat{f}(\xi) d\xi = (\hat{f})^\vee(x).$$

It follows that $f = (\hat{f})^\vee$ a.e., and similarly $(f^\vee)^\wedge = f$ a.e. Since $(\hat{f})^\vee$ and $(f^\vee)^\wedge$ are Fourier transforms of L^1 functions they are continuous and the proof is complete. $\square$

Cor 8.27: If $f \in L^1$ and $\hat{f} = 0$ then $f = 0$ a.e.

Cor 8.28: If $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ to itself.

Proof. We already know that $\mathcal{F}$ is continuous from $\mathcal{S}$ into itself. By the Fourier inversion theorem, $f \to f^\vee$ where $f^\vee(x) = \hat{f}(-x)$ is a continuous inverse, so $\mathcal{F}$ is an isomorphism. □

The Plancherel Theorem: If $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\mathcal{F}$ restricted to $L^1 \cap L^2$ extends to a unitary isomorphism on l^2.

Proof. Let $\mathcal{X} = \{f \in L^1 : \hat{f} \in L^1\}$. Since $\hat{f} \in L^1$ implies f is bounded, we have $\mathcal{X} \subset L^2$. $\mathcal{X}$ is dense in L^2 since it contains $\mathcal{S}$ and $\mathcal{S}$ is dense.

Given $f, g \in \mathcal{X}$, let $h = \overline{\hat{g}}$. By the inversion theorem,

$$\begin{aligned}
\hat{h}(\xi) &= \int e^{-2\pi i \xi \cdot x} \overline{\hat{g}(x)} dx \\
&= \int \overline{e^{2\pi i \xi \cdot x} \hat{g}(x)} dx \\
&= \overline{g(\xi)}
\end{aligned}$$

Thus

$$\int f \bar{g} = \int f \hat{h} = \int \hat{f} h = \int \hat{f} \overline{\hat{g}},$$

so $\mathcal{F}$ preserves the inner product on L^2. Thus it preserves norms (take $g = f$). Since $\mathcal{F}(\mathbf{x}) = \mathcal{X}$ by the inversion theorem, $\mathcal{F}$ extends to an unitary isomorphism of L^2

It remains only to show that this extension agrees with $\mathcal{F}$ on all of $L^1 \cap L^2$. But if $f \in L^1 \cap L^2$ and $g(x) = \exp(-\pi|x|^2)$, we have $f * g_t \in L^1$ by Young's inequality and $(f * g_t)^\wedge \in L^1$ because

$$(f * g_t)^\wedge(\xi) = e^{-\pi t|\xi|^2} \hat{f}(\xi)$$

and $\hat{f}$ is bounded. Hence $f * g_t \in \mathcal{X}$. By Theorem 8.14 $f * g_t \to f$ in both L^1 and L^2 norms. Thus $(f * g_t)^\wedge \to \hat{f}$ in both L^2 and uniform norms. This completes the proof. □

We have thus extended the domain of the Fourier transform from l^1 to $L^1 + L^2$.

Just as on $\mathbb{T}^n$, the Riesz- Thorin theorem yields the following result.

The Hausdorff-Young Inequality: Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent to p. If $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q(\mathbb{R}^n)$ and $\|\hat{f}\|_q \leq \|f\|_p$.

If $ftf \in L^1$ the inversion formula says f can be written as a superposition of the complex exponential functions

$$f(x) = \int \hat{f}(x) e^{2\pi \xi \cdot x} d\xi.$$

This is the **Fourier integral representation** of f.

This formula remains valid in spirit for L^2, although the integral (as well as the integral defining $\hat{f}$) may not converge pointwise. The interpretation of the inversion formula will be studied further in the next section.

Theorem 8.31: If $f \in L^1(\mathbb{R}^n)$, the series

$$\sum_{k\in\mathbb{Z}} f(x-k)$$

converges pointwise almost everywhere and in L^1 to a function Pf such that $\|Pf\|_1 \leq \|f\|_1$. Moreover, for $k \in \mathbb{Z}^n$, $(Pf)^\wedge(k) = \hat{f}(k)$, i.e., the Fourier series of the periodic function Pf equals the Fourier transform of f restricted to the integer lattice.

Proof. Let $Q = [-\frac{1}{2}, \frac{1}{2})^n$. Then $\mathbb{R}^n$ is disjoint union of integer translates of Q, so

$$\begin{aligned}\int_Q \sum_k |f(x-k)dx &= \sum_k \int_{Q+k} |f(x)|dx \\ &= \int_{\mathbb{R}^n} |f(x)|dx.\end{aligned}$$

Now apply Theorem 2.25 (if $\{f_k\} \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges a.e. to $f \in L^1$). This implies $\sum f(x-k)$ converges a.e. and in L^1 to a function

$Pf \in L^1$ such that $\|Pf\|_1 \leq \|f\|_1$. Moreover,

$$\begin{aligned}
(Pf)^\wedge(k) &= \int_Q \sum_k f(x-k)e^{-2\pi i k\cdot x}dx \\
&= \sum_k \int_{Q+k} f(x)e^{-2\pi i k\cdot(x+k)}dx \\
&= \sum_k \int_{Q+k} f(x)e^{-2\pi i k\cdot x}dx \\
&= \int_{\mathbb{R}^n} f(x)e^{-2\pi i k\cdot x}dx \\
&= \hat{f}(k).
\end{aligned}$$

$\square$

If we impose stronger conditions on f we get a better conclusion.

The Poisson Summation Formula: Suppose $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| < C(1+|x|)^{-n-\epsilon}$ and $|\hat{}f(x)| < C(1+|x|)^{-n-\epsilon}$ for some $C < \infty$ and $\epsilon > 0$. Then

$$\sum_f f(x+k) = \sum_k \hat{f}(k) e^{2\pi i k\cdot x}.$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$. In particular, taking $x = 0$,

$$\sum_f f(k) = \sum_k \hat{f}(k).$$

Proof. The absolute and uniform convergence of the series follows from the fact that $\sum_k (1+k)^{-n-\epsilon} < \infty$, which can be seen by comparing the latter series to the convergent integral $\int (1+|x|^{-n-\epsilon} dx$. Thus the function $P] = \sum_k f(x-k)$ is continuous on $\mathbb{T}^n$, hence bounded and in L^2. so Theorem 8.31 implies that the series $\sum \hat{f}(k) e^{2\pi i k\cdot x}$ converges in L^2 to Pf. Since it also converges uniformly, its sum equals Pf pointwise. □

Poisson's summation formula appears in Ramanujan's notebooks and can be used to prove some of his formulas, in particular it can be used to prove one of the formulas in Ramanujan's first letter to Hardy.

Siméon Denis Poisson (1781–1840)

Method of images: In partial differential equations, the Poisson summation formula provides a rigorous justification for the fundamental solution of the heat equation with absorbing rectangular boundary by the method of images.

Statistical study of time-series: if f is a function of time, then looking only at its values at equally spaced points of time is called "sampling." In applications, typically the function f is band-limited, meaning that there is some cutoff frequency such that the Fourier transform is zero for frequencies exceeding the cutoff. The summation information guarantees that no information is lost by sampling, since $\hat{f}$ can be reconstructed from these sampled values, then, by Fourier inversion, so can f. This leads to the NyquistShannon sampling theorem.

Ewald summation: Computationally, the Poisson summation formula is useful since a slowly converging summation in real space is guaranteed to be converted into a quickly converging equivalent summation in Fourier space.

Lattice points in a sphere: The Poisson summation formula may be used to derive Landau's asymptotic formula for the number of lattice points in a large Euclidean sphere.

Number theory: Poisson summation can also be used to derive a variety of functional equations including the functional equation for the Riemann zeta function.

Sphere packings: Cohn and Elkies (2003) proved an upper bound on the density of sphere packings using the Poisson summation formula, which subsequently led to a proof of optimal sphere packings in dimension 8 and 24.

Chapter 8.4: Summation of Fourier integrals and series

In last section we saw hop wf can be recovered from $\hat{f}$ if $\hat{f} \in L^1$.

• When is $\hat{f}$ in L^1? If f is smooth enough.

• How to recover f when $\hat{f} \notin L^1$? Approximate by smooth functions and take limit.

If $f \in L^1$ then $\hat{f}$ is bounded, so $\hat{f} \in L^1$ if it decays fast enough at infinity, i.e, $|x|^\alpha \hat{f}$ is bounded for $|\alpha|$ large enough. This happens if $\partial^\alpha f \in L^\cap C_0$.

Theorem 8.33: Suppose f is periodic and absolutely continuous on $\mathbb{R}$ and $f' \in L^p$ for some $p > 1$. Then $\hat{f} \in \ell^1(\mathbb{Z})$.

Proof. Since $p > 1$, set $C_p = \sum_{k=1}^{\infty} k^{-p} < \infty$. Since $L^p(\mathbb{T}) \subset L^2(\mathbb{T})$ for for $p > 2$, we may assume that $p \leq 2$. Integration by parts (Theorem 3.36) shows that

$$(f')^\wedge(k) = 2\pi i k \hat{f}(k).$$

By Hölder and Hausdorff-Young, if q is the conjugate exponent to p,

$$\begin{aligned}
\sum_{k\neq 0} |\hat{f}(k)| &\leq \left(\sum_{k\neq 0} (2\pi|k|)^{-p}\right)^{1/p} \cdot \left(\sum_{k\neq 0} (2\pi|k\hat{f}(k)|^q\right)^{1/q} \\
&= \frac{(2C_p)^{1/p}}{2\pi} \|(f')^\wedge\|_q \\
&\leq \frac{(2C_p)^{1/p}}{2\pi} \|f'\|_p
\end{aligned}$$

Adding $|f(0)|$ to both sides, we see that $\|\hat{f}\|_1 < \infty$. □

To recover f from $\hat{f}$ when $\hat{f} \notin L^1$, we convolve f with a smooth approximation of the identity ϕ_t. This convolution converges to f as $t \to 0$. When we take the Fourier transform of the convolution, we get the product of $\hat{t}$ and $\hat{\phi}$. Since ϕ is smooth the latter decays quickly, so the product is in L^1. So convolution can be recovered from its Fourier transform.

Lemma 8.34: If $f, g \in L^2(\mathbb{R}^n)$ then $(\hat{f} \cdot \hat{g})^\vee = f * g$.

Proof. $\hat{f} \cdot \hat{g} \in L^1$ by Plancherel's theorem and Hölder's inequality. Thus $(\hat{f} \cdot \hat{g})^\vee$ is defined.

Given $x \in \mathbb{R}^n$ let $h(y) = \overline{g(x-y)}$ Then

$$\hat{h}(\xi) = \overline{\hat{g}(\xi)} \exp(-2\pi i \xi \cot x).$$

Since $\mathcal{F}$ is unitary on L^2 (preserves norms)

$$f * g(x) = \int f\bar{h} = \int \hat{f}\overline{\hat{h}} = \int \hat{f}(\xi)\hat{g}(\xi)e^{2\pi i \xi \cdot x} = (\hat{f} \cdot \hat{g})^\wedge. \quad \square$$

Theorem 8.35: Suppose that $\Phi \in L^1 \cap C_0$, $\Phi(O) = 1$, and $\phi = \Phi^\vee \in L^1$. Given $f \in L^1 + L^2$, for $t > 0$ set

$$f^t(x) = \int \hat{f}(\xi)\Phi(t\xi)e^{2\pi i \xi \cdot x} dxi.$$

a. If $f \in L^p$, $1 \leq p < \infty$ then $f^t \in L^p$ and $\|f^t - f\|_p \to 0$ as $t \to O$.

b. If f is bounded and uniformly continuous, then so is f^t, and $f^t \to f$ uniformly as $t \to 0$.

c. Suppose also that $\phi(x) \leq C(1+|x|)^{-n-\epsilon}$ for some $C < \infty$ and $\epsilon > 0$. Then $f^t(x) \to f(x)$ for every x in the Lebesgue set of f.

Proof. We have $f = f_1 + f_2$ where $f_1 \in L^1$ and $f_2 \in L^2$. Since $\hat{f}_1 \in L^\infty$, $\hat{f}_2 \in L^2$ and $\Phi \in (L^1 \cap C_0) \subset (L^1 \cap L^2)$, the integral defining f^t converges absolutely for every x. Moreover, if $\phi_t(x) = t^{-n}\phi(x/t)$ we have

$$\Phi(t\xi) = (\phi_t^\wedge(\xi)$$

by the inversion theorem and Theorem 8.22.b (Properties of $\mathcal{F}$). Also $\int \phi(x)dx = \Phi(0) = 1$.

Since $\phi, \Phi \in L^1$ we have $f_1 * \phi \in L^1$ and $\hat{f}_1 \cdot \Phi \in L^1$, so by Theorem 8.22.c and the inversion theorem

$$\int \hat{f}_1(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot x}d\xi = f_1 * \phi(t).$$

Also, $\phi \in L^2$ by the Plancherel theorem, so by Lemma 8.34 $(\hat{f}\hat{g})^\vee = f * g)$,

$$\int \hat{f}_2(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot x}d\xi = f_2 * \phi(t).$$

Adding these, $f^t = f * \phi_t$, so the assertions follow from Theorems 8.14 and 8.15 (properties of approximations of the identity). □

Theorem 8.36: Suppose that $\Phi \in C(\mathbb{R}^n)$ satisfies

$$\Phi(\xi)| \leq \frac{C}{(1+|||)^{n+\epsilon}}, \qquad \Phi^{\vee}(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}},$$

and $\Phi(0) = 1$. Given $f \in L^1(\mathbb{T}^n)$, for $t > 0$ set

$$f^t(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\Phi(tk)e^{2\pi i \xi \cdot x},$$

(which converges absolutely since $\sum_k |\Phi(tk)| < \infty$). Then

a. If $f \in L^p(\mathbb{T}^n)$, $1 \leq p < \infty$, then $\|f^t - f\|_p \to 0$ as $t \to 0$ and if f is continuous, then $f^t \to f$ uniformly as $t \to 0$.

b. $f^t(x) \to f(x)$ at every Lebesgue point of f.

Proof of a: As before, let

$$\phi = \Phi^{\vee}, \qquad \phi(t) = t^{-n}\phi(x/t).$$

Then $(\phi_t)^{\wedge}(\xi) = \Phi(t\xi)$ and ϕ_t satisfies the hypothesis of the Poisson summation formula, so

$$\sum_{k\in\mathbb{Z}^n} \phi_t(x-k) = \sum_{k\in\mathbb{Z}^n} \Phi(tk)e^{2\pi i k\cdot x}.$$

Let us denote the common value of these sums by $\psi_t(k)$. Then

$$(f * \psi_t)^{\wedge}(k) = \hat{f}(k)\psi_t(k) = \hat{f}(k)\Phi(tk) = (f^t)^{\wedge}(k).$$

Hence $f^t = f * \psi_t$. Hence, by Young's inequality and Theorem 8.31 we have

$$\|f^t\|_p \leq \|f\|_p\|\psi_t\|_1 \leq \|f\|_p\|\phi_t\|_1 \leq \|f\|_p\|\phi\|_1$$

so the operators $f \to f^t$ are uniformly bounded on L^p, $1 \leq p \leq \infty$.

Suppose f is a trig polynomial (f is periodic and the Fourier transform has finitely many non-zero entries) Since Φ is continuous and $\Phi(0) = 1$ we have $f^t \to t$ uniformly and in L^p on $\mathbb{T}^n$. But trig polynomials are dense in $C(\mathbb{T}^n)$ by the Stone-Weierstrass theorem, so also dense in L^p for $1 \leq p < \infty$. Therefore (a) follows from Prop 5.17 (if we have uniformly bounded operators $\{T_n\}$ and $T_n x \to Tx$ on a dense set then $T_n \to T$ strongly).

Proof of b: Suppose x is in the Lebesgue set and, by translation, assume $x = 0$. Set $Q = [\frac{1}{2}, \frac{1}{2})^n$.

$$\begin{aligned} f^t(0) &= f * \psi_t(0) \\ &= \int_Q f(x)\phi_t(-x)dx + \sum_{k\neq 0} \int_Q f(x)\phi_t(-x+k)dx. \end{aligned}$$

Since

$$|\phi_t(x)| \leq Ct^{-n}(1+|x|/t)^{-n-\epsilon} \leq t^\epsilon |x|^{-n-\epsilon},$$

for $x \in Q$and $k \neq 0$, we have

$$\phi_t(-x+k) \leq C2^{n+\epsilon} t^\epsilon |k|^{-n-\epsilon},$$

and hence

$$\sum_{k\neq 0} \int_Q |f(x)\phi_t(-x+k)|dx \leq \left(C2^{n+\epsilon} \cdot \|f\|_1 \cdot \sum_{k\neq 0} k^{-n-\epsilon} \right) t^\epsilon$$

which vanishes as $t \to 0$.

On the other hand, if we define $g = f\chi_Q \in L^1(\mathbb{R}^n)$, then 0 is in the Lebesgue set of g (because 0 is in the interior of Q, and the condition that 0 be in the Lebesgue set of g depends only on the behavior of g near 0), so by Theorem 8.15 (convergence of approximation of identity on Lebesgue set),

$$\lim_{t\to 0} \int_Q f(x)\phi_t(-x)dx = \lim_{t\to 0} g * \phi_t(O) = g(O) = f(O).$$

Example - Gauss kernel:

$$\Phi(\xi) = e^{-\pi|\xi|^2}, \qquad \phi(x) = \Phi^\vee(x) = e^{-\pi|x|^2}.$$

This is connected to solution of the heat equation.

Example - Possion kernel: For $n = 1$,

$$\begin{aligned} \Phi(\xi) &= e^{-2\pi|\xi|}, \\ \phi(x) &= \int_{-\infty}^{0} e^{2\pi(1+ix)\xi} d\xi \\ &= \frac{1}{2\pi}\left[\frac{1}{1+ix} + \frac{1}{1-ix}\right] \\ &= \frac{1}{\pi(1+x^2)}. \end{aligned}$$

This is connected to solving Dirichlet problem in a half-plane.

Example - Able summation:

$$\Phi(\xi) = e^{-2\pi|\xi|},$$

Make substitution $r = e^{-2\pi t}$ and write $A_r f$ instead of f^t,

$$\begin{aligned}
A_r f(x) &= \sum_{k\in\mathbb{Z}} r^{|k|}\hat{f}(k)e^{-2\pi ikx} \\
&= \hat{f}(0) + \sum_{k=1}^{\infty} r^k[\hat{f}(k)e^{2\pi ikx} + \hat{f}(-k)e^{-2\pi ikx}] \\
&= \sum_{k=0}^{\infty} \hat{f}(k)z^k + \sum_{k=1}^{\infty} \hat{f}(-k)\overline{z}^k.
\end{aligned}$$

Harmonic extension of f to unit disk and we are taking radial limits as $r \nearrow 1$.

A series $\sum_k a_k$ is called **Able summable** if

$$\lim_{r\nearrow 1} r^k a_k$$

exists. Is a method for giving a value to a divergent series.

Example - Cesàro summation: A series $\sum a_k$ is Cesàro summable if

$$\frac{1}{n}\sum_{k=1}^{n}(\sum_{j=1}^{k} a_k)$$

has a limit (average of partial sums). Can be rewritten as

$$\sum_{k=1}^{n}(1-\frac{k}{n})a_k,$$

has a limit as $n \nearrow \infty$.

In terms of Fourier series we take the "tent" function:

$$\Phi(\xi) = \max(1-|\xi|, 0)$$

$$\begin{aligned}\phi(x) &= \int_{-1}^{0}(1+\xi)e^{2\pi i\xi\cdot x}d\xi + \int_{0}^{1}(1-\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \frac{e^{2\pi ix}+e^{-2\pi ix}-2}{(2\pi ix)^2} \\ &= \left(\frac{\sin \pi x}{\pi x}\right)^2.\end{aligned}$$

If we take $t = 1/(m+1)$ and write $\sigma_m f(x)$ for $f^{1/(m+1)}(x)$ we get

$$\begin{aligned}\sigma_m f(x) &= \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} \hat{f}(k) e^{2\pi i k x} \\ &= \hat{f}(0) + \sum_{k=1}^{m} \frac{m+1-k}{m+1} \left(\hat{f}(k) e^{2\pi i k x} + + \hat{f}(-k) e^{-2\pi i k x} \right).\end{aligned}$$

Chapter 8.5: Pointwise Convergence of Fourier Series

Suppose $f \in L^1(\mathbb{T})$ and define the partial sums

$$S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k) e^{2\pi i k x}.$$

When does $S_m f(x) \to f(x)$?

$$S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k) = f * D_m(x),$$

where D_m is the **Dirichlet kernel**

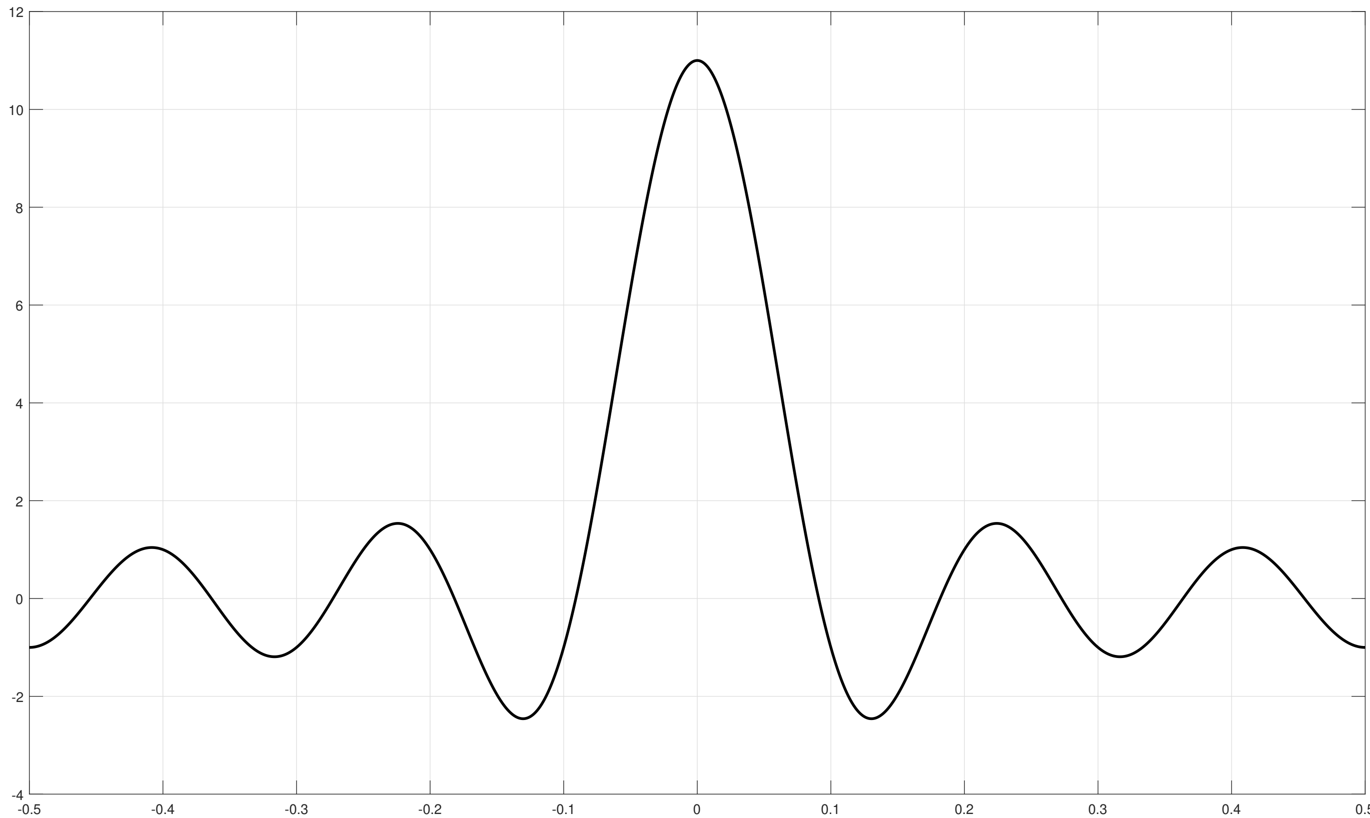
$$\begin{aligned} D_m(x) &= \sum_{k=-m}^{m} e^{2\pi ikx} = e^{-2\pi imx} \sum_{k=0}^{2m} e^{2\pi ikx} \\ &= e^{-2\pi imx} \frac{2^{2\pi i(2m+1)x} - 1}{e^{2\pi i} - 1} \\ &= \frac{\sin(2m+1)\pi x}{\sin \pi x} \end{aligned}$$

This is special case of Theorem 8.36 where

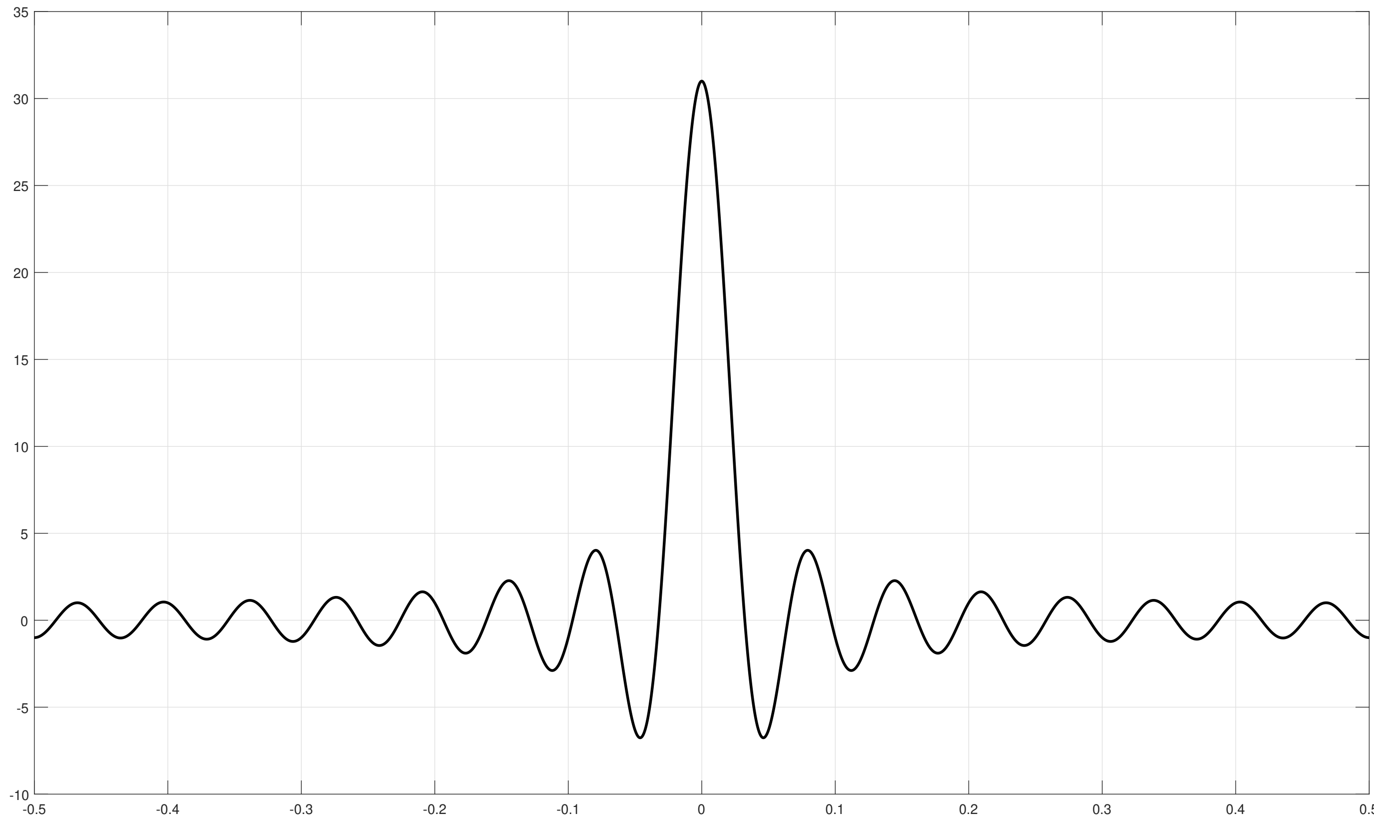
$$\Phi = \chi_{[-1,1]}, \qquad \phi(x) = \Phi^{\vee}(x) = \frac{\sin 2\pi x}{\pi x}.$$

But $\phi \notin L^1(\mathbb{R})$. Alternatively, $\|D_m\|_1 \simeq \log m \to \infty$.

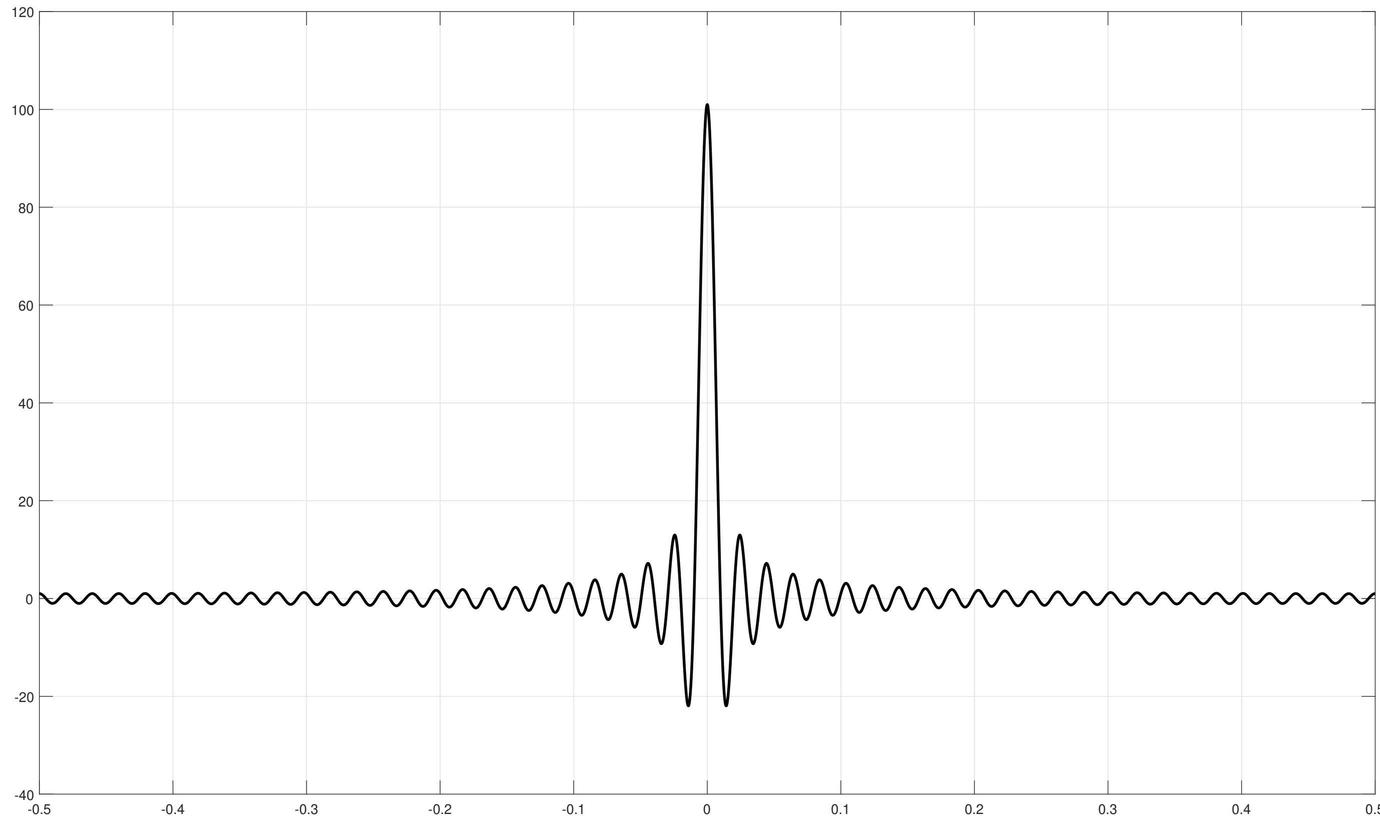
The partial sums need not converge to f for general continuous functions, but the Able or Cesàro sums do. Partial sums converge if f is bounded variation.



Dirichlet kernel D_m, $m = 5$



Dirichlet kernel D_m, $m = 15$



Dirichlet kernel D_m, $m = 50$

Johann Peter Gustav Lejeune Dirichlet (1805–1859)

Lemma 8.41: Let ϕ and ψ be real-valued functions on $[a, b]$,a, bJ. Suppose that ϕ is monotone and right continuous on $[a, b]$ and and ψ is continuous on [a, b]. Then there exists $\eta \in [a, b]$ such that

$$\int_1^b \phi(x)\psi(x)dx = \Phi(\eta) = \phi(a)\int_a^\eta \psi(x)dx + \phi(b)\int_\eta^b \psi(x)dx.$$

Proof. Adding a constant to ϕ changes both sides by the same amount, so we may assume $\phi(a) = 0$. We may also assume ϕ is increasing; otherwise replace ϕ by $-\phi$.

Since Φ is continuous, this is just the intermediate value theorem applied to Φ since

$$\Phi(a) \geq \int_1^b \phi(x)\psi(x)dx \geq \Phi(a) \geq \Phi(a)$$

. □

Lemma 8.42: There is a constant $C < \infty$ such that for every$m \geq 0$ and every $[a, b] \subset [-\frac{1}{2}, \frac{1}{2}]$

$$|\int_a^b D_m(x)dx| \leq C.$$

Moreover, for all m

$$\int_{-1/2}^{0} D_m(x)dx = \int_0^{1/2} D_m(x)dx = \frac{1}{2}$$

Proof. We know

$$D_m(x) \ = \ \frac{\sin(2m+1)\pi x}{\sin \pi x},$$

so

$$\begin{aligned}
\int_a^b D_m(x)dx \ &= \ \int_a^b \frac{\sin(2m+1)\pi x}{\sin \pi x}dx \\
&= \ \int_a^b \frac{\sin(2m+1)\pi x}{\pi x}dx \\
&\quad + \int_a^b \sin(2m+1)\pi x \left[[\frac{1}{\sin \pi x} - \frac{1}{\pi x}dx \right].
\end{aligned}$$

The second integrand is bounded, so the second integral is bounded (uniformly in m). A change of variable makes the first integal into

$$\int_{(2m+1)\pi a}^{(2m+1)\pi b} \frac{\sin(2m+1)\pi y}{\pi y} dy = \frac{1}{\pi}\mathrm{Si}((2m+1)\pi b)) - \mathrm{Si}((2m+1)\pi b))$$

where $\mathrm{Si}(x) = \int_0^x \frac{1}{y} \sin y dy$. But this function is bounded and approaches a finite limit as $x \to \pm\infty$. This is the first part of the lemma.

For the second part note that

$$\int_{-1/2}^{1/2} D_m(x)dx = 1$$

since D_m is a sum of mean zero exponential terms except for one constant term. Since D_m is even, each half-integral equals $1/2$. □

Theorem 8.43: If f is bounded variation on $\mathbb{T}$, that is, f is periodic on $\mathbb{R}$ and of bounded variation on $[-1/2, 1/2]$, then

$$\lim_{m\to\infty} S_m f(x)\frac{1}{2}\left[f(x+) + f(x-)\right]$$

for every x. In particular, $\lim S_m f(x) = f(x)$ at every x at which f is continuous.

Proof. We begin by making some reductions. In examining the convergence of $S_m f(x)$, we may assume that $x = 0$ (by replacing f with the translated function $\tau_x f$), that f is real-valued (by considering the real and imaginary parts separately), and that f is right continuous (since replacing $f(t)$ by $f(t+)$ affects neither $S_m f$ nor $f(O+) + f(O-)$.

In this case, by Theorem 3.27b, on the interval $[-1/2, 1/2)$ we can write f as the difference of two right continuous increasing functions g and h. If these functions are extended to $\mathbb{R}$ by periodicity, they are again of bounded variation, and it is enough to show that $S_m g(O) \to \frac{1}{2}[g(0+) + g(O-)]$ and likewise for h.

In short, it suffices to consider the case where $x = 0$ and f is increasing and right continuous on $[-1/2, 1/2)$.

Since D_m is even, we have

$$S_m f(0) = f * D_m(0) = \int_{-1/2}^{1/2} (x) D_m(x) dx,$$

so by Lemma 8.42,

$$S_m f(0) - \frac{1}{2}[f(0+) + f(0-)] \; = \; \int_0^{1/2} [f(x) - f(0+)] D_m(x) dx +$$

We shall show that the first integral on the right tends to zero as $m \to \infty$; a similar argument shows that the second integral also tends to zero, thereby completing the proof.

Given $\epsilon > 0$, choose $\delta > 0$ small enough so that $f(\delta) - f(0+) < \epsilon/C$ where C is as in Lemma 8.42. Then by Lemma 8.41, for some $\eta \in [0, \delta]$,

$$|\int_0^\delta [f(x) - f(0+)]D_m(x)dx| = [f(\delta) - f(0+)]|\int_\eta^\delta D_m(x)dx|,$$

which is less than ϵ. On the other hand, by (8.40),

$$\int_\delta^{1/2} [f(x) - f(0+)]D_m(x)dx = \int \chi_{[\delta,1/2]}(x)\frac{[f(x) - f(0+)]}{\sin \pi x} \sin((2m + 1x)dx$$

This tends to zero by the Riemann-Lebesgue lemma. Thus

$$\limsup_{m\to\infty} |\int_0^{1/2} [f(x) - f(0+)]D_m(x)dx| < \epsilon,$$

for every $\epsilon > 0$, and we are done. □

Theorem 8.44: If $f, g \in L^1(\mathbb{T})$ and $f = g$ on an opne interval I then $s_m f - S_m g \to 0$ uniformly on compact subsets of I.

Proof. See Folland. □

Cor. 8.34: Suppose $f \in L^1(\mathbb{T})$ and I is an open interval of length ≤ 1.

a. If f agrees on I with a function f such that $\hat{g} \in \ell^1(\mathbb{Z})$ then $S_m f \to f$ uniformly on compact subsets of I.

b. If f is absolutely continuous on I and $f' \in L^p(I)$ for some $p > 1$, then $S_m f \to f$ uniformly on compact subsets of I.

Proof. See Folland. □

Defn: f is **Dini continuous** at x if

$$\int_{-1}^{1} \frac{f(x+t) - f(x)}{t} < \infty.$$

This contains all α-Hölder functions $\alpha > 0$.

Theorem: If f is Dini continuous at x the $S_m f(x) \to f(x)$.

Chapter 8.6: Fourier Analysis of Measures

Defn: $M(\mathbb{R}^n)$ is space of complex Borel measures on $\mathbb{R}^n$.

Defn: If μ, ν are Radon measures on $\mathbb{R}^n$ we define the product

$$d(\mu \times \nu)(x, y) = \frac{d\mu}{d|\mu|}\frac{d\nu}{d|\nu|} d(|\mu| \times |\nu|)(x, y)$$

Defn: If μ, ν are Borel measures, then their convolution is

$$\mu * \nu(E) = \iint \chi_E(x + y) d\mu(x) d\nu(y).$$

The unit delta-mass at the origin is the identity for this.

Prop 8.4:

a. Convolution of measures is commutative and associative.

b. For any bounded Borel measurable function h,

$$\int hd(\mu * \nu) = \int\int h(x+y)d\mu(x)d\nu(y).$$

c. $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$. d. If $dm = fdm$ and $d\nu = 9dm$, then $d(\mu * \nu) = (f * g)dm$. In other words, on L^1, convolution of measures agrees with convolution of functions.

Proof of a: Commutativity is obvious from Fubini's theorem . To see ssociativity note that regardless of the order it is evaluted in, we get

$$\lambda * \mu * \nu(E) = \int\int\int \chi(x+y+z)d\lambda(x)d\mu(y)d\nu(z).$$

Proof of b: Left to reader (approximate h by simple function and use linearity and approximation).

Proof of c: Take $h = d|\mu * \nu|/d(\mu * \nu)$. Since $|h| = 1$,

$$\|\mu * \nu\| = \int hd(\mu * \nu) \leq \iint |f|d|\mu|d|\nu| = \|\mu\| \cdot \|\nu\|.$$

Proof of d: If $d\mu = fdm$ and $d\nu = gdm$ then for any bounded measurable h

$$\begin{aligned}
\int hd(\mu * \nu) &= \iint h(x+y)f(x)g(y)dxdy \\
&= \iint h(x)f(x-y)g(y)dxdy \\
&= \iint h(x)(f*g)(x)dxdy
\end{aligned}$$

whence $d(\mu * \nu) = (f * g)dm$.

We can also define convolutions of measures with functions in $L^p(\mathbb{R}^n)$.

Prop. 8.49 If $f \in L^p$, $1 \leq p \leq \infty$, and $\mu \in M(\mathbb{R}^n)$, then

$$f * \mu(x) = \int f(x-y) d\mu(y)$$

exists for a.e. x, and $f * \mu \in L^p$ with

$$\|f * \mu\|_p \leq \|f\|_p \cdot \|\mu\|.$$

Proof. If f and μ are nonnegative, then $f * \mu(x)$ exists (possibly being equal ∞ for every x, and by Minkowski's inequality for integrals,

$$\|f * \mu\|_p \leq \int \|f(\cdot - x\|_p d\mu \leq \|f\|_p \cdot \|\mu\|.$$

In particular, $f * \mu(x) < \infty$ for a.e. x. In the general case this argument applies to $|f|$ and μ and the result follows easily. □

This implies L^1 is an ideal in $M(\mathbb{R}^n)$, not just a subalgebra.

Defn: Fourier transform of a measure is

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi\cdot x} d\mu(x).$$

$\hat{\mu}$ is a bounded continuous function and $(\mu * mu)^{\wedge} = \hat{\mu} \cdot \hat{\nu}$.

Recall vague topology on $M(\mathbb{R}^n)$ is weak* topology relative to $C_0(\mathbb{R}^n)$.

Prop. 8.50: Suppose $\{\mu_n\}$ and μ are in $M(\mathbb{R}^n)$. If $\sup_k \|\mu_k\| \leq C < \infty$ and $\hat{\mu}_k \to \hat{\mu}$ pointwise, then $\mu_k \to \mu$ vaguely.

Proof. If $f \in \mathcal{S}$ then $f^{\vee} \in \mathcal{S}$ so by the Fourier inversion theorem

$$\int f d\mu_k = \iint f^{\vee}(y) e^{-2\pi i y\cdot x} dy d\mu_k(x) \int f^{\vee}(y)\hat{\mu}_k(y) dy.$$

Since $f^{\vee} \in L^1$ and $\|\mu_k\| \leq C$ the dominated convergence theorem implies $\int f d\mu_k \to \int f d\mu$. But $\mathcal{S}$ is dense in C_0 so by Prop 5.17 (boundedness and convergence on dense set implies strong convergence), $\int f d\mu_k \to \int f d\mu$ for all $f \in C_0$. Thus $\mu_k \to \mu$ vaguely. $\square$

Chapter 8.7: Applications to PDEs

The term differential operator means a linear partial differential operator with smooth coefficients, that is, an operator L of the form

$$Lf(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha f(x),$$

where $a_\alpha \in C^\infty$.

If the a's are constants, we call L a constant-coefficient operator.

In this case, if $f \in \mathcal{S}$,

$$(Lf)^\wedge(\xi) = \sum_{|\alpha|\leq m} a_\alpha(2\pi\xi)^\alpha \hat{f(x)},$$

It is convenient to write L in a slightly different form. Set

$$b_\alpha(2\pi i)^{|\alpha|}a_\alpha,$$

$$D^\alpha = (2\pi i)^{-|\alpha|}\partial^\alpha,$$

$$L = \sum_{|\alpha|\leq m} b_\alpha D^\alpha,$$

$$(Lf)^\wedge = \sum_{|\alpha|\le m} b_\alpha \xi^\alpha \hat{f}.$$

Given a polynomial $P = \sum_{|\alpha|\le m} b_\alpha \xi^\alpha$, we can define the operator

$$P(D) = \sum_{|\alpha|\le m} b_\alpha D^\alpha.$$

The polynomial P is called the symbol of the operator $P(D)$.

Formally, one can solve $P(D)u = f$ by taking

$$u = (\hat{f}/P)^\vee.$$

If $1/P$ is the Fourier transform of a function ϕ then $u = f * \phi$.

However, to make this work the Fourier transform needs to be defined on all these objects. This requires an extension, given in the theory of distributions.

Laplacian:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x + k^2} = -4\pi^2 \sum_{1}^{m} D_k^2 = P(D)$$

where $P(\xi) = -4\pi|\xi|^2$.

Theorem 8.51: A differential operator T satisfies $L(f \circ T) = (Lf) \circ T$ for all translations and rotations T iff there is a polynomial P in one variable such that $L = P(\Delta)$.

Proof. Clearly L is translation-invariant iff L has constant coefficients, in which case $L = Q(D)$ for some polynomial Q in n variables. Moreover, since $Lf)^{\wedge} = Q\hat{f}$ and the Fourier transform commutes with rotations, L commutes with rotations iff Q is rotation-invariant. Let $Q = \sum Q_j$ where Q_j is homogeneous of degree j; then it is easy to see that Q is rotation-invariant iff each Q_j is rotation-invariant. To prove this, use induction on j and the fact that

$$Q_j(\xi) = \lim_{r \to 0} r^{-j} \sum_{j}^{m} Q_i(r\xi).$$

But this means that $Q_j(\xi)$ depends only on $|\xi|$, so $Q_j(\xi) = c_j|\xi|$ by homogeneity. Moreover, $|\xi|^j$ is a polynomial precisely when j is even, so $c_j = 0$ for j odd. Setting $b_k = (-4\pi^2)^{-k}c_{2k}$, we have $Q(\xi) = \sum b_k(-4\pi^2|\xi|^2)^k$ or $L = \sum b_k\Delta^k$.

$\square$

Defn: Dirichlet problem: Given an open set $\Omega \subset \mathbb{R}^n$, and a function f on its boundary $\partial\Omega$, find a function u on Ω such that $\Delta u = 0$ on Ω and $u|_{\partial\Omega} = f$.

We can use Fourier transform to solve this on a half-space. Denote coordinates on $\mathbb{R}^{n+1}$ by $x_1, \ldots, x_n, t$. The Laplacian on $\mathbb{R}^{n+1}$ is $\Delta + \partial_t^2$.

Let $\Omega = \{(x_1, \ldots, x_n, t) : t > 0\}$. Then

$$(\Delta + \partial_t^2)u = 0,$$

becomes

$$(-4\pi^2|\xi|^2)\hat{u} = 0o.$$

The general solution is

$$\hat{u}(\xi, t)c_1(\xi)e^{-2\pi t|\xi|} + c_2(\xi)e^{2\pi t|\xi|}$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

This converts to

$$u(x, t) = (f * P_t)(x)$$

where

$$P_t(x) = \frac{ct}{(t^2 + |x|^2)^{-(n+1)/2}}.$$

Theorem 8.53: Suppose $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then the function

$$u(x,t) = (f * P_t)(x)$$

satisfies $(\Delta + \partial_t^2)u = 0$ on $\mathbb{R}^n \times (0,\infty)$ and

$$\lim_{t\to 0} u(x,t) = f,$$

for a.e. x, and for every x at which f is continuous. Moreover, $\lim_{t\to 0} \|u(\cdot,t) - f\|_p \to 0$.

Proof. See Folland. □

The same idea can be applied to the heat equation

$$(\partial_t \Delta)u = 0$$

on the upper half-space. The unique solution is

$$\hat{u}(\xi, t) = \hat{f} e^{-4\pi^2 t|\xi|^2},$$

or

$$u(x,t) = f * G_t(x), \qquad G_t(x) = (4\pi t)^{-n/2} e^{-\mathbf{x}|^2/4t}.$$

The wave equation is given by

$$(\partial_t^2 - \Delta)u = 0.$$

$$u(x,0) = f(x), \qquad \partial_t u(x,0) = g(x).$$

Applying the Fourier transform we get

$$(\partial_t^2 + 4\pi^2|\xi|^2)\hat{u}(\xi,t) = 0$$

$$\hat{u}(x,0) = \hat{f}(x), \qquad \partial_t \hat{u}(x,0) = \hat{g}(x).$$

This yields

$$\hat{u}(\xi,t,) = (\cos 2\pi t|\xi|)\hat{f}(x) + \frac{\sin 2\pi t|\xi|}{2\pi|\xi|}\hat{g}(x).$$

Since

$$\cos 2\pi t|\xi| = \frac{\partial}{\partial t}\left[\frac{\sin 2\pi t|\xi|}{2\pi|\xi|}\right],$$

we get

$$u(x,t) = f * \partial W_t(x) + g * W_t(x),$$

where

$$W_t(x) = \left[\frac{\sin 2\pi t|\xi|}{2\pi|\xi|}\right]^\vee.$$

However, this inverse Fourier transform only gives a function when $n = 1, 2$ and a measure when $n = 3$. In higher dimensions one needs to consider distributions.