A HIGH-SCHOOL ALGEBRA[1] , WALLET-SIZED PROOF, OF THE BIEBERBACH CONJECTURE [After L. Weinstein]

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Dedicated to Leonard Carlitz[3] *, master of formal mathematics.*

Weinstein's[2] brilliant short proof of de Branges'[1] theorem can be made yet much shorter(modulo routine calculations), completely elementary (modulo Löwner theory), self contained(no need for the esoteric Legendre polynomials' addition theorem), and motivated(ditto), as follows. Replace the text between p. 62, line 7 and p. 63, line 7, by Fact 1 below, and the text between the last line of p.63 and p.64, line 7, by Fact 2 below.

FACT 1: Let $f_t(z) = e^t z \exp(\sum_{k=1}^{\infty} c_k(t) z^k)$ where $c_k(t)$ are *formal* functions of t. Let z and w be related by $z/(1-z)^2 = e^t w/(1-w)^2$. The following formal identity holds. (For any formal Laurent series $f(z)$, $CT_z f(z)$ denotes the *Constant Term* of $f(z)$.)

$$(1+w)\frac{d}{dt}\{\sum_{k=1}^{\infty}(4/k - kc_k(t)\overline{c_k(t)})w^k\} =$$

$$(1-w)\sum_{k=1}^{\infty} \mathrm{Re}\, CT_z \left\{ \frac{\frac{\partial f_t(z)}{\partial t}}{\frac{z\partial f_t(z)}{\partial z}} \cdot \left(2(1+\sum_{r=1}^{k} rc_r(t)z^r) - kc_k(t)z^k\right) \cdot \left(2(1+\sum_{r=1}^{k} r\overline{c_r(t)}z^{-r}) - k\overline{c_k(t)}z^{-k}\right)\right\} w^k$$

Proof: Routine. (Even for a human.) □

FACT 2: The coefficients $A_{k,n}(c)$ in the formal power series (Laurent in w) expansion $(1 - z(2c + (1-c)(w+1/w)) + z^2)^{-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n}(c)(w^k + w^{-k})z^n$ are non-negative for $0 \le c \le 1$.

Proof: This follows immediately from the stronger fact that the coefficients $B_{k,n}(c)$, defined by the expansion $(1 - z(2c + (1-c)(w+1/w)) + z^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{k,n}(c)(w^k + w^{-k})z^n$ can be expressed as $L_{k,n}(c)^2$, for some double sequence $L_{k,n}$, $0 \le k \le n$, that is *real* for c in $[0,1]$.

First use Maple to output $B_{k,n}(c)$, for $0 \le k \le n \le 20$, factor them, and observe that for this range they are expressible as $L_{k,n}(c)^2$. Using the *gfun* Maple package[4] , Maple conjectures a certain second-order linear recurrence, in n, satisfied by the $L_{k,n}(c)$, and using *gfun* once again, it computes the linear recurrence satisfied by the squares of the terms of the solution-sequence of the previous recurrence. That new recurrence turns out to be identical with the third-order recurrence, in n, that the WZ method[5] outputs for $B_{k,n}$. It follows that indeed $B_{k,n} = L_{k,n}^2$, where *now* $L_{k,n}(c)$

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1 and high-school (purely formal) calculus.

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3 L. Carlitz was, for many years, editor of the Duke Journal, until he was relieved from his duties by the proponents of so-called "modern math", who despised formal, Carlitz-style, mathematics. This paper makes explicit the power of formal math, that is (too) subtly hidden in Weinstein's original presentation.

4 Developed by Salvy and Zimmerman, Available by anonymous ftp from `ftp.inria.fr` in directory lang/maple/INRIA/gfun .

5 Wilf and Zeilberger, Invent. Math. 108(1992), 575-633. The program, and the input file for this problem, are available via anonymous ftp to `math.temple.edu`, in directory `pub/zeilberger/programs`.

denotes the solution of the above-mentioned second-order recurrence, with the obvious initial values at $n = k, k+1$. $\square$

References

1. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154**(1985), 137-152.
2. L. Weinstein, *The Bieberbach Conjecture*, Inter. Math. Res. Notices (of the Duke J.) **3**(1991, No. 5), 61-64.

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