CHAPTER 8

Probability

1. Random variables and independence

2. Borel-Cantelli and 2nd moment method.

3. Law of large numbers

THEOREM 3.1 (Law of Large Numbers). *Let* $\{f_n\}$, $n = 1, 2 \dots$ *be a sequence of orthogonal functions on a probability space* $(X, d\nu)$ *and suppose* $E(f^2) = \int |f|^2 d\nu \leq 1$. *Then*

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{k=1}^{n} f_k \to 0,$$

a.e. (with respect to ν*) as* $n \to \infty$.

We begin with the simple observation that if $\{g_n\}$ is a sequence of functions on a probability space $(X, d\nu)$ such that

$$\sum_n \int |g_n|^2 d\nu < \infty,$$

then $\sum_n |g_n|^2 < \infty$ a.e. $(d\nu)$ and hence $g_n \to 0$ a.e. $(d\nu)$.

Using this, it is easy to verify the law of large numbers (LLN) for $n \to \infty$ along the sequence of squares. Namely,

$$\int (\frac{1}{n} S_n)^2 d\nu = \frac{1}{n^2} \int |S_n|^2 d\nu = \frac{1}{n^2} \sum_{k=1}^{n} \int |f_k|^2 d\nu \leq \frac{1}{n}.$$

Therefore if we set $g_n = \frac{1}{n^2} S_{n^2}$, we have

$$\int \frac{1}{n^2} |S_{n^2}|^2 d\nu \leq \frac{1}{n^2}.$$

Since the right hand side is summable, the observation above implies $g_n \to 0$ a.e. $(d\nu)$. This is the same as $\frac{1}{n^2} S_{n^2} \to 0$, a.e..

To deal with limit over all the integers take $m^2 \leq n < (m+1)^2$ and set $m(n) = \lfloor\sqrt{n}\rfloor$. Then

$$\begin{aligned}\int |\frac{1}{m^2}S_n - \frac{1}{m^2}S_{m^2}|^2 d\nu &= \frac{1}{m^4}\int |\sum_{k=m^2+1}^{n} f_k|^2 d\nu \\ &= \frac{1}{m^4}\int \sum_{k=m^2+1}^{n} |f_k|^2 d\nu \\ &\leq \frac{2}{m^3}\end{aligned}$$

since the sum has at most $2m$ terms, each of size at most 1. Put

$$g_n = \frac{S_n}{m(n)^2} - \frac{S_{m(n)^2}}{m(n)^2}.$$

Then since each $m = m(n)$ is associated to at most $2m+1$ different n's we get

$$\sum_{n=1}^{\infty}\int |g_n|^2 d\mu \leq \sum_{n=1}^{\infty}\frac{2}{m(n)^3} \leq \sum_m (2m+1)\frac{2}{m^3} < \infty,$$

so by the initial observation, $g_n \to 0$ a.e. with respect to ν. This implies $\frac{1}{m(n)^2}S_n \to 0$ a.e., which in turn implies $\frac{1}{n}S_n \to 0$ a.e., which is what we wanted.

This version is sometimes called the strong law of large numbers because it gives a.e. convergence, as opposed to the weak version which only says that $\frac{1}{n}S_n$ converges to 0 in L^2.

As a remark we should note that better estimates for the decay of S_n are possible if we assume that the functions $\{f_n\}$ are independent with respect to the measure ν. This means that for any n and any collection of measurable sets $\{A_1, \ldots, A_n\}$ we have

$$\nu(x \in X : f_j(x) \in A_j, j = 1, \ldots, n\}) = \prod_{j=1}^{n} \nu(\{x \in X : f_j(x) \in A_j\}).$$

Roughly, this says that knowing the values of any of the f_j's at x does not give us any information about the values of the remaining functions there.

By 1915 Hausdorff had proved that if $\{f_n\} \in L^2(\nu) \cap L^1(\nu)$ are independent, orthonormal (orthogonal and have L^2 norm 1) and satisfy $\int f_n d\nu = 0$ then

$$\lim_{N\to\infty}\frac{1}{N^{\frac{1}{2}+\epsilon}}\sum_{n=0}^{N} f_n(x) = 0, \text{ for a.e. } x$$

and for every $\epsilon > 0$. After that Hardy-Littlewood, and independently Khinchin, proved

$$\lim_{N\to\infty} \frac{1}{\sqrt{N \log N}} \sum_{n=0}^{N} f_n(x) = 0 \text{ for a.e. } x.$$

The "final" result, found by Khinchin for a special case in 1928 and proved in general by Hartman-Wintree in 1941 says

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \sum_{n=0}^{N} f_n(x) = 1 \text{ for a.e. } x.$$

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