Smooth Interpolation

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Contributions from

- Whitney (1930's)
- Glaeser (1950's)
- Brudnyi-Shvartsman (1980's-present)
- Bierstone-Milman-Pawlucki (2000's-present)
- Fefferman/Fefferman-Klartag (2003-present)
- Fefferman-I-Luli (2010-present)

Notation

Let $F : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth.

- For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$\partial^\alpha F(x) := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} F(x);$$

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

- For $k \geq 1$,

$$\nabla^k F(x) := \big(\partial^\alpha F(x)\big)_{|\alpha|=k}.$$

Notation

Let $F : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth.

- For $m \geq 1$,

$$\|F\|_{C^m} := \sup_{x \in \mathbb{R}^n} |\nabla^m F(x)|.$$

The Problem

Given:

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Compute a C-optimal interpolant: $F : \mathbb{R}^n \to \mathbb{R}$ with

(a) $F = f$ on E;

(b) $\|F\|_{C^m} \leq C \cdot \|G\|_{C^m}$ whenever $G = f$ on E.

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Side Questions:

- Estimate the nearly minimal norm $\|F\|_{C^m}$.
- How long do these computations take?

Theorem (Fefferman-Klartag ('09))

Can construct C_1-optimal interpolants in time $C_2 N \log(N)$.

A Variant Problem

For $m \geq 1$ and $p \geq 1$, let

$$\|F\|_{L^{m,p}} := \left(\int_{x \in \mathbb{R}^n} |\nabla^m F(x)|^p dx \right)^{1/p}.$$

Compute a C-optimal Sobolev interpolant: $F : \mathbb{R}^n \to \mathbb{R}$ with

- $F = f$ on E;
- $\|F\|_{L^{m,p}} \leq C \cdot \|G\|_{L^{m,p}}$ whenever $G = f$ on E.

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Can we prove this? Can we achieve $O(N \log(N))$?

Example I

Given:

- $t_1, \ldots, t_N \in \mathbb{R}$
- $p_1, \ldots, p_N \in \mathbb{R}$

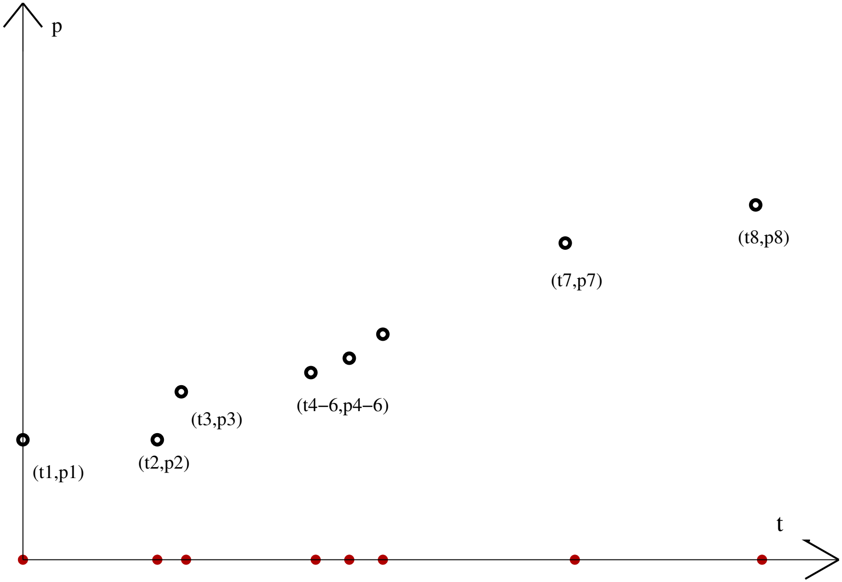
Construct $p : \mathbb{R} \to \mathbb{R}$ with

(a) $p(t_1) = p_1, \cdots, p(t_N) = p_N$;

(b) $\sup_{t \in \mathbb{R}} |p'(t)| \leq \sup_{t \in \mathbb{R}} |q'(t)|$, for any other interpolant q.

Estimate:

$$M = \sup_{t \in \mathbb{R}} |p'(t)|.$$


p
(t8,p8)
(t7,p7)
(t4–6,p4–6)
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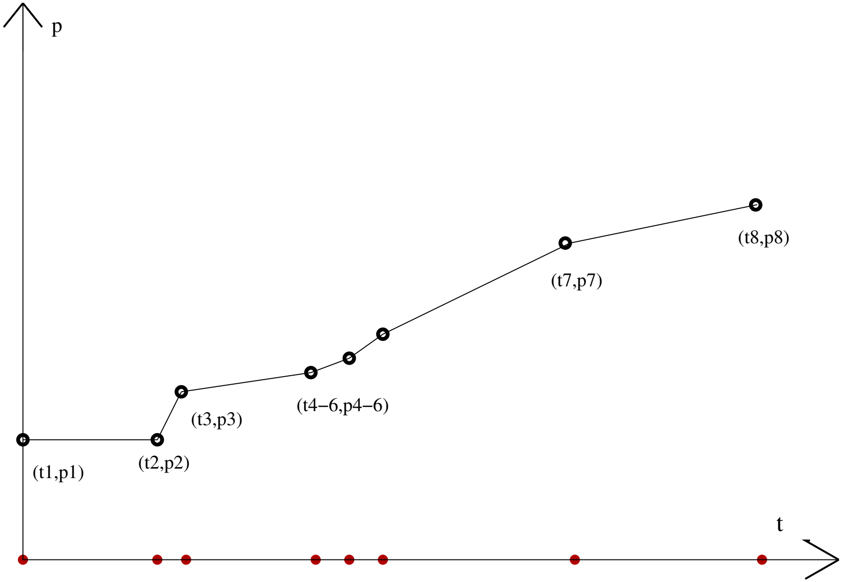
(t1,p1)
(t2,p2)
t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$$(1) \quad \sup |p'(t)| = \left| \frac{p_2 - p_3}{t_2 - t_3} \right|.$$

The competitor q interpolates the data, so MVT $\implies$

$$(2) \ \exists t^* \in [t_2, t_3] \quad \text{with} \quad q'(t^*) = \frac{p_2 - p_3}{t_2 - t_3}.$$

Finally, (1) and (2) $\implies$

$$(3) \sup |p'(t)| \leq C \sup |q'(t)|.$$

Example II

Given:

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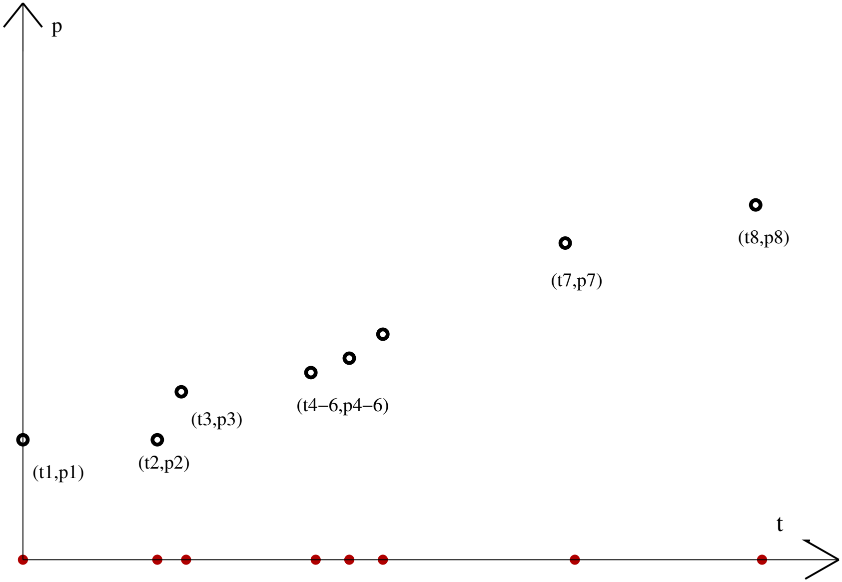
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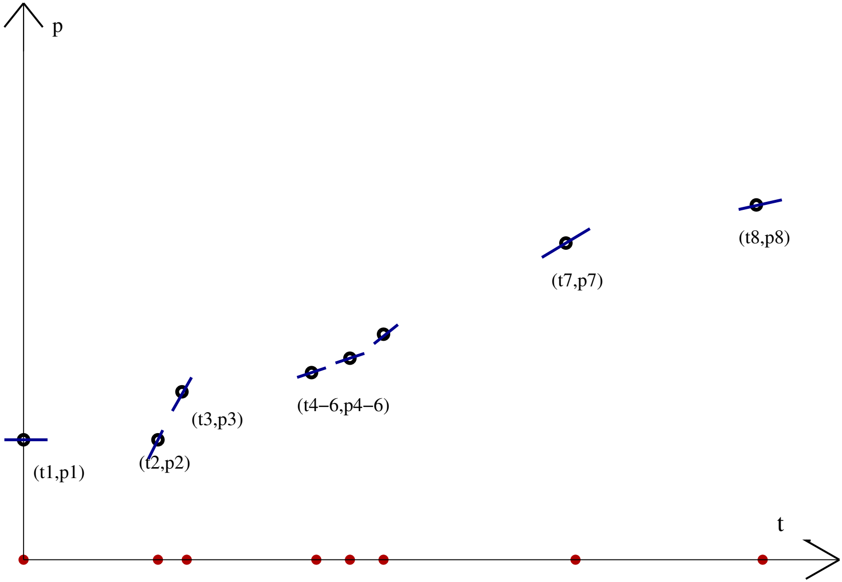
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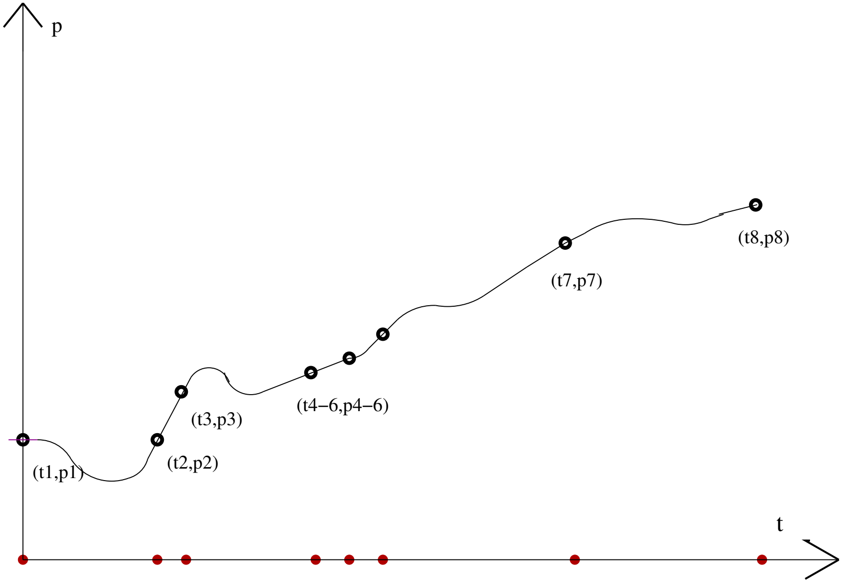
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Goal: Construct $F : [0,1]^2 \to \mathbb{R}$ with

$$F = f \text{ on } E;$$

$$|\nabla^2 F| \leq C \text{ on } [0,1]^2.$$

Two Examples

(a) E contained in a line.

(b) E contained in a smooth curve.

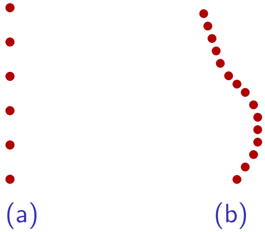

Figure: Sets with 1D structure

The Straight Line

Suppose that

$$E = \{(0, y_1), \ldots, (0, y_N)\};$$

$$f : E \to \mathbb{R}.$$

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$$g(y_k) = f(0, y_k) \quad \text{for} \quad k = 1, \ldots, N,$$

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$$|g''(y)| \leq C.$$

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and

$$|g''(y)| \leq C.$$

Step 2: Define $F(x, y) := g(y)$. Then

$$|\nabla^2 F(x, y)| = |g''(y)| \leq C \quad \text{for all} \quad (x, y).$$

The Smooth Curve

Suppose that

$$E \subset \{(\phi(y), x)\}, \quad \text{where} \quad |\phi''| \leq 1.$$

(a) (b)

Figure: Sets with 1D structure

- Consider the diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Phi(x, y) = (x - \phi(y), y).$$

The Smooth Curve

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(c)

(d)

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- Consider the diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$:

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(e) (f)

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- Consider the diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Phi(x, y) = (x - \phi(y), y).$$

- Note that Φ maps E onto a line segment.
- There is a $1-1$ correspondence between interpolation problems on E and on $\Phi(E)$.

Some Notation

$S(x, \delta) :=$ square with center x and sidelength δ.

$\delta(S) :=$ sidelength of the square S.

$A \cdot S :=$ A-dilate of S about its center.

Definition (Neat Squares)

A square S is neat if $3S \cap E$ lies on the graph of a function h with

$$|h''| \leq \delta(S)^{-1} \text{ uniformly.}$$

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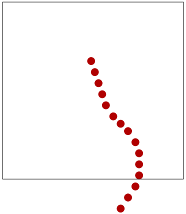
$$|h''| \leq 1 \text{ uniformly.}$$
- Small enough squares are neat.
- If S is neat and $S' \subset S$ then S' is neat.

Lemma

Suppose that S is neat. Then we can construct $F : 3S \to \mathbb{R}$ with $F = f$ on $E \cap 3S$ and $|\nabla^2 F| \leq C$ on $3S$.

(a) A Neat S...



(b) Rescaled

Definition (Messy Squares)

A square S is messy if S is not neat.

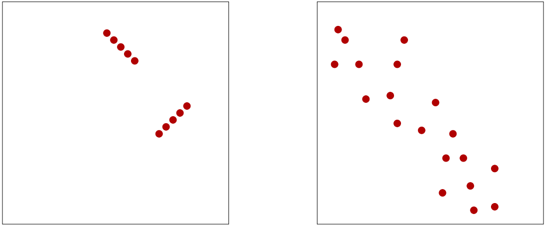
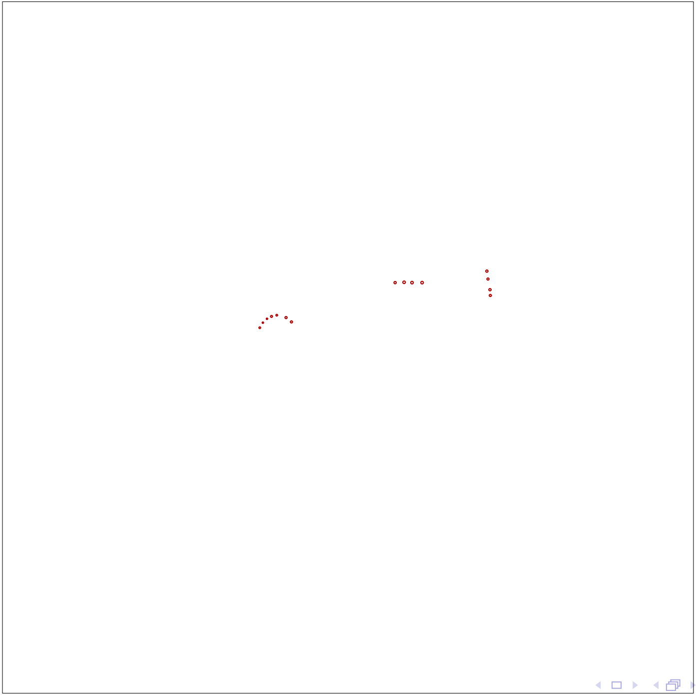
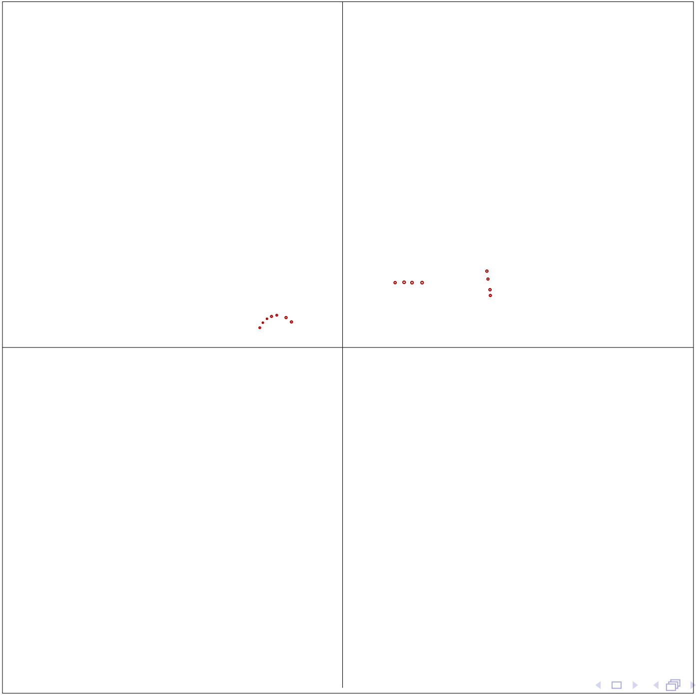


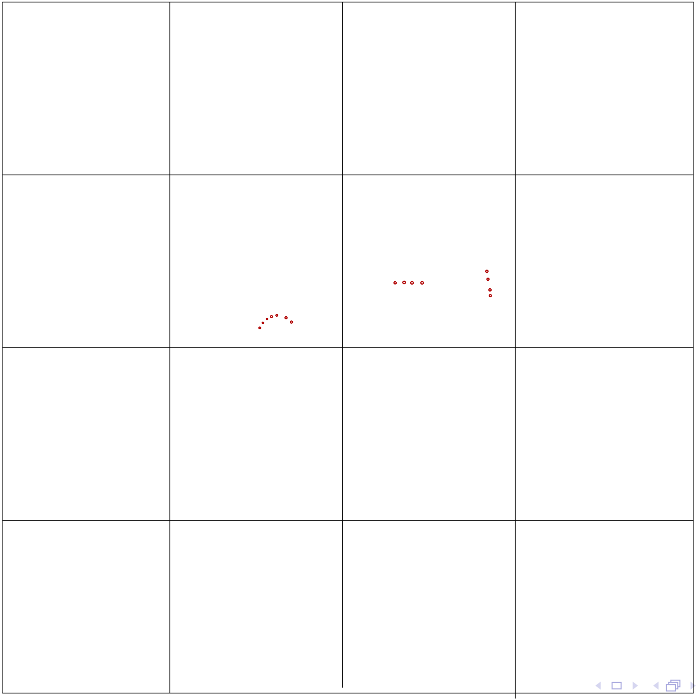
Figure: Some Messy Squares

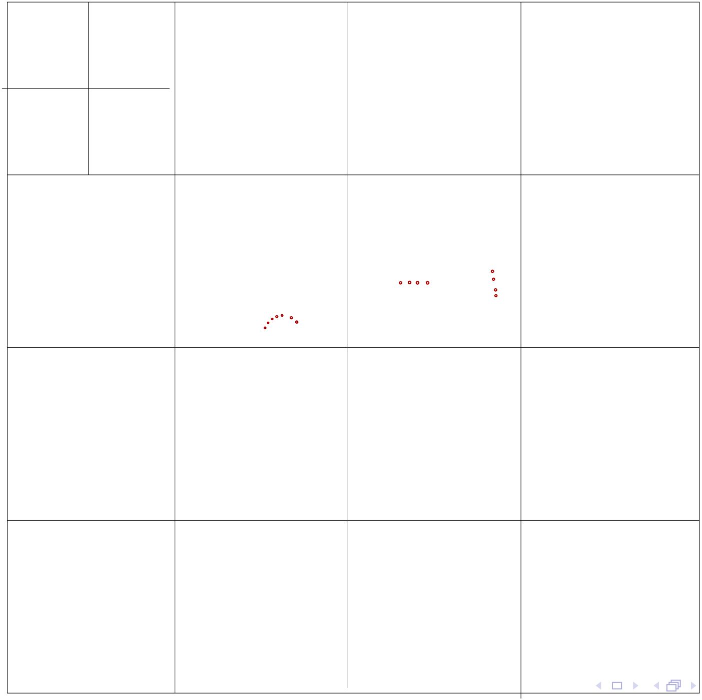
The CZ Decomposition

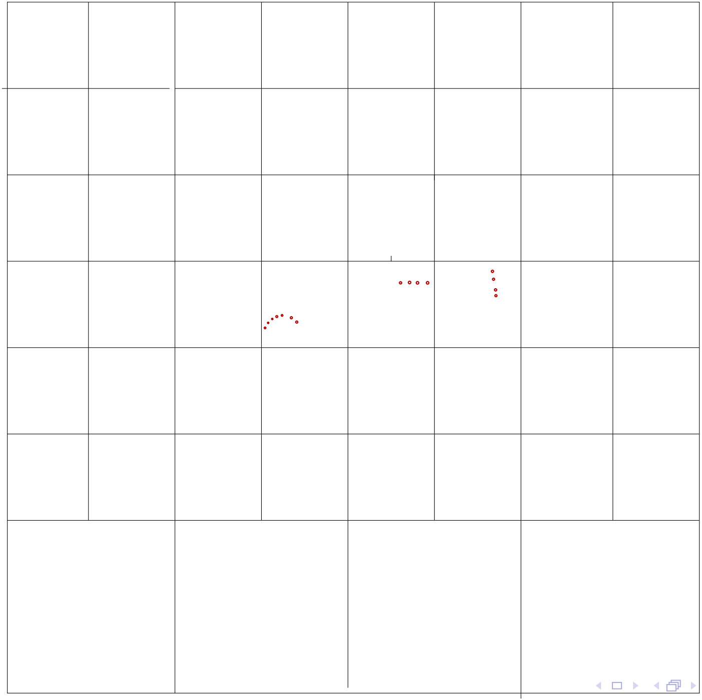
- Keep bisecting $S \subset [0,1]^2$ until S is neat.
- Define CZ as the collection of nonbisected squares.

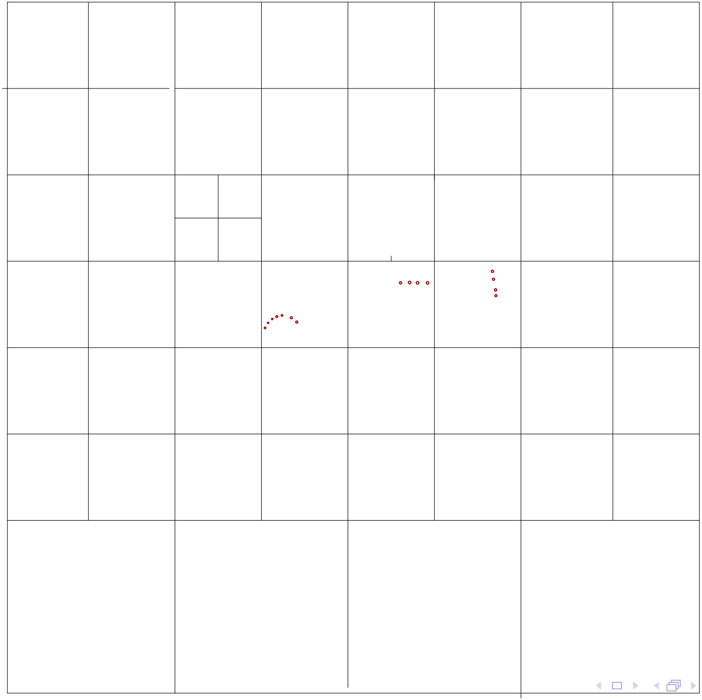


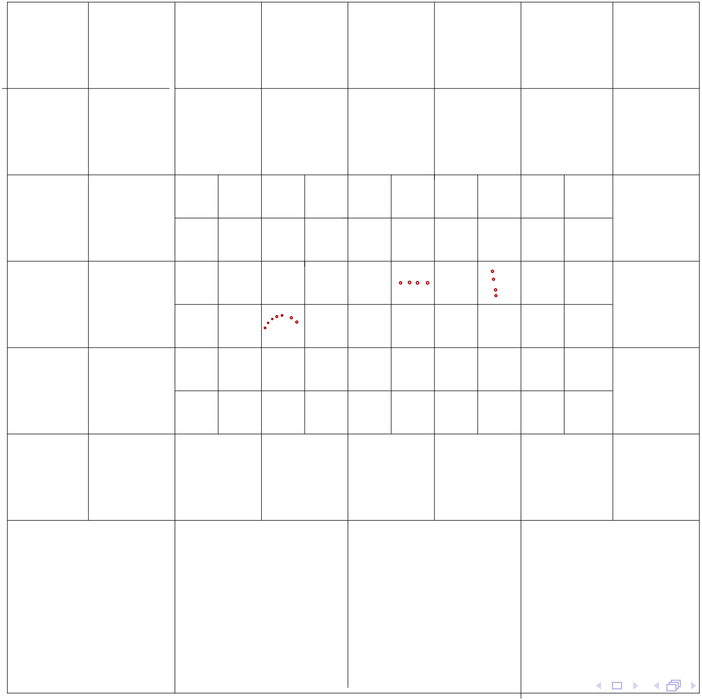












Properties of the CZ Decomposition

Note that $CZ = \{S_\nu\}$ partitions $[0,1]^2$.

(a) If $S \in CZ$, then S is neat.

(b) If $S \in CZ$, then $3S$ is messy.

(c) Good Geometry: If $S, S' \in CZ$ touch, then

$$\frac{1}{2}\delta(S') \leq \delta(S) \leq 2\delta(S').$$

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<u>One-Line Proofs:</u>

(a) That was our stopping rule!

(b) $3S$ contains the dyadic parent S^+.

(c) If $S, S' \in CZ$ touch and $\delta(S) \leq \delta(S')/4$, then $3S^+ \subset 3S'$.

The Naive Plan: Step 1

Construct local interpolants for the CZ squares:

- Functions $F_\nu : 3S_\nu \to \mathbb{R}$ that satisfy:

$$(a) \quad F_\nu = f \quad \text{on} \quad E \cap (1.1)S_\nu.$$

$$(b) \quad |\nabla^2 F_\nu| \le C \quad \text{on} \quad 3S_\nu.$$

The Naive Plan: Step 2

Introduce a partition of unity adapted to the CZ squares:

- Functions $\theta_\nu : [0,1]^2 \to \mathbb{R}$ that satisfy

$$
\begin{aligned}
&(a) \quad 0 \leq \theta_\nu \leq 1;\\
&(b) \quad \mathrm{supp}(\theta_\nu) \subset (1.1) S_\nu;\\
&(c) \quad |\nabla \theta_\nu| \leq C \cdot \delta(S_\nu)^{-1} \quad \text{and} \quad |\nabla^2 \theta_\nu| \leq C \cdot \delta(S_\nu)^{-2};\\
&(d) \quad \sum_\nu \theta_\nu = 1 \quad \text{on} \quad [0,1]^2.
\end{aligned}
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The Naive Plan: Step 3

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$$F = \sum_{\nu} \theta_{\nu} F_{\nu}.$$

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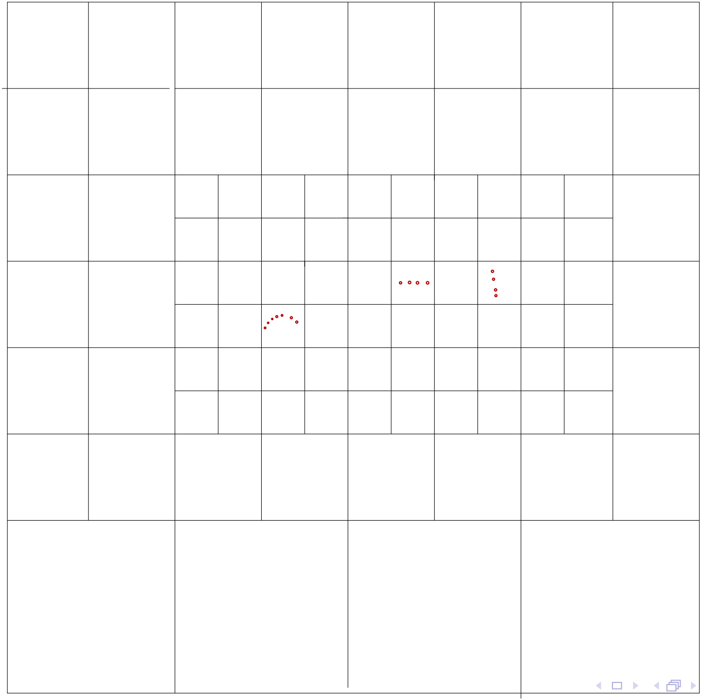
Question: Is

$$|\nabla^2 F| \leq C \qquad \text{on} \qquad [0,1]^2?$$

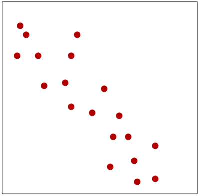


Arranging Consistency

Lemma

Let S be any messy square. Then there exists a "non-degenerate" triplet

$$T \subset E \cap 9S.$$



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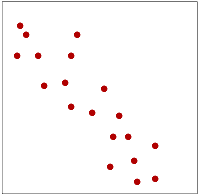

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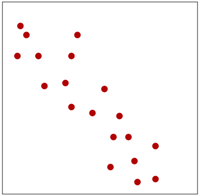

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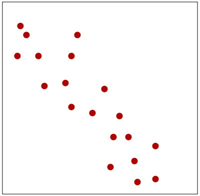

- Associate to each S_ν some "non-degenerate" triplet: $T_\nu \subset E \cap 9S_\nu$.
- Let L_ν be affine with $L_\nu = f$ on T_ν.
- This gives our rough guess for the affine structure of our interpolant.

- Let's check consistency:

Lemma

Suppose that S_ν and $S_{\nu'}$ are neighboring squares. Then

$$|\nabla L_\nu - \nabla L_{\nu'}| \leq C\delta(S_\nu)$$

and

$$|L_\nu - L_{\nu'}| \leq C\delta(S_\nu)^2 \quad on \quad 100S_\nu.$$

Need this version of Rolle's Theorem:

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Suppose that H vanishes on a "non-degenerate" triplet $T \subset S$ and $\|H\|_{C^2} \leq 1$. Then,

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Recall that

- $G = f$ on E and $\|G\|_{C^2} \leq 1$.
- $L_\nu = f$ on T_ν.
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For any $x \in 100S_\nu$,

$$|\nabla L_\nu - \nabla L_{\nu'}| \leq |\nabla L_\nu - \nabla G(x)| + |\nabla L_{\nu'} - \nabla G(x)| \leq C\delta(S),$$

Lemma

Suppose that S_ν *and* $S_{\nu'}$ *are neighboring squares. Then*

$$|\nabla L_\nu - \nabla L_{\nu'}| \leq C\delta(S_\nu)$$

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$$|(L_\nu - L_{\nu'})(x)| \leq C\delta(S_\nu)^2 \quad \text{on} \quad 100S_\nu.$$

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- Define $F_\nu := L_\nu$ whenever $E \cap (1.1)S_\nu = \emptyset$.

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- Do something similar for all other CZ squares.

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- Define $F_\nu := L_\nu$ whenever $E \cap (1.1)S_\nu = \emptyset$.
- Do something similar for all other CZ squares.
- Set

$$F = \sum_\nu F_\nu \theta_\nu.$$

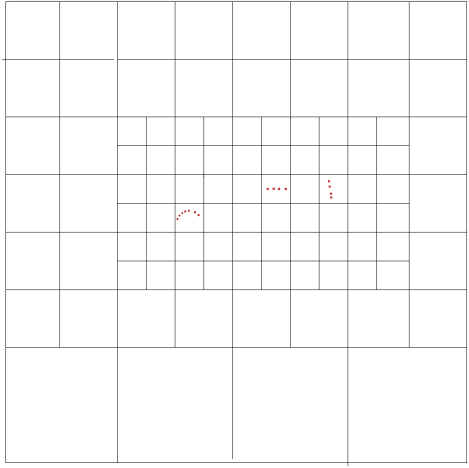
- We obtain

$$\|F\|_{C^2} \leq C'.$$

Keystone Squares

Definition

$S_\mu \in CZ$ is keystone if every CZ square that intersects $9S_\mu$ has sidelength larger than S_μ.



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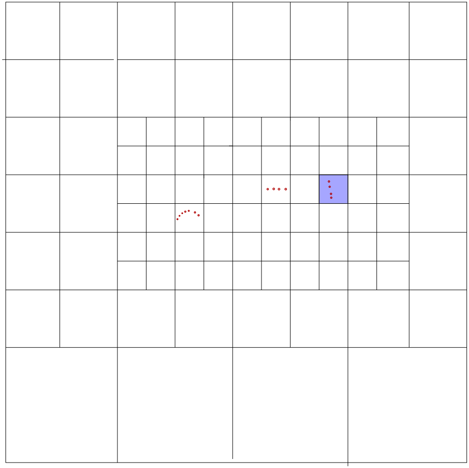
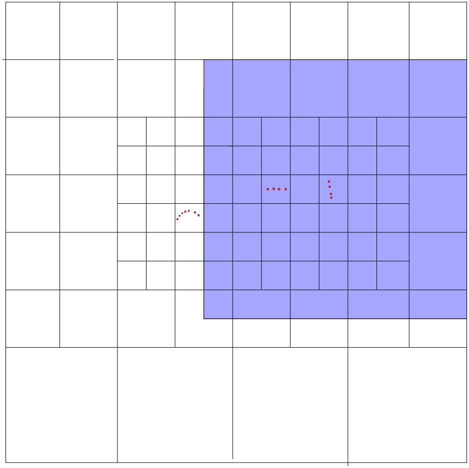


Definition

$S_\mu \in CZ$ is keystone if every CZ square that intersects $9S_\mu$ has sidelength larger than S_μ.



Diverging Paths