Mappings and Meshes

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ABSTRACT

In my talk I will attempt to draw some connections between complex analysis and computational geometry, particularly between conformal mappings, hyperbolic geometry, the medial axis and optimal meshing. The Riemann mapping theorem says that there is a conformal (angle preserving) map of the unit disk, $\mathbb{D}$, to the interior Ω of any simple n-gon. How much work is needed to compute this map?

Theorem 1: *[1] We can compute the conformal map $f : \mathbb{D} \to \Omega$ to within error ϵ in time $O(n \cdot \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.*

The proof uses an iteration that converges quadratically to f, but it needs a good initial guess that is close to the correct answer. The medial axis allows us to quickly construct a map $\Omega \to \mathbb{D}$ that is close to Riemann's map with precise estimates. This gives our starting point, and the time estimates in the theorem depend on time needed to compute the medial axis (linear by work of Chin, Snoeyink and Wang).

Conversely, the proof of Theorem 1 uses ideas from analysis that may be of interest in CG. For example, the dome of a planar domain Ω is the surface in $\mathbb{R}^3_+$ given by the upper envelope of all hemispheres whose base on $\mathbb{R}^2$ is a medial axis disk of Ω. Domes arise in the study of 3-manifolds, and a fundamental result of Dennis Sullivan about convex sets in hyperbolic 3-space lies at the heart of Theorem 1.

The proof of Theorem 1 also introduces the thick-thin decomposition of a polygon, inspired by the thick-thin decomposition of a hyperbolic manifold. The thin parts correspond to "long, narrow" pieces of the polygon (but not in the obvious way; the precise definition uses a conformal invariant called extremal length). This decomposition plays in important role in proving:

Theorem 2: *[2] Any simple n-gon has a $O(n)$ quadrilateral mesh with all angles $\leq 120°$ and all new angles $\geq 60°$.*

The sharp upper bound is due to Bern and Eppstein; the lower bound is the novel part here. The result is also true for PSLGs with at most n edges and vertices, although simple examples show the $O(n)$ must be replaced by $O(n^2)$:

Theorem 3: *[4] Every PSLG has an $O(n^2)$ quadrilateral mesh with all angles $\leq 120°$ and all new angles $\geq 60°$. Only $O(n/\epsilon)$ angles satisfy $|\theta - 90°| > \epsilon$, for any $\epsilon > 0$.*

Adding diagonals to the quadrilaterals gives a triangulation with all angles $\leq 120°$, improving results of S. Mitchell (157.5°) and Tan (132°). Applying thick-thin decompositions directly to triangulations gives an even better result: for any $\epsilon > 0$, a PSLG has a $O(n^2)$ triangulation with angles $\leq 90° + \epsilon$ (but the constant blows up as $\epsilon \searrow 0$). For $\epsilon = 0$, a more intricate construction shows:

Theorem 4: *[3] Every PSLG has a $O(n^{2.5})$ nonobtuse triangulation.*

No polynomial bound was previously known, and I strongly suspect $n^{2.5}$ can be improved to n^2 (known sharp). Some consequences of Theorem 4 and its proof include

- Every PSLG has a $O(n^{2.5})$ conforming Delaunay triangulation (improves $O(n^3)$ by Edelsbrunner and Tan).
- Any triangulation of an n-gon has a $O(n^2)$ nonobtuse refinement (improves $O(n^4)$ by Bern and Eppstein).
- There is a point set of size $O(n^{2.5})$ whose Voronoi diagram conforms to a given PSLG of size n.

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G.m [**Mathematics of Computing**]: Miscellaneous

Keywords

Conformal maps, hyperbolic geometry, quadrilateral meshes, nonobtuse triangulation, conforming Delaunay triangulation

Bio

I graduated from Michigan State, did Part III at Cambridge and was a Ph.D. student of Peter W. Jones at the University of Chicago. I work in geometric function theory and its connections to hyperbolic geometry, quasiconformal maps, probability, dynamics and (most recently) computational geometry. My path from classical analysis to meshing is described in the essay [5]. See the 'papers' link at `http://www.math.sunysb.edu/~bishop`.

1. REFERENCES

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