We note that if we reverse the order, then the composition $f \circ g$ is given by the formula

$$(f \circ g)(x) = 1 - x,$$

but only for those x in the domain $D(g) = \{x : x \geq 0\}$. □

We now give the relationship between composite functions and inverse images. The proof is left as an instructive exercise.

1.1.14 Theorem *Let $f : A \to B$ and $g : B \to C$ be functions and let H be a subset of C. Then we have*

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)).$$

Note the *reversal* in the order of the functions.

Restrictions of Functions

If $f : A \to B$ is a function and if $A_1 \subset A$, we can define a function $f_1 : A_1 \to B$ by

$$f_1(x) := f(x) \quad \text{for} \quad x \in A_1.$$

The function f_1 is called the **restriction of f to A_1**. Sometimes it is denoted by $f_1 = f|A_1$.

It may seem strange to the reader that one would ever choose to throw away a part of a function, but there are some good reasons for doing so. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the **squaring function:**

$$f(x) := x^2 \quad \text{for} \quad x \in \mathbb{R},$$

then f is not injective, so it cannot have an inverse function. However, if we restrict f to the set $A_1 := \{x : x \geq 0\}$, then the restriction $f|A_1$ is a bijection of A_1 onto A_1. Therefore, this restriction has an inverse function, which is the **positive square root function.** (Sketch a graph.)

Similarly, the trigonometric functions $S(x) := \sin x$ and $C(x) := \cos x$ are not injective on all of $\mathbb{R}$. However, by making suitable restrictions of these functions, one can obtain the **inverse sine** and the **inverse cosine** functions that the reader has undoubtedly already encountered.

Exercises for Section 1.1

1. Let $A := \{k : k \in \mathbb{N}, k \leq 20\}$, $B := \{3k - 1 : k \in \mathbb{N}\}$, and $C := \{2k + 1 : k \in \mathbb{N}\}$. Determine the sets:
   (a) $A \cap B \cap C$,
   (b) $(A \cap B) \backslash C$,
   (c) $(A \cap C) \backslash B$.
2. Draw diagrams to simplify and identify the following sets:
   (a) $A \backslash (B \backslash A)$,
   (b) $A \backslash (A \backslash B)$,
   (c) $A \cap (B \backslash A)$.
3. If A and B are sets, show that $A \subseteq B$ if and only if $A \cap B = A$.
4. Prove the second De Morgan Law [Theorem 1.1.4(b)].
5. Prove the Distributive Laws:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

6. The **symmetric difference** of two sets A and B is the set D of all elements that belong to either A or B but not both. Represent D with a diagram.
   (a) Show that $D = (A \backslash B) \cup (B \backslash A)$.
   (b) Show that D is also given by $D = (A \cup B) \backslash (A \cap B)$.
7. For each $n \in \mathbb{N}$, let $A_n = \{(n+1)k : k \in \mathbb{N}\}$.
   (a) What is $A_1 \cap A_2$?
   (b) Determine the sets $\cup\{A_n : n \in \mathbb{N}\}$ and $\cap\{A_n : n \in \mathbb{N}\}$.
8. Draw diagrams in the plane of the Cartesian products $A \times B$ for the given sets A and B.
   (a) $A = \{x \in \mathbb{R} : 1 \le x \le 2 \text{ or } 3 \le x \le 4\}$, $B = \{x \in \mathbb{R} : x = 1 \text{ or } x = 2\}$.
   (b) $A = \{1, 2, 3\}$, $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$.
9. Let $A := B := \{x \in \mathbb{R} : -1 \le x \le 1\}$ and consider the subset $C := \{(x, y) : x^2 + y^2 = 1\}$ of $A \times B$. Is this set a function? Explain.
10. Let $f(x) := 1/x^2$, $x \neq 0$, $x \in \mathbb{R}$.
    (a) Determine the direct image $f(E)$ where $E := \{x \in \mathbb{R} : 1 \le x \le 2\}$.
    (b) Determine the inverse image $f^{-1}(G)$ where $G := \{x \in \mathbb{R} : 1 \le x \le 4\}$.
11. Let $g(x) := x^2$ and $f(x) := x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h := g \circ f$.
    (a) Find the direct image $h(E)$ of $E := \{x \in \mathbb{R} : 0 \le x \le 1\}$.
    (b) Find the inverse image $h^{-1}(G)$ of $G := \{x \in \mathbb{R} : 0 \le x \le 4\}$.
12. Let $f(x) := x^2$ for $x \in \mathbb{R}$, and let $E := \{x \in \mathbb{R} : -1 \le x \le 0\}$ and $F := \{x \in \mathbb{R} : 0 \le x \le 1\}$. Show that $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$, while $f(E) = f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. What happens if 0 is deleted from the sets E and F?
13. Let f and E, F be as in Exercise 12. Find the sets $E \backslash F$ and $f(E) \backslash f(f)$ and show that it is *not* true that $f(E \backslash F) \subseteq f(E) \backslash f(F)$.
14. Show that if $f : A \to B$ and E, F are subsets of A, then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$.
15. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.
16. Show that the function f defined by $f(x) := x/\sqrt{x^2 + 1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.
17. For $a, b \in \mathbb{R}$ with $a < b$, find an explicit bijection of $A := \{x : a < x < b\}$ onto $B := \{y : 0 < y < 1\}$.
18. (a) Give an example of two functions f, g on $\mathbb{R}$ to $\mathbb{R}$ such that $f \neq g$, but such that $f \circ g = g \circ f$.
    (b) Give an example of three functions f, g, h on $\mathbb{R}$ such that $f \circ (g + h) \neq f \circ g + f \circ h$.
19. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.
    (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.
20. (a) Suppose that f is an injection. Show that $f^{-1} \circ f(x) = x$ for all $x \in D(f)$ and that $f \circ f^{-1}(y) = y$ for all $y \in R(f)$.
    (b) If f is a bijection of A onto B, show that f^{-1} is a bijection of B onto A.
21. Prove that if $f : A \to B$ is bijective and $g : B \to C$ is bijective, then the composite $g \circ f$ is a bijective map of A onto C.
22. Let $f : A \to B$ and $g : B \to C$ be functions.
    (a) Show that if $g \circ f$ is injective, then f is injective.
    (b) Show that if $g \circ f$ is surjective, then g is surjective.
23. Prove Theorem 1.1.14.
24. Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

[This result can also be proved without using Mathematical Induction. If we let $s_n := 1 + r + r^2 + \cdots + r^n$, then $rs_n = r + r^2 + \cdots + r^{n+1}$, so that

$$(1 - r)s_n = s_n - rs_n = 1 - r^{n+1}.$$

If we divide by $1 - r$, we obtain the stated formula.]

(g) Careless use of the Principle of Mathematical Induction can lead to obviously absurd conclusions. The reader is invited to find the error in the "proof" of the following assertion.

Claim: If $n \in \mathbb{N}$ and if the maximum of the natural numbers p and q is n, then $p = q$.

"Proof." Let S be the subset of $\mathbb{N}$ for which the claim is true. Evidently, $1 \in S$ since if $p, q \in \mathbb{N}$ and their maximum is 1, then both equal 1 and so $p = q$. Now assume that $k \in S$ and that the maximum of p and q is $k + 1$. Then the maximum of $p - 1$ and $q - 1$ is k. But since $k \in S$, then $p - 1 = q - 1$ and therefore $p = q$. Thus, $k + 1 \in S$, and we conclude that the assertion is true for all $n \in \mathbb{N}$.

(h) There are statements that are true for *many* natural numbers but that are not true for *all* of them.

For example, the formula $p(n) := n^2 - n + 41$ gives a prime number for $n = 1, 2, \ldots,$ 40. However, $p(41)$ is obviously divisible by 41, so it is not a prime number.

Another version of the Principle of Mathematical Induction is sometimes quite useful. It is called the "Principle of Strong Induction," even though it is in fact equivalent to 1.2.2.

1.2.5 Principle of Strong Induction *Let S be a subset of $\mathbb{N}$ such that*

(1″) $1 \in S$.

(2″) *For every $k \in \mathbb{N}$, if $\{1, 2, \ldots, k\} \subseteq S$, then $k + 1 \in S$.*

Then $S = \mathbb{N}$.

We will leave it to the reader to establish the equivalence of 1.2.2 and 1.2.5.

Exercises for Section 1.2

1. Prove that $1/1 \cdot 2 + 1/2 \cdot 3 + \cdots + 1/n(n+1) = n/(n+1)$ for all $n \in \mathbb{N}$.
2. Prove that $1^3 + 2^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$ for all $n \in \mathbb{N}$.
3. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.
4. Prove that $1^2 + 3^2 + \cdots + (2n-1)^2 = (4n^3 - n)/3$ for all $n \in \mathbb{N}$.
5. Prove that $1^2 - 2^2 + 3^2 + \cdots + (-1)^{n+1}n^2 = (-1)^{n+1}n(n+1)/2$ for all $n \in \mathbb{N}$.
6. Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.
7. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.
8. Prove that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.
9. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.
10. Conjecture a formula for the sum $1/1 \cdot 3 + 1/3 \cdot 5 + \cdots + 1/(2n-1)(2n+1)$, and prove your conjecture by using Mathematical Induction.
11. Conjecture a formula for the sum of the first n odd natural numbers $1 + 3 + \cdots + (2n - 1)$, and prove your formula by using Mathematical Induction.
12. Prove the Principle of Mathematical Induction 1.2.3 (second version).

13. Prove that $n < 2^n$ for all $n \in \mathbb{N}$.
14. Prove that $2^n < n!$ for all $n \geq 4$, $n \in \mathbb{N}$.
15. Prove that $2n - 3 \leq 2^{n-2}$ for all $n \geq 5$, $n \in \mathbb{N}$.
16. Find all natural numbers n such that $n^2 < 2^n$. Prove your assertion.
17. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.
18. Prove that $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$ for all $n \in \mathbb{N}$, $n > 1$.
19. Let S be a subset of $\mathbb{N}$ such that (a) $2^k \in S$ for all $k \in \mathbb{N}$, and (b) if $k \in S$ and $k \geq 2$, then $k - 1 \in S$. Prove that $S = \mathbb{N}$.
20. Let the numbers x_n be defined as follows: $x_1 := 1$, $x_2 := 2$, and $x_{n+2} := \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in \mathbb{N}$. Use the Principle of Strong Induction (1.2.5) to show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

Section 1.3 Finite and Infinite Sets

When we count the elements in a set, we say "one, two, three, . . . ," stopping when we have exhausted the set. From a mathematical perspective, what we are doing is defining a bijective mapping between the set and a portion of the set of natural numbers. If the set is such that the counting does not terminate, such as the set of natural numbers itself, then we describe the set as being infinite.

The notions of "finite" and "infinite" are extremely primitive, and it is very likely that the reader has never examined these notions very carefully. In this section we will define these terms precisely and establish a few basic results and state some other important results that seem obvious but whose proofs are a bit tricky. These proofs can be found in Appendix B and can be read later.

1.3.1 Definition **(a)** The empty set $\emptyset$ is said to have 0 **elements**.

(b) If $n \in \mathbb{N}$, a set S is said to have n **elements** if there exists a bijection from the set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ onto S.

(c) A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathbb{N}$.

(d) A set S is said to be **infinite** if it is not finite.

Since the inverse of a bijection is a bijection, it is easy to see that a set S has n elements if and only if there is a bijection from S onto the set $\{1, 2, \ldots, n\}$. Also, since the composition of two bijections is a bijection, we see that a set S_1 has n elements if and only if there is a bijection from S_1 onto another set S_2 that has n elements. Further, a set T_1 is finite if and only if there is a bijection from T_1 onto another set T_2 that is finite.

It is now necessary to establish some basic properties of finite sets to be sure that the definitions do not lead to conclusions that conflict with our experience of counting. From the definitions, it is not entirely clear that a finite set might not have n elements for *more than one* value of n. Also it is conceivably possible that the set $\mathbb{N} := \{1, 2, 3, \ldots\}$ might be a finite set according to this definition. The reader will be relieved that these possibilities do not occur, as the next two theorems state. The proofs of these assertions, which use the fundamental properties of $\mathbb{N}$ described in Section 1.2, are given in Appendix B.

1.3.2 Uniqueness Theorem *If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.*

Exercises for Section 1.3

1. Prove that a nonempty set T_1 is finite if and only if there is a bijection from T_1 onto a finite set T_2.
2. Prove parts (b) and (c) of Theorem 1.3.4.
3. Let $S := \{1, 2\}$ and $T := \{a, b, c\}$.
   (a) Determine the number of different injections from S into T.
   (b) Determine the number of different surjections from T onto S.
4. Exhibit a bijection between $\mathbb{N}$ and the set of all odd integers greater than 13.
5. Give an explicit definition of the bijection f from $\mathbb{N}$ onto $\mathbb{Z}$ described in Example 1.3.7(b).
6. Exhibit a bijection between $\mathbb{N}$ and a proper subset of itself.
7. Prove that a set T_1 is denumerable if and only if there is a bijection from T_1 onto a denumerable set T_2.
8. Give an example of a countable collection of finite sets whose union is not finite.
9. Prove in detail that if S and T are denumerable, then $S \cup T$ is denumerable.
10. (a) If (m, n) is the 6th point down the 9th diagonal of the array in Figure 1.3.1, calculate its number according to the counting method given for Theorem 1.3.8.
    (b) Given that $h(m, 3) = 19$, find m.
11. Determine the number of elements in $\mathcal{P}(S)$, the collection of all subsets of S, for each of the following sets:
    (a) $S := \{1, 2\}$,
    (b) $S := \{1, 2, 3\}$,
    (c) $S := \{1, 2, 3, 4\}$.
    Be sure to include the empty set and the set S itself in $\mathcal{P}(S)$.
12. Use Mathematical Induction to prove that if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.
13. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.