Answer Key

MAT 320 Fall 2021, Final Exam, 8:00am - 10:45am Tuesday, December 14, 2021

Name	ID	Section

1-20	21	22	23	24	total

THIS EXAM IS WORTH 100 POINTS. THERE ARE 20 TRUE/FALSE QUESTIONS WORTH 2 POINTS EACH, AND 4 PROOFS WORTH 15 POINTS EACH. NO BOOKS OR NOTES ARE ALLOWED. THERE ARE SIX PRINTED PAGES AND TWO BLANK PAGES.

(1)-(20) TRUE/FALSE: put a T or F in each box.

(1) $\boxed{T}$ A continuous function on $\mathbb{R}$ that only takes rational values must be constant.

(2) $\boxed{T}$ If f is strictly increasing and continuous on $\mathbb{R}$, then its inverse f^{-1} is also strictly increasing and continuous.

(3) $\boxed{F}$ There is an increasing function on $\mathbb{R}$ that is discontinuous at every point.

(4) $\boxed{T}$ If f is differentiable everywhere on $[a, b]$ and $f'(x) > 0$ everywhere, then f is strictly increasing on $[a, b]$.

(5) $\boxed{F}$ If f is differentiable everywhere on $[a, b]$ then f' bounded on $[a, b]$.

(6) $\boxed{F}$ If f is Riemann integrable on $[a, b]$ and $\int_c^d f = 0$ for every $a \leq c < d \leq b$, then f is the constant zero function.

(7) $\boxed{T}$ A countable union of zero length sets also has zero length.

(8) $\boxed{T}$ If f is Riemann integrable on $[a, b]$ and g is continuous on $\mathbb{R}$, then $g \circ f$ is Riemann integrable on $[a, b]$.

(9) $\boxed{\text{T}}$ If $\{f_n\}$ converges uniformly on $[a, b]$ to a continuous function f on $[a, b]$ then the $\{f_n\}$ are bounded, i.e., there is an $M < \infty$ so that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [a, b]$.

(10) $\boxed{\text{F}}$ If $\{f_n\}$ are Riemann integrable on $[a, b]$ and converge pointwise everywhere on $[a, b]$ to f, then f is Riemann integrable on $[a, b]$.

(11) $\boxed{\text{F}}$ If $f_1 \geq f_2 \geq f_3 \geq \ldots$ are continuous and converge pointwise everywhere to f, then f is continuous.

(12) $\boxed{\text{F}}$ If $\sum x_n$ converges, then $\sum x_{2n}$ converges.

(13) $\boxed{\text{T}}$ If $\sum y_n$ converges, then $\sum \frac{1}{n} y_n$ also converges.

(14) $\boxed{\text{T}}$ $f(x) = \sum_{n=1}^{\infty} n^{-2} \sin(n^2 x)$ converges for all $x \in \mathbb{R}$.

(15) $\boxed{\text{T}}$ There is a re-arrangement of $\sum_1^{\infty} (-1)^n / n$ that converges to -2.

(16) $\boxed{\text{T}}$ Every non-empty open set in $\mathbb{R}$ contains a rational number.

(17) $\boxed{\text{T}}$ A set $K \subset \mathbb{R}$ is compact iff it is closed and bounded.

(18) $\boxed{\text{F}}$ Any intersection of open sets in $\mathbb{R}$ is open.

(19) $\boxed{\text{T}}$ A non-empty subset of a metric space is always a metric space with the restricted metric.

(20) $\boxed{\text{F}}$ $d(x, y) = |x - y|^2$ is a metric on $\mathbb{R}$.

In problems 21-24, you may use a result from the class or textbook without proof if you name it or quote it correctly.

(21) Suppose (a_n) is a real sequence so that $\sum na_n$ converges. Prove that $\sum a_n$ also converges.

Let $x_n = \frac{1}{n}$.

Let $y_n = na_n$.

Then $\{x_n\}$ is monotone and convergent and $\sum y_n$ is convergent so by Abel's Test (Thm 9.3.5)

$$\sum a_n = \sum \frac{1}{n} \cdot n \cdot a_n = \sum x_n y_n$$

is convergent. $\square$

(22) Suppose $\{f_n\}$ are continuous functions on a closed, bounded interval $[a, b]$ and $f_1(x) \leq f_2(x) \leq \dots$ for all $x \in [a, b]$. If f is continuous and $f_n(x) \to f(x)$ for all $x \in [a, b]$ prove that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

By Dini's theorem (Thm 8.2.6), $f_n \to f$ uniformly. Since $\{f_n\}$, f are continuous on a bounded interval they are Riemann integrable and uniform convergence of integrable functions implies the integrals converge, i.e.,

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

by Theorem 8.2.4. □ (you don't have to quote the theorem number).

(23) Show that if f, g are both continuous functions on $\mathbb{R}$, then $\{x : f(x) > g(x)\}$ is an open set.

Since f and g are continuous, so ~~or~~ is $h = f - g$. Then

$$\{x : f(x) - g(x)\} = \{x : h(x) > 0\}$$
$$= h^{-1}((0, \infty)).$$

Finally, the inverse image of an open set ~~undera~~ under a continuous function is ~~con~~ open. $\square$

(24) Suppose $A \subset \mathbb{R}$ is closed. Show there is a countable subset $B \subset A$ so that every point of A is the limit point of a sequence in B.

The set of intervals $I = (r, s)$ with rational endpoints is countable, since $\mathbb{Q} \times \mathbb{Q}$ is countable.

For each such interval I choose a point $b \in I \cap A$ if the intersection is non-empty. Let B be the union of chosen points. Clearly B is a countable subset of A.

Suppose $x \in A$. If $n \in \mathbb{N}$ then $(x - \frac{1}{n}, x)$ contains a rational number r and $(x, x + \frac{1}{n})$ contains a rational number s. Since $I = (r, s)$ contains $x \in A$, it contains some point $b \in$ ~~$I \cap A$~~ B. Let $x_n = b$.

Then $\{x_n\} \subset B$ and $|x - x_n| < \frac{1}{n}$ so $x_n \to x$. Thus B has the desired properties. $\square$