PROBLEM SET 5

1. Does \{\sin(nx)\} converge in the \(L^1\) norm?

2. Give an example of a sequence of functions \(\{f_n\}\) which converges to the constant zero function in \(L^1\), but so that \(f_n(x)\) does not converge to zero at any point of \([0, 1]\).

3. If \(f_n \to f\) in the \(L^1\) norm, show that there is a subsequence \(f_{n_k}\) which converges a.e. to \(f\).

4. Prove that the set of continuous functions of compact support is dense in \(L^1\).

5. For each \(1 \leq p \leq \infty\) give an example of a function which is in \(L^p\) but not in \(L^q\) for any \(q \neq p\).

6. If \(f \in L^1\) show that \(X_f = \{g \in L^1 : |g| \leq |f|\}\) is a compact set in the \(L^1\) topology (i.e., a sequence in this set has a subsequence converging to a point of the set).

7. If \(f \in L^2\) let \(T_n f\) be the \(n\)th partial sums of the Fourier series. Then \(T_n f \to f\) in the \(L^2\) norm. It is also true that \(T_n f \to f\) a.e., but this is very hard to prove (one of the most famous theorems proven in the 20th century).