MAT 324, Fall 2015
PROBLEM SET 7, Due Tuesday, December 1
Product Measures

(1) Suppose $E \subset \mathbb{R}^2$ is measurable and that $E_y = \{x : (x,y) \in E\} \subset \mathbb{R}^1$ has measure zero for almost every y. Show that E has measure zero in $\mathbb{R}^2$.

(2) If f and g are measurable on $\mathbb{R}$, show that $h(x,y) = f(x)g(y)$ is measurable on $\mathbb{R}^2$.

(3) Show

$$\int_{\mathbb{R}^n} e^{-|x|^2} dm = \pi^{n/2}.$$

(Hint: For $n = 1$ use

$$(\int e^{-x^2} dx)^2 = \int\int e^{-x^2-y^2} dxdy,$$

and polar coordinates. For $n > 1$, use $|x|^2 = x_1^2 + \cdots + x_n^2$ and Fubini's theorem to reduce to the $n = 1$ case.)

(4) The convolution $f * g$ of two functions $f, g \in L^1(\mathbb{R})$ is defined as the function

$$f * g(y) = \int_{\mathbb{R}} f(y-x)g(x)dx.$$

Show that $f * g$ is in $L^1(\mathbb{R})$ and

$$\int f * g(y)dy = (\int_{\mathbb{R}} f(x)dx)(\int_{\mathbb{R}} g(x)dx).$$

(5) If $E \subset \mathbb{R}$ is closed, let $\delta(y) = \mathrm{dist}(y, E) = \inf\{|x-y| : x \in E\}$. Show that

$$M(x) = \int_0^1 \frac{\delta^\alpha(y)dy}{|x-y|^{1+\alpha}} < \infty,$$

for almost every $x \in E$. (Hint: integrate M over E.)