Smooth Interpolation

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Contributions from

- Whitney (1930’s)
- Glaeser (1950’s)
- Brudnyi-Shvartsman (1980’s-present)
- Bierstone-Milman-Pawlucki (2000’s-present)
- Fefferman/Fefferman-Klartag (2003-present)
- Fefferman-I-Luli (2010-present)
Let $F : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth.

- For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,
  \[
  \partial^\alpha F(x) := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} F(x);
  \]
  \[
  |\alpha| := \alpha_1 + \cdots + \alpha_n.
  \]

- For $k \geq 1$,
  \[
  \nabla^k F(x) := (\partial^\alpha F(x))_{|\alpha|=k}.
  \]
Let $F : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth.

For $m \geq 1$,

$$\|F\|_{C^m} := \sup_{x \in \mathbb{R}^n} |\nabla^m F(x)|.$$
The Problem

Given:

- Finite subset $E \subset \mathbb{R}^n$ with cardinality $N$;
- Function $f : E \rightarrow \mathbb{R}$.
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Compute a $C$-optimal interpolant: $F : \mathbb{R}^n \to \mathbb{R}$ with

(a) $F = f$ on $E$;

(b) $\|F\|_{C^m} \leq C \cdot \|G\|_{C^m}$ whenever $G = f$ on $E$. 

Side Questions:
- Estimate the nearly minimal norm $\|F\|_{C^m}$.
- How long do these computations take?
The Problem

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Theorem (Fefferman-Klartag ('09))

*Can construct* $C_1$-*optimal interpolants in time* $C_2N \log(N)$.
For $m \geq 1$ and $p \geq 1$, let

$$\|F\|_{L^{m,p}} := \left( \int_{x \in \mathbb{R}^n} |\nabla^m F(x)|^p \, dx \right)^{1/p}.$$ 

Compute a $C$-optimal Sobolev interpolant: $F : \mathbb{R}^n \rightarrow \mathbb{R}$ with

- $F = f$ on $E$;
- $\|F\|_{L^{m,p}} \leq C \cdot \|G\|_{L^{m,p}}$ whenever $G = f$ on $E$. 
Theorem (Fefferman-I-Luli ('11))

Can construct $C$-optimal Sobolev interpolants.

Plausible running-time bound is $O(m, n, p(N \log(\Delta)))$, where

$$\Delta := \max \{ |x - y| : x, y \in E \}$$

$$\min \{ |x - y| : x, y \in E \}$$

Can we prove this? Can we achieve $O(N \log(N))$?
Theorem (Fefferman-I-Luli ('11))

Can construct C-optimal Sobolev interpolants.

Plausible running-time bound is $O_{m,n,p}(N \log(\Delta')$, where

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\Delta := \frac{\max\{|x - y| : x, y \in E\}}{\min\{|x - y| : x, y \in E\}}
\]
Theorem (Fefferman-I-Luli ('11))

*Can construct C-optimal Sobolev interpolants.*

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Can we prove this? Can we achieve $O(N \log(N))$?
Example I

Given:
- \( t_1, \ldots, t_N \in \mathbb{R} \)
- \( p_1, \ldots, p_N \in \mathbb{R} \)

Construct \( p : \mathbb{R} \to \mathbb{R} \) with

(a) \( p(t_1) = p_1, \ldots, p(t_N) = p_N \);

(b) \( \sup_{t \in \mathbb{R}} |p'(t)| \leq \sup_{t \in \mathbb{R}} |q'(t)| \), for any other interpolant \( q \).

Estimate:

\[
M = \sup_{t \in \mathbb{R}} |p'(t)|.
\]
\[ (1) \quad \sup |p'(t)| = \left| \frac{p_2 - p_3}{t_2 - t_3} \right|. \]

The competitor \( q \) interpolates the data, so MVT \( \Rightarrow \)

\[ (2) \quad \exists t^* \in [t_2, t_3] \text{ with } q'(t^*) = \frac{p_2 - p_3}{t_2 - t_3}. \]

Finally, (1) and (2) \( \Rightarrow \)

\[ (3) \quad \sup |p'(t)| \leq C \sup |q'(t)|. \]
Example II

Given:

- $t_1, \ldots, t_N \in \mathbb{R}$
- $p_1, \ldots, p_N \in \mathbb{R}$

Construct $p : \mathbb{R} \rightarrow \mathbb{R}$ with

(a) $p(t_1) = p_1, \ldots, p(t_N) = p_N$;

(b) $\sup_{t \in \mathbb{R}} |p''(t)| \leq \sup_{t \in \mathbb{R}} |q''(t)|$, for any other interpolant $q$.

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There's a Competitor: $G : \mathbb{R}^2 \to \mathbb{R}$ with

$$G = f \text{ on } E;$$

$$|\nabla^2 G| \leq 1 \text{ on } \mathbb{R}^2.$$
Higher Dimensions

Given:
- Finite subset $E \subset [0, 1]^2$;
- Function $f : E \to \mathbb{R}$

There’s a Competitor: $G : \mathbb{R}^2 \to \mathbb{R}$ with

$$G = f \text{ on } E;$$

$$|\nabla^2 G| \leq 1 \text{ on } \mathbb{R}^2.$$

Goal: Construct $F : [0, 1]^2 \to \mathbb{R}$ with

$$F = f \text{ on } E;$$

$$|\nabla^2 F| \leq C \text{ on } [0, 1]^2.$$
Two Examples

(a) $E$ contained in a line.
(b) $E$ contained in a smooth curve.

Figure: Sets with 1D structure
Suppose that

\[ E = \{(0, y_1), \ldots, (0, y_N)\}; \]

\[ f : E \rightarrow \mathbb{R}. \]
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**Step 1:** Let \( g : \mathbb{R} \to \mathbb{R} \) be the cubic spline with

\[ g(y_k) = f(0, y_k) \quad \text{for} \quad k = 1, \ldots, N, \]

and

\[ |g''(y)| \leq C. \]
Suppose that

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**Step 1:** Let \( g : \mathbb{R} \to \mathbb{R} \) be the cubic spline with

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and

\[ |g''(y)| \leq C. \]

**Step 2:** Define \( F(x, y) := g(y) \). Then

\[ |\nabla^2 F(x, y)| = |g''(y)| \leq C \quad \text{for all} \quad (x, y). \]
Suppose that

\[ E \subset \{ (\phi(y), x) \}, \quad \text{where} \quad |\phi''| \leq 1. \]

Consider the diffeomorphism \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \):

\[ \Phi(x, y) = (x - \phi(y), y). \]
The Smooth Curve

Suppose that

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**Figure:** Sets with 1D structure

- Consider the diffeomorphism \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \):

  \[ \Phi(x, y) = (x - \phi(y), y). \]

- Note that \( \Phi \) maps \( E \) onto a line segment.
Suppose that

\[ E \subset \{ (\phi(y), x) \}, \quad \text{where} \quad |\phi''| \leq 1. \]

Figure: Sets with 1D structure

- Consider the diffeomorphism \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2: \)

\[ \Phi(x, y) = (x - \phi(y), y). \]

- Note that \( \Phi \) maps \( E \) onto a line segment.
- There is a 1–1 correspondence between interpolation problems on \( E \) and on \( \Phi(E) \).
$S(x, \delta) :=$ square with center $x$ and sidelength $\delta$.

$\delta(S) :=$ sidelength of the square $S$.

$A \cdot S :=$ $A$-dilate of $S$ about its center.
Definition (Neat Squares)

A square $S$ is neat if $3S \cap E$ lies on the graph of a function $h$ with

$$|h''| \leq \delta(S)^{-1} \text{ uniformly.}$$
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- Equivalently, \( S \) neat when \( \delta(S)^{-1} \cdot (3S \cap E) \) lies on the graph of a function \( h \) with

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- Equivalently, $S$ is neat when $\delta(S)^{-1} \cdot (3S \cap E)$ lies on the graph of a function $h$ with
  $$|h''| \leq 1 \text{ uniformly.}$$
- Small enough squares are neat.
- If $S$ is neat and $S' \subset S$ then $S'$ is neat.
Lemma

Suppose that $S$ is neat. Then we can construct $F : 3S \to \mathbb{R}$ with $F = f$ on $E \cap 3S$ and $|\nabla^2 F| \leq C$ on $3S$.
Definition (Messy Squares)

A square $S$ is messy if $S$ is not neat.

Figure: Some Messy Squares
The CZ Decomposition

- Keep bisecting $S \subset [0, 1]^2$ until $S$ is neat.
- Define $CZ$ as the collection of nonbisected squares.
Properties of the CZ Decomposition

Note that $CZ = \{S_\nu\}$ partitions $[0, 1]^2$.

(a) If $S \in CZ$, then $S$ is neat.

(b) If $S \in CZ$, then $3S$ is messy.

(c) Good Geometry: If $S, S' \in CZ$ touch, then

$$\frac{1}{2} \delta(S') \leq \delta(S) \leq 2\delta(S').$$
Properties of the CZ Decomposition

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One-Line Proofs:

(a) That was our stopping rule!

(b) \( 3S \) contains the dyadic parent \( S^+ \).

(c) If \( S, S' \in CZ \) touch and \( \delta(S) \leq \delta(S')/4 \), then \( 3S^+ \subset 3S' \).
The Naive Plan: Step 1

Construct local interpolants for the CZ squares:

- Functions $F_\nu : 3S_\nu \to \mathbb{R}$ that satisfy:

(a) $F_\nu = f$ on $E \cap (1.1)S_\nu$.

(b) $|\nabla^2 F_\nu| \leq C$ on $3S_\nu$. 
Introduce a partition of unity adapted to the CZ squares:

- Functions $\theta_\nu : [0, 1]^2 \to \mathbb{R}$ that satisfy

  (a) $0 \leq \theta_\nu \leq 1$;
  
  (b) $\text{supp}(\theta_\nu) \subset (1.1)S_\nu$;
  
  (c) $|\nabla \theta_\nu| \leq C \cdot \delta(S_\nu)^{-1}$ and $|\nabla^2 \theta_\nu| \leq C \cdot \delta(S_\nu)^{-2}$;
  
  (d) $\sum_\nu \theta_\nu = 1$ on $[0, 1]^2$. 
Define:

\[ F = \sum_{\nu} \theta_{\nu} F_{\nu}. \]
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By Local Interpolation and support properties of the partition of unity,

\[ F = f \quad \text{on} \quad E. \]
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**Question:** Is

\[ |\nabla^2 F| \leq C \quad \text{on} \quad [0, 1]^2? \]
Lemma

Let $S$ be any messy square. Then there exists a "non-degenerate" triplet $T \subset E \cap 9S$. 

![Image of a square with points scattered within it]
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Associate to each $S_\nu$ some “non-degenerate” triplet: $T_\nu \subset E \cap 9S_\nu$.
Lemma

Let $S$ be any messy square. Then there exists a “non-degenerate” triplet $T \subset E \cap 9S$.

- Associate to each $S_\nu$ some “non-degenerate” triplet: $T_\nu \subset E \cap 9S_\nu$.
- Let $L_\nu$ be affine with $L_\nu = f$ on $T_\nu$. 
Lemma

Let $S$ be any messy square. Then there exists a “non-degenerate” triplet

$$T \subset E \cap 9S.$$ 

- Associate to each $S_\nu$ some “non-degenerate” triplet: $T_\nu \subset E \cap 9S_\nu$.
- Let $L_\nu$ be affine with $L_\nu = f$ on $T_\nu$.
- This gives our rough guess for the affine structure of our interpolant.
Let’s check consistency:

**Lemma**

Suppose that $S_\nu$ and $S_{\nu'}$ are neighboring squares. Then

$$|\nabla L_\nu - \nabla L_{\nu'}| \leq C \delta(S_\nu)$$

and

$$|L_\nu - L_{\nu'}| \leq C \delta(S_\nu)^2 \quad \text{on} \quad 100S_\nu.$$
Need this version of Rolle’s Theorem:

**Lemma**

Suppose that $H$ vanishes on a “non-degenerate” triplet $T \subset S$ and $\|H\|_{C^2} \leq 1$. Then,

$$|\nabla H| \leq C\delta(S) \quad \text{and} \quad |H| \leq C\delta(S)^2 \quad \text{on} \quad S.$$
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Recall that

- $G = f$ on $E$ and $\|G\|_{C^2} \leq 1$.
- $L_\nu = f$ on $T_\nu$.
- $L_{\nu'} = f$ on $T_{\nu'}$. 
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Recall that

- $G = f$ on $E$ and $\|G\|_{C^2} \leq 1$.
- $L_\nu = f$ on $T_\nu$.
- $L_\nu' = f$ on $T_\nu'$.

For any $x \in 100S_\nu$,

$$|\nabla L_\nu - \nabla L_\nu'| \leq |\nabla L_\nu - \nabla G(x)| + |\nabla L_\nu' - \nabla G(x)| \leq C\delta(S),$$
Lemma

Suppose that $S_\nu$ and $S_{\nu'}$ are neighboring squares. Then

$$|\nabla L_\nu - \nabla L_{\nu'}| \leq C\delta(S_\nu)$$

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$$|(L_\nu - L_{\nu'})(x)| \leq C\delta(S_\nu)^2 \quad \text{on} \quad 100S_\nu.$$
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- Define $F_\nu := L_\nu$ whenever $E \cap (1.1)S_\nu = \emptyset$. 
Lemma

Suppose that $S_\nu$ and $S_{\nu'}$ are neighboring squares. Then

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- Define $F_\nu := L_\nu$ whenever $E \cap (1.1)S_\nu = \emptyset$.
- Do something similar for all other CZ squares.
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and

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- Define $F_\nu := L_\nu$ whenever $E \cap (1.1)S_\nu = \emptyset$.
- Do something similar for all other CZ squares.
- Set

$$F = \sum_\nu F_\nu \theta_\nu.$$

- We obtain

$$\|F\|_{C^2} \leq C'.$$
Definition

\( S_\mu \in \text{CZ} \) is keystone if every CZ square that intersects \( 9S_\mu \) has sidelength larger than \( S_\mu \).
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