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HARMONIC MEASURE AND CONFORMAL LENGTH

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ABSTRACT. Let $f(z)$ be any univalent function that maps the unit disc onto a domain Ω. We prove that for any line L the length of $f^{-1}(\Omega \cap L)$ is less than 4π.

Let D be the open unit disc. Throughout this paper $f(z)$ is a univalent function that maps D onto a simply connected domain Ω. z lives in D and w lives in Ω. $f(z_n) = w_n$. The pseudohyperbolic metric in D is defined by

$$\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right|,$$

where ρ is conformally invariant.

In Ω, ρ can be defined by $\rho(w_1, w_2) = \rho(z_1, z_2)$. In Π^+, the upper halfplane,

$$\rho(w_1, w_2) = \left| \frac{w_1 - w_2}{w_1 - \overline{w}_2} \right|.$$

The length of a curve K is denoted $|K|$. We will prove

Theorem. *If L is any line, $|f^{-1}(\Omega \cap L)|$ add an absolute value sign $< 4\pi$.*

The theorem without the constant is due to Hayman and Wu [5]. See also [4]. The best previously known constant is $4\pi^2$ (see [3]). It is known that the constant cannot be less than

$$C = 8 \int_0^1 \frac{dx}{\sqrt{1 + x^4}},$$

and it has been conjectured that the best constant is C. For enlightening discussions and generalizations of the Hayman-Wu theorem, see [1] and [4]. Our proof of the theorem depends on the following

Lemma. *Assume $\{w_n\} \subset L \cap \Omega$ satisfies $\rho(w_n, w_m) \geq \delta$ for $n \neq m$. Then* $\sum(1 - |z_n|) \leq \frac{2\pi}{\arctan \delta}$.

Proof. Note that $\rho(w_1, w_2) = \sup\{|f(w_1)| \colon f \in \operatorname{ball} H^\infty(\Omega),\ f(w_2) = 0\}$. Therefore ρ increases when Ω decreases. We may assume that $L = \mathbb{R}$ and

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that Ω is a Jordan domain. If Ω is not, approximate Ω by the domains $f_r(D)$ where $f_r(z) = f(rz)$.

Let L_k be the components of $L \cap \Omega$, and let Ω_k be the component of $\Omega \cap \{z : \bar{z} \in \Omega\}$ that contains L_k. $\{\Omega_k\}$ is a disjoint family of Jordan domains. If $k \neq s$, $\partial\Omega_k$ and $\partial\Omega_s$ are essentially disjoint. They have at most one point in common.

Assume that w_n, $w_m \in L_k$. Let φ map Ω_k conformally onto Π^+ such that $\varphi(L_k) = i\mathbb{R}^+$. $\varphi(w_r) = iy_r$. Since ρ is conformally invariant we have

$$(*) \qquad \left|\frac{y_n - y_m}{y_n + y_m}\right| \geq \delta .$$

Let

$$K_r = \left[y_r \frac{1-\delta}{1+\delta}, y_r\right] \cup \left[-y_r, -\frac{1-\delta}{1+\delta} y_r\right] .$$

K_n and K_m are essentially disjoint by $(*)$. Interpreting harmonic measure in the upper halfplane as normalized angles we see that $\omega(iy_r, K_r, \Pi^+) = (2/\pi)\arctan\delta$. Let $K_r^* = \{\zeta \in K_r : \varphi^{-1}(\zeta) \in \partial\Omega\}$. By the symmetry of Ω_k and the choice of φ we have $\zeta \in K_r$, $\zeta \notin K_r^*$ implies $-\zeta \in K_r^*$. Therefore $\omega(iy_r, K_r^*, \Pi^+) \geq (\arctan\delta)/\pi = \delta'$. Let $C_r = \varphi^{-1}(K_r^*)$. Conformal invariance gives $\omega(w_r, C_r, \Omega_k) \geq \delta'$ and, by the maximum principle, $\omega(w_r, C_r, \Omega) \geq \delta'$. If $r \neq s$, C_r and C_s are essentially disjoint. Let $E_r = f^{-1}(C_r)$, and let P_{z_r} be the Poisson kernel of z_r. We have

$$\delta' \leq \omega(z_r, E_r, D) = \frac{1}{2\pi}\int_{E_r} P_{z_r} \leq \frac{1}{2\pi} \cdot \frac{1-|z_r|^2}{(1-|z_r|)^2}|E_r| .$$

Hence $1 - |z_r| < |E_r|/\arctan\delta$. This proves the lemma since $\sum |E_r| \leq 2\pi$.

Proof of the theorem. For $\delta > 0$ let $D_\rho(w, \delta) = \{w' : \rho(w, w') \leq \delta\}$. By Theorem 2.13 in [2] $D_\rho(w, \delta)$ is (euclidean) convex if $\delta < 2 - \sqrt{3}$. Therefore for small $\delta > 0$ we can choose $\{w_n\}$ in $L \cap \Omega$ such that

(i) if w_n and w_m are neighbours $\rho(w_n, w_m) = \delta$,
(ii) $\bigcup D_\rho(w_n, \delta)$ covers $L \cap \Omega$ twice.

Since f is a pseudohyperbolic isometry, $f^{-1}(D_\rho(w_n, \delta))$ is a euclidean disc whose diameter is easily computed to be

$$2\delta \frac{1-|z_n|^2}{1-\delta^2|z_n|^2} .$$

Every univalent function in the unit disc satisfies $|f''(z)/f'(z)| < 6/(1-|z|^2)$. Integration leads to $|\arg f'(z') - \arg f'(z'')| < K\rho(z', z'')$ if $\rho(z', z'') < \delta_0 < 1$. Therefore $(f^{-1})'(w)$ satisfies the same inequalities. Hence

$$\begin{aligned} |f^{-1}(L \cap D_\rho(w_n, \delta))| &= \int_{L\cap D_\rho(w_n, \delta)} |(f^{-1})(f^{-1})'(w)||dw| \\ &\leq (1 + o(1))\left|\int_{L\cap D_\rho(w_n, \delta)} (f^{-1})'(w)\, dw\right| \\ &\leq 2\delta\frac{1-|z_n|^2}{1-\delta^2|z_n|^2}(1+o(1)) \leq \frac{4\delta}{1-\delta^2}(1+o(1))(1-|z_n|) \end{aligned}$$

uniformly in n.

We now apply the lemma,

$$\begin{aligned} 2|f^{-1}(L)| &\leq \frac{4\delta}{1-\delta^2}(1+o(1))\sum(1-|z_n|) \\ &\leq \frac{4\delta}{1-\delta^2}(1+o(1))\frac{2\pi}{\arctan\delta} \to 8\pi \quad \text{when } \delta \to 0. \end{aligned}$$

We have used the crude inequality $1+|z_n|<2$. Since this holds uniformly on a large part of $f^{-1}(L)$, we have strict inequality.

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