ABSTRACT. Let $f(z)$ be any univalent function that maps the unit disc onto a domain $\Omega$. We prove that for any line $L$ the length of $f^{-1}(\Omega \cap L)$ is less than $4\pi$.

Let $D$ be the open unit disc. Throughout this paper $f(z)$ is a univalent function that maps $D$ onto a simply connected domain $\Omega$. $z$ lives in $D$ and $w$ lives in $\Omega$. $f(z_n) = w_n$. The pseudohyperbolic metric in $D$ is defined by

$$\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right|,$$

where $\rho$ is conformally invariant.

In $\Omega$, $\rho$ can be defined by $\rho(w_1, w_2) = \rho(z_1, z_2)$. In $\Pi^+$, the upper halfplane,

$$\rho(w_1, w_2) = \frac{|w_1 - w_2|}{|w_1 - \overline{w}_2|}.$$

The length of a curve $K$ is denoted $|K|$. We will prove

**Theorem.** If $L$ is any line, $|f^{-1}(\Omega \cap L)|$ add an absolute value sign $< 4\pi$.

The theorem without the constant is due to Hayman and Wu [5]. See also [4]. The best previously known constant is $4\pi^2$ (see [3]). It is known that the constant cannot be less than

$$C = 8 \int_0^1 \frac{dx}{\sqrt{1 + x^4}},$$

and it has been conjectured that the best constant is $C$. For enlightening discussions and generalizations of the Hayman-Wu theorem, see [1] and [4]. Our proof of the theorem depends on the following

**Lemma.** Assume $\{w_n\} \subset L \cap \Omega$ satisfies $\rho(w_n, w_m) \geq \delta$ for $n \neq m$. Then

$$\sum (1 - |z_n|) \leq \frac{2\pi}{\arctan \delta}.$$

**Proof.** Note that $\rho(w_1, w_2) = \sup \{|f(w_1)| : f \in \text{ball} H^\infty(\Omega), f(w_2) = 0\}$. Therefore $\rho$ increases when $\Omega$ decreases. We may assume that $L = \mathbb{R}$ and
that \( \Omega \) is a Jordan domain. If \( \Omega \) is not, approximate \( \Omega \) by the domains \( f_s(D) \) where \( f_s(z) = f(sz) \).

Let \( L_k \) be the components of \( L \cap \Omega \), and let \( \Omega_k \) be the component of \( \Omega \cap \{ z : z \in \Omega \} \) that contains \( L_k \). \( \{ \Omega_k \} \) is a disjoint family of Jordan domains. If \( k \neq s \), \( \partial \Omega_k \) and \( \partial \Omega_s \) are essentially disjoint. They have at most one point in common.

Assume that \( w_n, w_m \in L_k \). Let \( \varphi \) map \( \Omega_k \) conformally onto \( \Pi^+ \) such that \( \varphi(L_k) = i\mathbb{R}^+ \). \( \varphi(w) = iy \). Since \( \rho \) is conformally invariant we have

\[
\left| \frac{y_n - y_m}{y_n + y_m} \right| \geq \delta.
\]

Let \( K_\rho = \left[ \frac{1 - \delta}{1 + \delta}, y_r \right] \cup \left[ -y_r, -\frac{1 - \delta}{1 + \delta} y_r \right] \).

\( K_n \) and \( K_m \) are essentially disjoint by \((*)\). Interpreting harmonic measure in the upper halfplane as normalized angles we see that \( \omega(iy, K_\rho, \Pi^+) = \frac{2}{\pi} \arctan \delta \). Let \( K_\rho^* = \{ \zeta \in K_\rho : \varphi^{-1}(\zeta) \in \partial \Omega \} \). By the symmetry of \( \Omega_k \) and the choice of \( \varphi \) we have \( \zeta \in K_\rho \), \( \zeta \notin K_\rho^* \) implies \( -\zeta \notin K_\rho^* \). Therefore \( \omega(iy, K_\rho^*, \Pi^+) \geq \frac{(\arctan \delta)}{\pi} = \delta' \). Let \( C_r = \varphi^{-1}(K_\rho^*) \). Conformal invariance gives \( \omega(w_r, C_r, \Omega_k) \geq \delta' \) and, by the maximum principle, \( \omega(w_r, C_r, \Omega) \geq \delta' \). If \( r \neq s \), \( C_r \) and \( C_s \) are essentially disjoint. Let \( E_r = f^{-1}(C_r) \), and let \( P_z \) be the Poisson kernel of \( z \). We have

\[
\delta' \leq \omega(z_r, E_r, D) = \frac{1}{2\pi} \int_{E_r} |P_z| \leq \frac{1}{2\pi} \frac{1 - |z_r|^2}{(1 - |z_r|)^2} |E_r|.
\]

Hence \( 1 - |z_r| < |E_r|/\arctan \delta \). This proves the lemma since \( \sum |E_r| \leq 2\pi \).

**Proof of the theorem.** For \( \delta > 0 \) let \( D_\rho(w, \delta) = \{ w' : \rho(w, w') \leq \delta \} \). By Theorem 2.13 in [2] \( D_\rho(w, \delta) \) is (euclidean) convex if \( \delta < 2 - \sqrt{3} \). Therefore for small \( \delta > 0 \) we can choose \( \{ w_n \} \) in \( L \cap \Omega \) such that

(i) if \( w_n \) and \( w_m \) are neighbours \( \rho(w_n, w_m) = \delta \),

(ii) \( \bigcup D_\rho(w_n, \delta) \) covers \( L \cap \Omega \) twice.

Since \( f \) is a pseudohyperbolic isometry, \( f^{-1}(D_\rho(w_n, \delta)) \) is a euclidean disc whose diameter is easily computed to be

\[
2\delta \frac{1 - |z_n|^2}{1 - \delta^2 |z_n|^2}.
\]

Every univalent function in the unit disc satisfies \( |f''(z)/f'(z)| < 6/(1 - |z|^2) \). Integration leads to \( |\arg f'(z') - \arg f'(z'')| < K\rho(z', z'') \) if \( \rho(z', z'') < \delta_0 < 1 \). Therefore \( (f^{-1})'(w) \) satisfies the same inequalities. Hence

\[
|f^{-1}(L \cap D_\rho(w_n, \delta))| = \int_{L \cap D_\rho(w_n, \delta)} |(f^{-1})'(w)| \, |dw| \\
\leq (1 + o(1)) \int_{L \cap D_\rho(w_n, \delta)} (f^{-1})'(w) \, |dw| \\
\leq 2\delta \frac{1 - |z_n|^2}{1 - \delta^2 |z_n|^2} (1 + o(1)) \leq \frac{4\delta}{1 - \delta^2} (1 + o(1))(1 - |z_n|)
\]

uniformly in \( n \).
We now apply the lemma,

\[
2|f^{-1}(L)| \leq \frac{4\delta}{1 - \delta^2} (1 + o(1)) \sum (1 - |z_n|) \\
\leq \frac{4\delta}{1 - \delta^2} (1 + o(1)) \frac{2\pi}{\arctan \delta} \to 8\pi \quad \text{when} \ \delta \to 0.
\]

We have used the crude inequality \( 1 + |z_n| < 2 \). Since this holds uniformly on a large part of \( f^{-1}(L) \), we have strict inequality.

References


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