Discrete Conformality and Graph Embedding

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Connections between Analysis and Computational Geometry
University of North Carolina

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Outlines

- Combinatorics begets Geometry
- Circle Packing
  - Spontaneous and Wonderful Geometry
  - This Geometry conjures Conformality
  - Conformality recruits Theory
  - Theory pays Dividends

Technical/Summary

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Graph Embedding
Combinatorics begets Geometry
Combinatorics begets Geometry — Circle Packing
Combinatorics begets Geometry — Circle Packing

Spontaneous and Wonderful Geometry
Outline

- *Combinatorics begets Geometry* — Circle Packing
- *Spontaneous and Wonderful Geometry*
- *This Geometry conjures Conformality*
Combinatorics begets Geometry — Circle Packing

Spontaneous and Wonderful Geometry

This Geometry conjures Conformality

Conformality recruits Theory
Combinatorics begets Geometry — Circle Packing

Spontaneous and Wonderful Geometry

This Geometry conjures Conformality

Conformality recruits Theory

Theory pays Dividends
Combinatorics begets Geometry — Circle Packing

Spontaneous and Wonderful Geometry

This Geometry conjures Conformality

Conformality recruits Theory

Theory pays Dividends

Technical/Summary
Basic Geometry
Circle Packing: imposing geometry on Combinatorics

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Graph Embedding
Circle Packing: imposing geometry on Combinatorics
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Graph Embedding
Circle Packing

Definition: A circle packing is a configuration of circles with a specified pattern of tangencies.

Key Theorem (Koebe-Andreev-Thurston): For any triangulation $K$ of a sphere, there exists an associated univalent circle packing $P_K$ of the Riemann sphere, unique up to Möbius transformations.

Cautionary Note: Circle packing is NOT 2D sphere packing.
Circle Packing

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Existence/Uniqueness

Theorem: Given any triangulation $K$ of an oriented topological surface $S$, there is an essentially unique conformal structure on $S$ supporting a circle packing $P$ having the combinatorics of $K$ and 'filling' $S$. 

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Graph Embedding
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Wonderful Geometry

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Graph Embedding
Summary: Embedding

- Euclidean, hyperbolic, or spherical geometry
- Simultaneous embedding of the dual
- Rigidity, but with flexible boundary controls
- Controlled distortion
- Symmetry preserving
Euclidean, hyperbolic, or spherical geometry
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Euclidean, hyperbolic, or spherical geometry

Simultaneous embedding of the dual

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Euclidean, hyperbolic, or spherical geometry

Simultaneous embedding of the dual

Rigidity, but with flexible boundary controls

Controlled distortion

Symmetry preserving
Discrete Conformal Mapping

Graph Embedding

harmonic measure

extremal length = L/H
Convergence

Graph Embedding
Conformal Flow

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Graph Embedding
Conformal Welding

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Graph Embedding
Summary: Conformality

- Discrete Analytic Function Theory
Summary: Conformality

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- Harmonic Measure
Summary: Conformality

- Discrete Analytic Function Theory
- Harmonic Measure
- Random Walks
Summary: Conformality

- Discrete Analytic Function Theory
- Harmonic Measure
- Random Walks
- Extremal Length
Summary: Conformality

- Discrete Analytic Function Theory
- Harmonic Measure
- Random Walks
- Extremal Length
- Curvature Flow
Discrete Analytic Function Theory

Harmonic Measure

Random Walks

Extremal Length

Curvature Flow

Conformal Moduli
Synergies
Synergies — e.g. Conformal Chair Tiling
Synergies — e.g. Conformal Chair Tiling
Chair substitution
Chair substitution tiling
Conformal Chairs

Make each tile into a combinatorial octagon
Make each tile into a combinatorial octagon
Conformal Chairs

The tiling itself
Conformal Chairs

The tiling itself

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Graph Embedding
Conformal Chairs

The underlying circle packing

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Conformal Chairs

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Graph Embedding
Detail: converging to a conformally regular tiling of octagons
Summary: Synergies

Grid generation
Conformal tiling
Emergent conformality
Spontaneity — surprises in nearly every experiment

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Graph Embedding
Grid generation
Summary: Synergies

- Grid generation
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- Emergent conformality
Summary: Synergies

- Grid generation
- Conformal tiling
- Emergent conformality
- Spontaneity
• Grid generation

• Conformal tiling

• Emergent conformality

• Spontaneity — surprises in nearly every experiment
Resources and Acknowledgements

CirclePack, Java, cross-platform, open source

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Graph Embedding
CirclePack, Java, cross-platform, open source

For a general overview discrete analyticity via circle packing, see Notices of the AMS, December 2003, cover article.

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