On Genus-1 Gromov-Witten Invariants of Complete Intersection 3-Folds

- Description and Dfn of GW-Invariants
- Broad Ques., Partial Answers, Extensions
- Outline of Proof of Main Theorem

GW-Invariants

(X, ω, J) (almost) Kahler cmpt mnfd

$$\rightsquigarrow \quad \boxed{\mathrm{GW}_{g,k}^{\omega,A}(\mu) \in \mathbb{Q}}$$

$$g, k \geq 0, \quad A \in H_2(X), \quad \mu = (\mu_1, \ldots, \mu_k) \in H^*(X)^{\oplus k}$$

$$\underset{\text{roughly}}{\overset{\text{very}}{\sim}} \quad \text{counts of } J\text{-holomor. } \texttt{curves in } X$$

$$C \subset X, \ \dim_{\mathbb{R}} C = 2, \ J\,TC = TC \subset TX$$

$\underset{\text{roughly}}{\sim}$ counts of J-holomor. maps into X

$$u\colon \underbrace{(\Sigma, j)}_{\substack{\text{Riemann surf.}\\ \text{of genus } g}} \longrightarrow (X,J), \qquad \underbrace{\bar{\partial}_J u \equiv du + J\circ du \circ j \;=\; 0}_{\text{Cauchy-Riemann eqn.}}$$

$\underset{\text{really}}{\sim}$ integrals over $\overline{\mathfrak{M}}_{g,k}(X,A;J)$

moduli space of
$J-$holomor. maps

$$\boxed{\langle \mathrm{ev}_1^*\mu_1 \cup \ldots \cup \mathrm{ev}_k^*\mu_k, [\overline{\mathfrak{M}}_{g,k}(X,A;J)]^{\mathtt{vir}}\rangle}$$

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,k}(X,A;J) \longrightarrow X, \quad (\Sigma, x_1, \ldots, x_k; u) \longrightarrow u(x_i)$$

Standard Fact from Topology

$$
\begin{tikzcd}
\mathbb{C}^n \arrow[r] & V \arrow[d] \\
& M^m \arrow[u, bend right, "\varphi"']
\end{tikzcd}
$$

If φ is generic,

$$\boxed{\left[\varphi^{-1}(0)\right] = \mathrm{PD}_X e(V) \in H_{m-2n}(M)}$$

Dfn of $\mathrm{GW}_{g,k}^{\omega,A}$ or $\left[\overline{\mathfrak{M}}_{g,k}(X,A;J)\right]^{\mathbf{vir}}$

$$\begin{tikzcd} \Gamma_J^{0,1} \arrow[d] \\ \mathfrak{X}_{g,k}(X,A) \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd} \qquad \Gamma_J^{0,1}(\Sigma,u)=\{\eta\colon T\Sigma\longrightarrow u^*TX \mid J\eta=-\eta j\}$$

$$\mathfrak{X}_{g,k}(X,A) = \{\text{smooth maps } u\colon \Sigma\longrightarrow X\}$$

may have simple nodes

$$\mathrm{GW}_{g,k}^{\omega,A} = \text{(PD of) "Euler class" of } \Gamma_J^{0,1}, \text{ w.r.t. } \bar\partial_J$$

$$\boxed{\mathrm{GW}_{g,k}^{\omega,A} \equiv \left[\{\bar\partial_J+\nu\}^{-1}(0)\right] \in H_{\mathrm{ind}\,\bar\partial_J}\big(\underbrace{\mathfrak{X}_{g,k}(X,A)}\,;\mathbb{Q}\big)}$$

small generic perturbation

can replace by small neighb. of $\bar\partial_J^{-1}(0)=\overline{\mathfrak{M}}_{g,k}(X,A;J)$

$$\text{strata of } \overline{\mathfrak{M}}_{g,k}(X,A;J) \sim \begin{array}{l}\text{parametr. of } J\text{-holomor. curves}\\ \text{in } X \text{ of various types}\end{array}$$

$$\Big\{\lim_{t\longrightarrow 0}\{\bar{\partial}_J+t\nu\}^{-1}(0)\Big\}\cap \text{stratum} \sim \begin{array}{l}\text{relates GW-inv. (of } \omega)\\ \text{to enum. inv. (of } J)\end{array}$$

Broad Question

$Y \subset X$ sympl. submnfd

$$\rightsquigarrow \textit{Relate } \mathrm{GW}^Y_{*,*} \text{ to } \mathrm{GW}^X_{*,*}$$

Main Example

$X = \mathbb{P}^4, \quad Y^3 =$ hypersurface of degree $a \in \mathbb{Z}^+$

$$\begin{tikzcd} \mathcal{L} = \gamma^{*\otimes a} \arrow[d] \\ \mathbb{P}^4 \arrow[u, bend right, "s"'] \end{tikzcd}$$

s holom. section

$\gamma \longrightarrow \mathbb{P}^4$ taut. line bundle

$$\boxed{Y = s^{-1}(0)}$$

$$\overline{\mathfrak{M}}_{g,k}(Y,d) = \Big\{(\Sigma,u)\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^4,d)\colon\ \underbrace{u(\Sigma)\subset Y}_{s\circ u=0\in H^0(\Sigma;u^*\mathcal{L})}\Big\}$$

$$\begin{tikzcd} \mathcal{V}_{g,k} \arrow[d] \\ \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^4,d) \arrow[u, bend right, "\tilde{s}"'] \end{tikzcd}$$

$$\mathcal{V}_{g,k}(\Sigma,u) = H^0(\Sigma;u^*\mathcal{L})$$

$$\tilde{s}(\Sigma,u) = s\circ u$$

$$\boxed{\overline{\mathfrak{M}}_{g,k}(Y,d)=\tilde{s}^{-1}(0)}$$

Suggests

$$\big[\overline{\mathfrak{M}}_{g,k}(Y,d)\big]^{\mathrm{vir}} = \mathrm{PD}e(\mathcal{V}_{g,k}) \in H_*\big(\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^4,d)\big)$$

Fact 1: true if $g\!=\!0$

$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d)$ smooth; $\mathcal{V}_{0,k}\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d)$ v.b.

Implications of Fact 1

(a) compute $\mathrm{GW}^Y_{0,k}$ by Atiyah-Bott localiz. thm.

(b) verify $g\!=\!0$ mirror symmetry prediction for curves in Calabi-Yau 3-folds
(Givental, Lian-Liu-Yau, etc.)

Fact 2: if $g=0$, true for general $Y\subset X$

with appropriate interpretations

$$\big[\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4,d)\big] \rightsquigarrow \big[\overline{\mathfrak{M}}_{0,k}(X,A;J)\big]^{\mathrm{vir}}$$

$$\mathrm{PD}e(\mathcal{V}_{0,k}) \rightsquigarrow \mathrm{PD}e(\mathcal{V}_{0,k})^{\mathrm{vir}} \equiv e(H^0-H^1)\equiv \mathrm{ind}\,\bar{\partial}_{\mathcal{L}}$$

Fact 3: false if $g\ge 1$

$$\big[\overline{\mathfrak{M}}_{g,k}(Y,d)\big]^{\mathrm{vir}} \neq \mathrm{PD}e(\mathcal{V}_{g,k})^{\mathrm{vir}}$$

with interpretations as above

Theorem A (J. Li, Z.-)

$$\boxed{\mathrm{GW}_{1,k}^{Y,d}(\mu) = \tfrac{2+(a-5)d}{24}\mathrm{GW}_{0,k}^{Y,d}(\mu) + \big\langle \mu e(\mathcal{V}_{1,k}), \big[\overline{\mathfrak{M}}_{1,k}^{0}(\mathbb{P}^4,d)\big]\big\rangle}$$

$$\begin{aligned}\overline{\mathfrak{M}}_{1,k}^{0}(\mathbb{P}^4,d) &= \text{main comp. of } \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4,d)\\ &= \overline{\big\{u : \text{smooth } \Sigma \longrightarrow \mathbb{P}^4\big\}}\end{aligned}$$

Theorem (Z.-)

(1) $\overline{\mathfrak{M}}_{1,k}^{0}(X,A;J)$ defined for all (X,ω,J)
algebraic-genus compactification

(2) if $J\approx J_0$, $[\overline{\mathfrak{M}}_{1,k}^{0}(\mathbb{P}^l,d;J)]$ carries fund. class

(3) if $J\approx J_0$, PD of euler class of $\mathcal{V}_{1,k}\longrightarrow\overline{\mathfrak{M}}_{1,k}^{0}(\mathbb{P}^l,d;J)$
is well-defined: zero set of a generic section

Theorem B (R. Vakil, Z.-)

(1) $\exists$ desing. $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)$

(2) $\exists$ desing. $\tilde{\mathcal{V}}_{1,k} \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)$ of

$\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d) \rightsquigarrow \boxed{e(\tilde{\mathcal{V}}_{1,k}) \sim e(\mathcal{V}_{1,k})}$

(3) localiz. data similar to genus-0 case

Results for Quintic 3-Fold

d	1	2	3	4
$\langle\ldots\rangle$	0	$\frac{2{,}875}{32}$	$\frac{49{,}355{,}000}{81}$	$\frac{952{,}691{,}384{,}375}{256}$
$N_1(d)$	$\frac{2{,}875}{12}$	$\frac{407{,}125}{8}$	$\frac{243{,}388{,}750}{9}$	$\frac{366{,}163{,}353{,}125}{16}$
$n_1(d)$	0	0	609,250	$3,721,431,625$

$N_1(d) = \mathrm{GW}^{Y,d}_{1,0}(1)$ $\quad n_1(d) =$ "# of genus-1 deg.-d curves"

table agrees with predictions of Bershadsky, etc.'93

Hope

Thms A,B $\rightsquigarrow$ verify $g=1$ mirror symm. predic.
for curves in Calabi-Yau 3-folds

Extensions of Theorem A

(1) applies to all $Y^3 = s^{-1}(0) \subset X^{3+n}$ `regular`

(2) should generalize to all $Y = s^{-1}(0) \subset X$ `regular` as

$$\mathrm{GW}_{1,k}^{Y,A}(\mu) = \mathrm{GW}_{0,k}^{Y,A}(\psi, \mu) + \left\langle \mu e(\mathcal{V}_{1,k}), \left[\overline{\mathfrak{M}}_{1,k}^0(X, A)\right] \right\rangle$$

$$\psi = \psi\big(\dim Y, c(TY)\big)$$

Conjectural Extensions

(1) applies to all $Y = s^{-1}(0) \subset X$ via $\mathrm{PD}e(\mathcal{V}_{1,k})^{\mathrm{vir}}$

(2) can define $\overline{\mathfrak{M}}^0_{g,k}(X, A; J)$ for all g

(3) if J is `regular`, PD of euler class of
$\mathcal{V}_{g,k} \longrightarrow \overline{\mathfrak{M}}^0_{g,k}(X, A; J)$ is well defined

(4) $Y^3 = s^{-1}(0) \subset X^{3+n}$ `regular`

$$\mathrm{GW}^{Y,A}_{g,k}(\mu) = \sum_{g'=0}^{g-1} C^g_{g'}\big(\langle c_1(TY), A\rangle\big)\mathrm{GW}^{Y,A}_{g',k}(\mu) + \big\langle \mu e(\mathcal{V}_{g,k}), \big[\overline{\mathfrak{M}}^0_{g,k}(X, A; J)\big]\big\rangle$$

$C^g_{g'}\big(\langle c(TY), A\rangle\big) \in \mathbb{Q}$ determined by $H^*(\overline{\mathcal{M}}_{g'',k'})$

related to Pandharipande's #'s for degenerate contributions for 3-folds

Outline of Pf

$$\boxed{\mathrm{GW}^{Y,d}_{1,k}(\mu) = \ {}^{\pm}\big|\{\bar\partial+\nu\}^{-1}(0)\big|}$$

$$\begin{tikzcd} \Gamma^{0,1}_J \arrow[d] \\ \mathfrak{X}_{1,k}(Y,\mu) \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd}$$

$$\mathfrak{X}_{1,k}(Y,\mu) = \big\{(\Sigma, x_1, \ldots, x_k; u) \in \mathfrak{X}_{1,k}(Y,A) \colon u(x_l) \in \mu_l\big\}$$

ν small $\Longrightarrow$ $\{\bar\partial_J+\nu\}^{-1}(0)$ close to $\bar\partial_J^{-1}(0) = \overline{\mathfrak{M}}_{1,k}(Y,d;J)$

General Approach

determine # of $\{\bar\partial_J+\nu\}^{-1}(0)$ close to each stratum of $\overline{\mathfrak{M}}_{1,k}(Y,d;J)$

$\rightsquigarrow$ `contribution` of stratum to $\mathrm{GW}^{Y,d}_{1,k}(\mu)$

do this for `good` $J \approx J_0$

Assumptions on $J \approx J_0$ on $Y \subset \mathbb{P}^4$

\# of genus-0,1 J-holomor. curves in Y thr. $\mu_1, \ldots, \mu_k$
is `finite` (to 1st order)
all such curves are smooth

Implications: Y non-CY ($a \neq 5$)

If $(\Sigma, u) \in \mathfrak{X}_{1,k}(Y, \mu)$ is J-holomor.,

- Σ has one or two irred. components
- u non-const on exactly one comp.
- u is $1\!:\!1$ on that comp.

Implications: Y CY ($a = 5$)

If $(\Sigma, u) \in \mathfrak{X}_{1,k}(Y, \mu)$ J-holomor., either

- u is covering of smooth genus-1 curve, or
- image of u is smooth genus-0 curve

Admissibility of Assumptions: Y non-CY

$g = 0$: OK by standard arguments
$g = 1$: OK by structure of $\overline{\mathfrak{M}}^0_{1,k}(Y, d; J)$

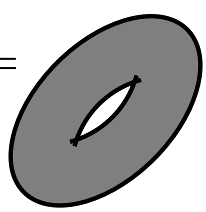
Admissibility of Assumptions: Y general

Ionel-Parker'02: OK for all g;   *proof incomplete*

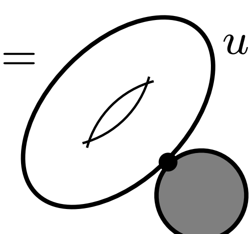
Solution

use perturbed (in a restricted way) J-holomor. maps
$\rightsquigarrow$ makes proof same in all cases

Strata of $\overline{\mathfrak{M}}_{1,k}(Y, d; J)$

$\Sigma =$  u 1:1

$\mathfrak{M}^0_{1,k}(Y, \mu)$

$\Sigma =$  u const, u 1:1

$\overline{\mathfrak{M}}^1_{1,k}(Y, \mu)$

Choice of ν

$$\begin{tikzcd} \Gamma_J^{0,1} \arrow[d] \\ \mathfrak{X}_{1,k}(\mathbb{P}^4,\mu) \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd} \supset \mathfrak{X}_{1,k}(Y,\mu)$$

ν supported near $\overline{\mathfrak{M}}_{1,k}(Y,\mu;J)$

Contribution from $\mathfrak{M}^0_{1,k}(Y,\mu;J)$

\# of solutions $\exp_u\xi$ of

$$\begin{cases} \bar\partial_J \exp_u\xi + \nu(\exp_u\xi) = 0 \\ s\circ\exp_u\xi = 0 \end{cases} \qquad \begin{aligned} & u\in\mathfrak{M}^0_{1,k}(\mathbb{P}^4,d;J) \\ & \xi\in T_u\mathfrak{X}_{1,k}(\mathbb{P}^4,\mu) \end{aligned}$$

$$\| $$

$$\underbrace{\ker D_{J,u}}_{\text{linear. of } \bar\partial_J}\oplus\Gamma_+(u)$$

has unique solution $\xi_+(u)\in\Gamma_+(u)$

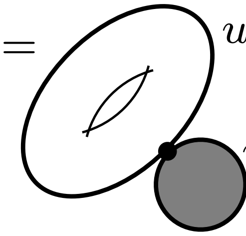
$\rightsquigarrow$ $s\circ\exp_u\xi_+(u)=0\in\mathcal{V}_{1,k}$

$\rightsquigarrow$ small perturb. of $\tilde s$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^4,\mu;J)$:

$$\boxed{\mathrm{Contr}\big(\mathfrak{M}^0_{1,k}(Y,\mu;J)\big) = \big\langle e(\mathcal{V}_{1,k}),\,\big[\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^4,\mu;J)\big]\big\rangle}$$

Contribution from $\overline{\mathfrak{M}}^1_{1,k}(Y,\mu)$

$\Sigma =$



u const

$\approx \overline{\mathcal{M}}_{1,1} \times \mathcal{K}$

u 1:1 onto $\mathcal{K} \approx \mathbb{P}^1 \subset Y$

$$u = (\Sigma_1, z_1; x_0) \in \overline{\mathcal{M}}_{1,1} \times \mathcal{K}$$

find coker of $D_{J,u} = d\bar{\partial}_J : \underbrace{T_u \mathfrak{X}_{1,k}(Y,\mu)}_{\| \; T_{z_1}\Sigma_1 \otimes T_{x_0}\mathcal{K} \oplus \Gamma(u)} \longrightarrow \Gamma_J^{0,1}(u)$

$\rightsquigarrow$ find coker of

$$\overline{D}_{J,u}\colon T_{z_1}\Sigma_1 \otimes T_{x_0}\mathcal{K} \longrightarrow \Gamma_J^{0,1}(u) / D_{J,u}\big(\Gamma(u)\big) = \mathcal{H}_{\Sigma}^{0,1} \otimes T_{x_0}Y$$

$$\approx \mathbb{E}^*_{z_1} \otimes T_{x_0}Y$$

$(0,1)$-harmonic forms

$\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{1,1}$ Hodge line bndl

Fact: $\overline{D}_{J,u}v = v \in \mathbb{E}^*_{z_1} \otimes T_{x_0}\mathcal{K} \subset \mathbb{E}^*_{z_1} \otimes T_{x_0}Y$

$$\rightsquigarrow \qquad \operatorname{coker} D_{J,u} = \operatorname{coker} \overline{D}_{J,u} = \mathbb{E}^*_{z_1} \otimes N_Y\mathcal{K}|_{x_0}$$

$$\begin{aligned}\rightsquigarrow \quad & \#\big\{\{\bar\partial_J + \nu\}(f) = 0 \colon f \text{ close to } \overline{\mathcal{M}}_{1,1} \times \mathcal{K}\big\} \\ & = \big\langle e(\mathbb{E}^* \otimes N_Y\mathcal{K}), \big[\overline{\mathcal{M}}_{1,1} \times \mathcal{K}\big]\big\rangle \\ & = -\frac{\langle c_1(N_Y\mathcal{K}), d\ell\rangle}{24} = \frac{2+d(a-5)}{24}\end{aligned}$$

Note: # of such $\mathcal{K}$'s $= \mathrm{GW}^{Y,d}_{0,k}(\mu)$

$$\rightsquigarrow \quad \boxed{\mathrm{Contr}\big(\overline{\mathfrak{M}}^1_{1,k}(Y,\mu;J)\big) = \tfrac{2+d(a-5)}{24}\mathrm{GW}^{Y,d}_{0,k}(\mu)}$$

Theorem

euler class of $\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l, d)$ is well-defined

Reminder

$$\begin{array}{c} \mathcal{L} = \gamma^{*\otimes a} \\ \Big\downarrow \\ \mathbb{P}^4 \end{array} \qquad a \in \mathbb{Z}^+$$

$$\mathcal{V}_{1,k}|_{(\Sigma,u)} = H^0(\Sigma; u^*\mathcal{L})$$

Outline of Proof

$$m = \dim_{\mathbb{C}} \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l, d) \equiv \dim_{\mathbb{C}} \mathfrak{M}^0_{1,k}(\mathbb{P}^l, d)$$

$$n = \dim_{\mathbb{C}} H^0(\Sigma; u^*\mathcal{L}) - \underbrace{\dim_{\mathbb{C}} H^1(\Sigma; u^*\mathcal{L})}_{0 \ \text{ or } \ 1} = \mathrm{ind}_{\mathbb{C}} \bar{\partial}_{\mathcal{L},u}$$

if $(\Sigma, u) \in \mathfrak{M}^0_{1,k}(\mathbb{P}^l, d) \equiv \{(\Sigma, u) \in \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^l, d) \colon \Sigma \text{ smooth}\}$

(1) find section $\varphi \colon \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l, d) \longrightarrow \mathcal{V}_{1,k}$ s.t.

$$\dim_{\mathbb{R}} \varphi^{-1}(0) \cap \underbrace{\partial \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l, d)}_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l, d) - \mathfrak{M}^0_{1,k}(\mathbb{P}^l, d)} \leq 2(m-n)-2$$

(2) show space of such sections is path-connected

φ generic $\Longrightarrow$ $\dim_{\mathbb{R}} \varphi^{-1}(0) \cap \mathfrak{M}^0_{1,k}(\mathbb{P}^l, d) = 2(m-n)$

$+$ bound. cond. $\Longrightarrow$ $[\varphi^{-1}(0)] \in H_{2(m-n)}\big(\mathfrak{M}^0_{1,k}(\mathbb{P}^l, d)\big)$

Key Proposition

$\exists$ stratification $\partial\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)=\bigsqcup_{\alpha\in A}\mathcal{S}_\alpha$ and for $\alpha\in A$ neighb. $\mathcal{U}_\alpha$ of $\mathcal{S}_\alpha$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)$ and v.b. $V_\alpha\longrightarrow\mathcal{U}_\alpha$ s.t.

(a) $V_\alpha\subset\mathcal{V}_{1,k}$

(b) $V_\beta|_{\mathcal{S}_\beta}\subset V_\alpha|_{\mathcal{S}_\beta}$ if $\overline{\mathcal{S}}_\alpha\cap\mathcal{S}_\beta\neq\emptyset$

(c) $\mathrm{rk}_{\mathbb{C}}\mathcal{V}_\alpha>\dim_{\mathbb{C}}S_\alpha-(m-n)$

$\Longrightarrow$ if φ is a generic section of $\mathcal{V}_{1,k}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)$, then $\dim_{\mathbb{R}}\varphi^{-1}(0)\cap\mathcal{S}_\alpha\leq 2(m-n)-2$

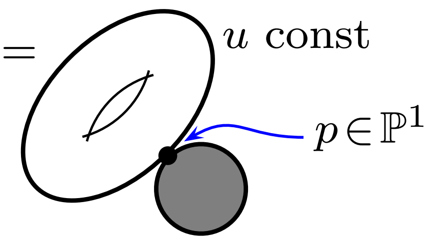
Meaning of Proposition

$\mathrm{rk}_{\mathbb{C}}\mathcal{V}_{1,k}|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^l,d)}=n$

if $\dim_{\mathbb{C}}H^0(\Sigma;u^*\mathcal{L})=n+1$, can find large-dim subspace of sections that extend over a neighb. in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d)$

Example 1

$\Sigma =$



$$(\Sigma,u)\in\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^l,d) \Longrightarrow du|_p=0$$

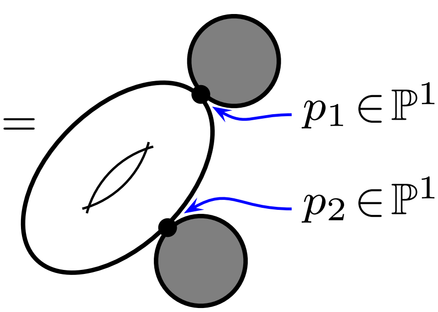
$$\Longrightarrow \dim_{\mathbb{C}}\mathcal{S}_\alpha = m-1$$

$$V_\alpha|_{(\Sigma,u)} = \big\{\xi\in H^0(\Sigma;u^*\mathcal{L}) \colon \underbrace{\nabla^{\mathcal{L}}\xi|_p}_{\text{indep. of } \nabla^{\mathcal{L}} \text{ in } \mathcal{L}\longrightarrow\mathbb{P}^l \text{ b/c } du|_p=0} = 0\big\}$$

$$\mathrm{rk}_{\mathbb{C}} V_\alpha|_{(\Sigma,u)} = (n+1)-1 = n \quad \surd$$

Example 2

$\Sigma =$



$$\mathcal{S}_\alpha \equiv \{du|_{p_1}=0,\ du|_{p_2}=0\}$$

$$\Longrightarrow \dim_{\mathbb{C}}\mathcal{S}_\alpha = m-2-l$$

$$V_\alpha|_{(\Sigma,u)} = \big\{(\xi_1,\xi_2)\in H^0(\Sigma;u^*\mathcal{L}) \colon \nabla^{\mathcal{L}}\xi_i|_{p_i}=0\big\}$$

$$\mathrm{rk}_{\mathbb{C}} V_\alpha|_{(\Sigma,u)} = (n+1)-2 = n-1 \quad \surd$$