Pseudocycles and Integral Homology

(Version B)

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Abstract

We describe a natural isomorphism between the set of equivalence classes of pseudocycles and the integral homology groups of a smooth manifold. Our arguments generalize to settings well-suited for applications in enumerative algebraic geometry and for construction of the virtual fundamental class in the Gromov-Witten theory.

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1 Introduction

In his seminal paper [G], Gromov initiated the study of pseudoholomorphic curves in symplectic manifolds and demonstrated their usefulness by proving a number of important results in symplectic topology. In [McSa] and [RT], pseudoholomorphic curves are used to define invariants of semipositive manifolds. In particular, it is shown in [McSa] and [RT] that for every compact semipositive symplectic manifold (X,ω), homology class $A\in H_2(X;\mathbb{Z})$, integers $k\ge 3$ and $N\ge 0$, and generic compatible almost complex structure J on X, there exists a smooth oriented manifold $\mathcal{M}_{k,N}(A,J)$ and a smooth map

$$\mathrm{ev}_{k,N}^{A,J}\colon \mathcal{M}_{k,N}(A,J)\longrightarrow X^{k+N}$$

such that the "boundary" of $\mathrm{ev}_{k,N}^{A,J}$ is small; see below. Such a smooth map is called a *pseudocycle* and determines a homomorphism $H_*(X^{k+N};\mathbb{Z})\longrightarrow\mathbb{Z}$, which turns out to be an invariant of $(X,\omega;A,k,N)$.

In general, if X is a smooth manifold, subset Z of X is said to have *dimension at most* k if there exists a k-dimensional manifold Y and a smooth map $h\colon Y\longrightarrow X$ such that the image of h contains Z. If $f: M\longrightarrow X$ is a continuous map between topological spaces, the *boundary* of f is the set

$$\mathrm{Bd}\, f = \bigcap_{K\subset M \text{ cmpt}} \overline{f(M-K)}.$$

A smooth map $f: M\longrightarrow X$ is a *k-pseudocycle* if M is an oriented k-manifold, $f(M)$ is a pre-compact[1] subset of X, and the dimension of $\mathrm{Bd} f$ is at most $k-2$. Two k-pseudocycles $f_0\colon M_0\longrightarrow X$ and $f_1\colon M_1\longrightarrow X$ are *equivalent* if there exists a smooth oriented manifold $\tilde{M}$ and a smooth map $\tilde{f}\colon \tilde{M}\longrightarrow X$ such that the image of $\tilde{f}$ is a pre-compact subset of X,

$$\dim \mathrm{Bd}\, \tilde{f} \le k-1, \quad \partial\tilde{M} = M_1 - M_0, \quad \tilde{f}|_{M_0} = f_0, \quad \text{and} \quad \tilde{f}|_{M_1} = f_1.$$

We denote the set of equivalence classes of pseudocycles into X by $\mathcal{H}_*(X)$. This set is naturally a $\mathbb{Z}$-graded module over $\mathbb{Z}$. In this paper, we prove

Theorem 1.1 *If X is a smooth manifold, there exist natural[2] homomorphisms of graded $\mathbb{Z}$-modules*

$$\Psi_*\colon H_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}_*(X) \quad \textit{and} \quad \Phi_*\colon \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z}),$$

such that $\Phi_*\circ\Psi_* = \mathrm{Id}$ and $\Psi_*\circ\Phi_* = \mathrm{Id}$.

Remark 1: In [McSa] and [RT], a pseudocycle is not explicitly required to have a pre-compact image. As [McSa] and [RT] work with compact manifolds, this condition is automatically satisfied. However, this requirement is essential in the non-compact case. As observed in [K], there is no surjective homomorphism from $H_*(X;\mathbb{Z})$ to $\mathcal{H}_*(X)$ if X is not compact and pseudocycles are not required to have pre-compact images.

Remark 2: It is sufficient to require that a pseudocycle map be continuous, as long as the same condition is imposed on pseudocycle equivalences. All arguments in this paper go through for continuous pseudocycles. In fact, Lemma 2.1 is no longer necessary. However, smooth pseudocycles are useful in symplectic topology for defining invariants as intersection numbers.

In order to define symplectic invariants, [McSa] and [RT] observe that every element of $\mathcal{H}_*(X)$ defines a homomorphism $H_*(X;\mathbb{Z})\longrightarrow\mathbb{Z}$, or equivalently an element of $H_*(X;\mathbb{Z})/\mathrm{Tor}\big(H_*(X;\mathbb{Z})\big)$. Thus, Theorem 1.1 leads to the symplectic invariants that may be strictly strongly than the GW-invariants defined in [McSa] and in [RT]. In fact, these invariants are as good as the maps $\mathrm{ev}_{k,N}^{A,J}$ can give:

Corollary 1.2 *If (X,ω_1) and (X,ω_2) are semipositive symplectic manifolds that have the same GW-invariants, viewed as a collection of integral homology classes, then the corresponding collections of evaluation maps from products of moduli spaces of pseudoholomorphic maps and Riemannian surfaces are equivalent as pseudocycles.*

[1] i.e. its closure is compact

[2] In other words, Ψ_* and Φ_* are natural transformations of functors $H_*(\cdot;\mathbb{Z})$ and $\mathcal{H}_*(\cdot)$ from the category of smooth compact manifolds and maps.

This corollary is immediate from Theorem 1.1.

The natural homomorphisms

$$\Psi_*\colon H_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}_*(X) \qquad \text{and} \qquad \Phi_*\colon \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})$$

of Theorem 1.1 are constructed in Subsections 3.1 and 3.2, respectively. In Subsection 3.3, it is shown that these maps are mutual inverses. The homomorphism

$$\Phi_*\colon \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})$$

of Subsection 3.2 induces the linear map

$$\mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})/\mathrm{Tor}\big(H_*(X;\mathbb{Z})\big)$$

described in [McSa] and [RT]. However, our construction of Φ_* differs from that of the induced map in [McSa] and [RT]. Indeed, the latter is constructed via the homomorphism Ψ_* and a natural intersection pairing on $\mathcal{H}_*(X)$. The construction of Φ_* in Subsection 3.2 is more direct. We use Proposition 2.2, which describes a topological property of "small" subsets of smooth manifolds, and Proposition 2.9, which allows us to replace the singular chain complex $S_*(X)$ by a quotient complex $\bar{S}_*(X)$. The advantage of the latter complex is that cycles can be constructed more easily.

This paper was begun while the author was a graduate student at MIT and then put on the back burner. The aim of this paper was to clarify relations between $H_*(X;\mathbb{Z})$ and $\mathcal{H}_*(X)$ that were hinted at in [McSa] and stated without a proof in [RT]. Since then, this issue has been explored in [K] and in [Sc]. The views taken in [K] and in [Sc] differ significantly from the present paper. In particular, while non-compact manifolds are considered in [K], pseudocycles in [K] are not required to have pre-compact images. Theorem 1.1 fails for such pseudocycles. The arguments in the present paper are rather direct and use no advanced techniques, beyond standard algebraic topology. In a sense they implement an outline proposed in Section 7.1 of [McSa]. However, a fully rigorous implementation of this outline requires non-trivial technical facts obtained in Subsections 2.2-2.4 of this paper.

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2 Preliminaries

2.1 Notation

If A is a finite subset of $\mathbb{R}^k$, we denote by $\mathrm{CH}(A)$ and $\mathrm{CH}^0(A)$ the (closed) convex hull of A and the open convex hull of A, respectively, i.e.

$$CH(A) = \Big\{\sum_{v\in A} t_v v\colon t_v\in[0,1];\ \sum_{v\in A} t_v = 1\Big\} \qquad \text{and}$$
$$CH^0(A) = \Big\{\sum_{v\in A} t_v v\colon t_v\in(0,1);\ \sum_{v\in A} t_v = 1\Big\}.$$

For each $p=1,\ldots,k$, let e_p be the pth coordinate vector in $\mathbb{R}^k$. Put $e_0=0\in\mathbb{R}^k$. Denote by

$$\Delta^k = \mathrm{CH}\big(e_0,e_1,\ldots,e_k\big) \qquad \text{and} \qquad \mathrm{Int}\,\Delta^k = \mathrm{CH}^0\big(e_0,e_1,\ldots,e_k\big)$$

the standard k-simplex and its interior. Let

$$b_k = \frac{1}{k+1}\bigg(\sum_{q=0}^{q=k} e_q\bigg) = \Big(\frac{1}{k+1},\ldots,\frac{1}{k+1}\Big) \in \mathbb{R}^k.$$

be the barycenter of Δ^k.

For each $p=0,1,\ldots,k$, let

$$\Delta^k_p = \mathrm{CH}\big(\{e_q\colon q\in\{0,1,\ldots,k\}-p\}\big) \qquad \text{and} \qquad \mathrm{Int}\,\Delta^k_p = \mathrm{CH}^0\big(\{e_q\colon q\in\{0,1,\ldots,k\}-p\}\big)$$

denote the pth face of Δ^k and its interior. Define a linear map[3]

$$\iota_{k,p}\colon \Delta^{k-1} \longrightarrow \Delta^k_p \subset \Delta^k \qquad \text{by} \qquad \iota_{k,p}(e_q) = \begin{cases} e_q, & \text{if } q<p;\\ e_{q+1}, & \text{if } q\ge p.\end{cases}$$

We also define a projection map

$$\tilde{\pi}^k_p\colon \Delta^k-\{e_p\} \longrightarrow \Delta^k_p \qquad \text{by} \qquad \tilde{\pi}^k_p\bigg(\sum_{q=0}^{q=k} t_q e_q\bigg) = \frac{1}{1-t_p}\bigg(\sum_{q\neq p} t_q e_q\bigg).$$

Put

$$b_{k,p} = \iota_{k,p}(b_{k-1}), \qquad b'_{k,p} = \frac{1}{k+1}\bigg(b_k + \sum_{q\neq p} e_q\bigg).$$

The points $b_{k,p}$ and $b'_{k,p}$ are the barycenters of the $(k-1)$-simplex Δ^k_p and of the k-simplex spanned by b_k and the vertices of Δ^k_p. Define a neighborhood of $\mathrm{Int}\,\Delta^k_p$ in Δ^k by

$$U^k_p = \Big\{t_p b'_{k,p} + \sum_{0\le q\le k;\, q\neq p} t_q e_q\colon t_p\ge 0,\ t_q>0\ \forall q\neq p;\ \sum_{q=0}^{q=k} t_q=1\Big\}.$$

These neighborhoods will be used to construct pseudocycles out of homology cycles.

If $p,q=0,1,\ldots,k$ and $p\neq q$, let

$$\Delta^k_{p,q} \equiv \Delta^k_p\cap\Delta^k_q \qquad \text{and}$$
$$\iota_{k,(p,q)} \equiv \iota_{k,p}\circ\iota_{k-1,\iota^{-1}_{k,p}(q)} = \iota_{k,q}\circ\iota_{k-1,\iota^{-1}_{k,q}(p)}\colon \Delta^{k-2} \longrightarrow \Delta^k_{p,q} \tag{2.1}$$

[3] A map $f\colon \Delta^m \longrightarrow \Delta^k$ is called *linear* if

$$f(t_0e_0+\ldots+t_me_m) = t_0f(e_0)+\ldots+t_mf(e_m) \qquad \forall\ (t_0,\ldots,t_m)\in[0,1]^{m+1} \ \text{ s.t. } \ t_0+\ldots+t_m=1.$$

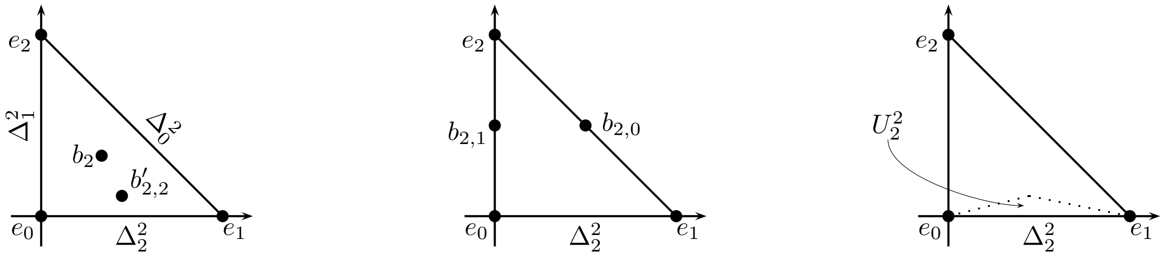

Figure 1: The Standard 2-Simplex and Some of its Distinguished Subsets

be the corresponding codimension-two simplex and the inclusion map. Define neighborhoods of $\operatorname{Int}\Delta^k_{p,q}$ in Δ^k by

$$\tilde{U}^k_{p,q}=\Big\{t_pb_{k,p}+t_qb_{k,q}+\sum_{r\neq p,q}t_re_r:\ t_p,t_q\geq 0,\ \ t_r>0\ \forall\, r\neq p,q;\ \ \sum_{r=0}^{r=k}t_r=1\Big\},$$

$$U^k_{p,q}=\Big\{t_p\iota_{k,p}(b'_{k-1,\iota_{k,\iota^{-1}_{k,p}(q)}})+t_q\iota_{k,q}(b'_{k-1,\iota_{k,\iota^{-1}_{k,q}(p)}})+\sum_{r\neq p,q}t_re_r:\ t_p,t_q\geq 0,\ t_r>0\,\forall r\neq p,q;\ \sum_{r=0}^{r=k}t_r=1\Big\}.$$

These sets will be used to construct pseudocycle equivalences out of singular cycles. Note that

$$\tilde{U}^k_{p_1,q_1}\cap\tilde{U}^k_{p_2,q_2}=\emptyset\qquad\text{if}\quad\{p_1,q_1\}\neq\{p_2,q_2\}.\tag{2.2}$$

Define a projection map

$$\tilde{\pi}^k_{p,q}\colon\Delta^k-\mathrm{CH}(e_p,e_q)\longrightarrow\Delta^k_{p,q}\qquad\text{by}\qquad\tilde{\pi}^k_{p,q}\Big(\sum_{r=0}^{r=k}t_re_r\Big)=\frac{1}{1-t_p-t_q}\Big(\sum_{r\neq p,q}t_re_r\Big).$$

Finally, let $\mathcal{S}_k$ denote the group of permutations of the set $\{0,\ldots,k\}$. The set $\mathcal{S}_k$ can be viewed as a subset of $\mathcal{S}_{k+1}$; if $\tau\in\mathcal{S}_k$, put $\tau(k+1)=k+1$. If $p^-,p^+=0,1,\ldots,k+1$, define

$$\tau_{k,(p^-,p^+)}\in\mathcal{S}_k\qquad\text{by}\qquad\tau_{k,(p^-,p^+)}(q)=\begin{cases}\iota^{-1}_{k+1,p^-}(\iota_{k+1,p^+}(q)), & \text{if } q\neq\iota^{-1}_{k+1,p^+}(p^-);\\ \iota^{-1}_{k+1,p^-}(p^+), & \text{if } q=\iota^{-1}_{k+1,p^+}(p^-).\end{cases}\tag{2.3}$$

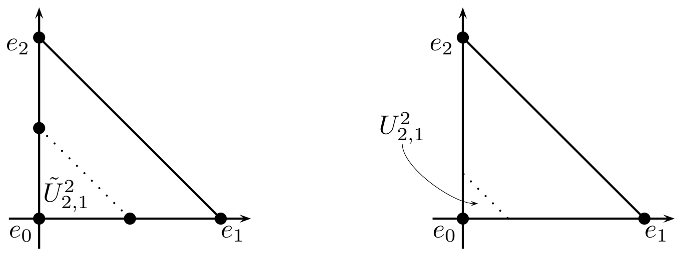

Figure 2: The Standard 2-Simplex and Some of its Distinguished Subsets

We note that

$$\iota_{k+1,p^-}\circ\tau_{k,(p^-,p^+)}=\iota_{k+1,p^+}:\Delta^k_{\iota^{-1}_{k+1,p^+}(p^-)}\longrightarrow\Delta^{k+1}_{p^-,p^+}\qquad\Longrightarrow \tag{2.4}$$

$$\tau_{k,(p^-,p^+)}\circ\iota_{k,\iota^{-1}_{k+1,p^+}(p^-)}=\iota_{k,\iota^{-1}_{k+1,p^-}(p^+)}:\Delta^{k-1}\longrightarrow\Delta^k_{\iota^{-1}_{k+1,p^-}(p^+)}; \tag{2.5}$$

the second equality follows from the first using (2.1). Furthermore,

$$\operatorname{sign}\tau_{k,(p^-,p^+)}=-(-1)^{p^-+p^+}. \tag{2.6}$$

For any $\tau\in\mathcal{S}_k$, define

$$\tau\colon\Delta^k\longrightarrow\Delta^k\qquad\text{by}\qquad\tau(e_q)=e_{\tau(q)}\quad\forall\,q=0,1,\ldots,k.$$

Lemma 2.1 *If $k\geq1$, Y is the $(k-2)$-skeleton of Δ^k, and $\tilde{Y}$ is the $(k-2)$-skeleton of Δ^{k+1}, there exist continuous functions*

$$\varphi_k\colon\Delta^k\longrightarrow\Delta^k\qquad\textit{and}\qquad\tilde\varphi_{k+1}\colon\Delta^{k+1}\longrightarrow\Delta^{k+1}$$

such that

(i) φ_k is smooth outside of Y and $\tilde\varphi_{k+1}$ is smooth outside of $\tilde{Y}$;

(ii) for all $p=0,\ldots,k$ and $\tau\in\mathcal{S}_k$,

$$\varphi_k|_{U^k_p}=\tilde\pi^k_p|_{U^k_p}\qquad\textit{and}\qquad\varphi_k\circ\tau=\tau\circ\varphi_k; \tag{2.7}$$

(iii) for all $p,q=0,\ldots,k+1$ with $p\neq q$ and $\tilde\tau\in\mathcal{S}_{k+1}$,

$$\tilde\varphi_{k+1}|_{U^{k+1}_{p,q}}=\tilde\pi^{k+1}_{p,q}|_{U^{k+1}_{p,q}},\qquad\tilde\varphi_{k+1}\circ\tilde\tau=\tilde\tau\circ\tilde\varphi_{k+1},\quad\textit{and}\quad\tilde\varphi_{k+1}\circ\iota_{k+1,p}=\iota_{k+1,p}\circ\varphi_k. \tag{2.8}$$

Proof: (1) Choose a smooth function

$$\tilde\eta_{0,1}\colon\Delta^{k+1}-\Delta^{k+1}_{0,1}\cap\tilde{Y}\longrightarrow[0,1]$$

such that $\tilde\eta_{0,1}=1$ on $U^{k+1}_{0,1}$, $\tilde\eta_{0,1}=0$ outside of $\tilde{U}^{k+1}_{0,1}$, and $\tilde\eta_{0,1}$ is invariant under any permutation $\tilde\tau\in\mathcal{S}_{k+1}$ that preserves the set $\{0,1\}$. If $\tilde\tau\in\mathcal{S}_{k+1}$ is any permutation, let

$$\tilde\eta_{\tilde\tau(0),\tilde\tau(1)}=\tilde\eta_{0,1}\circ\tilde\tau^{-1}\colon\Delta^{k+1}-\Delta^{k+1}_{\tilde\tau(0),\tilde\tau(1)}\cap\tilde{Y}\longrightarrow[0,1].$$

By the assumptions on $\tilde\eta_{0,1}$, $\tilde\eta_{p,q}$ is a well-defined smooth function such that $\tilde\eta_{p,q}=1$ on $U^{k+1}_{p,q}$, $\tilde\eta_{p,q}=0$ outside of $\tilde{U}^{k+1}_{p,q}$, and

$$\tilde\eta_{\tilde\tau(p),\tilde\tau(q)}=\tilde\eta\circ\tilde\tau^{-1} \tag{2.9}$$

for all $\tilde\tau\in\mathcal{S}_{k+1}$ and distinct $p,q=0,\ldots,k+1$.

(2) Define

$$\tilde\varphi_{k+1}\colon\Delta^{k+1}\longrightarrow\Delta^{k+1}\qquad\text{by}\qquad\tilde\varphi_{k+1}(x)=x+\sum_{0\leq p<q\leq k+1}\tilde\eta_{p,q}(x)\cdot\big(\tilde\pi^{k+1}_{p,q}(x)-x\big).$$

Since $\tilde{\eta}_{p,q}$ vanishes on a neighborhood of $\mathrm{CH}(e_p,e_q)$ and $\tilde{\pi}^{k+1}_{p,q}$ restricts to the identity on $\Delta^{k+1}_{p,q}$, the function $\tilde{\eta}$ is well-defined, continuous everywhere, and smooth on $\Delta^{k+1}-\tilde{Y}$. By (2.2), $\tilde{\varphi}_{k+1}=\tilde{\pi}^{k+1}_{p,q}$ on $U^{k+1}_{p,q}$. By (2.9), for every $\tilde{\tau}\in\mathcal{S}_{k+1}$

$$\begin{aligned}\tilde{\varphi}_{k+1}\circ\tilde{\tau} &= \tilde{\tau}+\sum_{0\le p<q\le k+1}\tilde{\eta}_{p,q}\circ\tilde{\tau}\cdot\big(\tilde{\pi}^{k+1}_{p,q}\circ\tilde{\tau}-\tilde{\tau}\big)\\ &= \tilde{\tau}+\sum_{0\le p<q\le k+1}\tilde{\eta}_{\tilde{\tau}^{-1}(p),\tilde{\tau}^{-1}(q)}\cdot\big(\tilde{\tau}\circ\tilde{\pi}^{k+1}_{\tilde{\tau}^{-1}(p),\tilde{\tau}^{-1}(q)}-\tilde{\tau}\big)\\ &= \tilde{\tau}+\sum_{0\le p<q\le k+1}\tilde{\eta}_{p,q}\cdot\big(\tilde{\tau}\circ\tilde{\pi}^{k+1}_{p,q}-\tilde{\tau}\big)=\tilde{\tau}\circ\tilde{\varphi}_{k+1}.\end{aligned}$$

Thus, $\tilde{\varphi}_{k+1}$ satisfies the first two conditions in (2.8), as well as (i) above.

(3) We define φ_k by the third condition in (2.8). The function φ_k is independent of the choice of p and satisfies the second condition in (2.7). To see this, suppose $p,q=0,\ldots,k+1$ and $\tau\in\mathcal{S}_k$. Let $\tilde{\tau}\in\mathcal{S}_{k+1}$ be defined by

$$\tilde{\tau}\circ\iota_{k+1,p}=\iota_{k+1,q}\circ\tau.$$

If $\varphi_{k,p}$ and $\varphi_{k,q}$ are the functions corresponding to p and q via the third equation in (2.8), then by the second equation in (2.8)

$$\begin{aligned}\iota_{k+1,q}\circ\tau\circ\varphi_{k,p}=\tilde{\tau}\circ\iota_{k+1,p}\circ\varphi_{k,p}=\tilde{\tau}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,p}&=\tilde{\varphi}_{k+1}\circ\tilde{\tau}\circ\iota_{k+1,p}\\ &=\tilde{\varphi}_{k+1}\circ\iota_{k+1,q}\circ\tau=\iota_{k+1,q}\circ\varphi_{k,q}\circ\tau.\end{aligned}$$

We conclude that

$$\tau\circ\varphi_{k,p}=\varphi_{k,q}\circ\tau\qquad\forall\ p,q=0,\ldots,k+1,\ \ \tau\in\mathcal{S}_k.$$

The function φ_k satisfies the first condition in (2.7) because

$$\begin{gathered}\iota_{k+1,p}\big(U^k_p\big)=U^{k+1}_{p,p+1}\cap\Delta^{k+1}_{p,p+1}\qquad\text{and}\\ \iota_{k+1,p}\circ\varphi_k=\tilde{\varphi}_{k+1}\circ\iota_{k+1,p}=\tilde{\pi}^{k+1}_{p,p+1}\circ\iota_{k+1,p}=\iota_{k+1,p}\circ\tilde{\pi}^k_p\quad\text{on}\quad U^k_p.\end{gathered}$$

Finally, φ_k satisfies (i) because $\tilde{\varphi}_{k+1}$ does.

2.2 Homology of Neighborhoods of Smooth Maps

In this subsection, we prove

Proposition 2.2 *If $h\colon Y\longrightarrow X$ is a smooth map and W is an open neighborhood of a subset A of* $\mathrm{Im}\,h$ *in X, there exists a neighborhood U of A in W such that*

$$H_l(U)=0\qquad\textit{if}\ \ l>\dim Y.$$

Note that it may not be true that

$$H_l(A) = 0 \qquad \text{if } \ l > \dim Y.$$

For example, let A be the subset of $X = \mathbb{R}^N$ consisting of countably many k-spheres of radii tending to 0 and having a single point in common. If $k \geq 2$, the set A has infinitely many nonzero homology groups, as shown in [BM].

If $h\colon Y \longrightarrow X$ is a smooth map and k is a nonnegative integer, put

$$N_k(h) = \big\{ y \in Y : \operatorname{rk} dh|_y \leq k \big\}.$$

Proposition 2.2 follows from Lemma 2.4 applied with X replaced by W, Y by $h^{-1}(W)$, and k by $\dim Y$.

One of the ingredients in the proof of Lemma 2.4 is Lemma 2.3. For the purposes of this paper, a *triangulation* of a smooth manifold X is a pair $T = (K, \eta)$ consisting of a simplicial complex and a homeomorphism $\eta\colon |K| \longrightarrow X$, where $|K|$ is a geometric realization of K in $\mathbb{R}^N$ in the sense of Section 3 in [Mu], such that $\eta|_{\mathrm{Int}\,\sigma}$ is smooth for every simplex $\sigma \in K$.

Lemma 2.3 *If X, Y are smooth manifolds and $h : Y \longrightarrow X$ is a smooth map, there exists a triangulation $T = (K, \eta)$ of X such that h is transverse to $\eta|_{\mathrm{Int}\,\sigma}$ for every simplex $\sigma \in K$.*

This lemma is clear. In fact, we can start with any triangulation of X and obtain a desired one by an arbitrary small generic perturbation.

Lemma 2.4 *If $h : Y \longrightarrow X$ is a smooth map, for every nonnegative integer k there exists a neighborhood U of $h\big(N_k(h)\big)$ in X such that*

$$H_l(U) = 0 \qquad \textit{if } \ l > k.$$

Proof: By Lemma 2.3, there exists a triangulation $T = (K, \eta)$ of X such that the smooth map h is transversal to $\eta|_{\mathrm{Int}\,\sigma}$ for all $\sigma \in K$. In particular,

$$h\big(N_k(h)\big) \subset \bigcup_{\sigma \in K, \dim \sigma \geq n-k} \eta(\mathrm{Int}\,\sigma) = \bigcup_{\sigma \in K, \dim \sigma \geq n-k} \eta\big(\mathrm{St}(b_\sigma, \mathrm{sd}\,K)\big),$$

where $n = \dim X$, $\mathrm{sd}\,K$ is the barycentric subdivision of K, and $\mathrm{St}(b_\sigma, \mathrm{sd}\,K)$ is the star of b_σ in $\mathrm{sd}\,K$.[4] Note that

$$\mathrm{St}(b_\sigma, \mathrm{sd}\,K) \cap \mathrm{St}(b_{\sigma'}, \mathrm{sd}\,K) = \emptyset$$

unless $\sigma \subset \sigma'$ or $\sigma' \subset \sigma$. Furthermore, if $\sigma_1 \subset \ldots \subset \sigma_m$,

$$\mathrm{St}(b_{\sigma_1}, \mathrm{sd}\,K) \cap \ldots \cap \mathrm{St}(b_{\sigma_m}, \mathrm{sd}\,K) = \mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_m}, \mathrm{sd}\,K);$$

the last set is contractable. Put

$$U'_m = \bigcup_{\sigma \in K, \dim \sigma = m} \mathrm{St}(b_\sigma, \mathrm{sd}\,K).$$

[4]If K is a simplicial complex and σ is a simplex in K, *the star of σ in K* is the union of the subsets Int σ' taken over the simplices $\sigma' \in K$ such that $\sigma \subset \sigma'$; see Section 62 in [Mu].

Then $U'_{l_m}\cap\ldots\cap U'_{m_j}$ is a disjoint union of contractable open sets in $|K|$. Let

$$U_m=\eta(U'_m),\quad m=n-k,\ldots,n;\qquad U=\bigcup_{m=n-k}^{n}U_m.$$

Since $\eta\colon|K|\longrightarrow X$ is a homeomorphism, $U_{m_1}\cap\ldots\cap U_{m_j}$ is a disjoint union of contractable open subsets of X. It follows from Lemma 2.5 below that $H_l(U)=0$ if $l>k$. Furthermore, by the above $h\big(N_k(h)\big)\subset U$.

Lemma 2.5 *Let $\{U_m\}_{m=0}^{m=k}$ be a collection of open sets in X and $U=\bigcup_{m=0}^{m=k}U_m$. If*

$$H_l\big(U_{m_1}\cap\ldots\cap U_{m_j};\mathbb{Z}\big)=0\qquad\forall\ l>0,\ m_1,\ldots,m_j=0,\ldots,k,$$

then $H_l(U)=0$ for all $l>k$.

This lemma follows by induction from Mayer-Vietoris Theorem; see [Mu, p186].

2.3 Oriented Homology Groups

If X is a simplicial complex, the standard singular chain complex $S_*(X)$ most naturally corresponds to the *ordered* simplicial chain complex of X; see Section 13 in [Mu]. In this subsection, we define a singular chain complex $\bar{S}_*(X)$ which corresponds to the standard, or *oriented*, simplicial chain complex. In particular, its homology is the same as the homology of the ordinary singular chain complex; see Proposition 2.9. On the other hand, it is much easier to construct cycles in $\bar{S}_*(X)$ than in $S_*(X)$.

If X is a topological space, let $\big(S_*(X),\partial_X\big)$ denote its singular chain complex, i.e. the free abelian group on the set

$$\bigcup_{k=0}^{\infty}\mathrm{Hom}(\Delta^k,X)$$

of all continuous maps from standard simplices to X, along with a map ∂_X of degree -1. Let $S'_k(X)$ denote the free subgroup of $S_*(X)$ spanned by the set

$$\big\{f-(\mathrm{sign}\,\tau)f\circ\tau\colon f\in\mathrm{Hom}(\Delta^k,X);\ \tau\in\mathcal{S}_k;\ k=0,1,\ldots\big\}.$$

If $\tau\in\mathcal{S}_k-\{\mathrm{id}\}$, put

$$\tilde{\tau}=\mathrm{Id}_{\Delta^k}-(\mathrm{sign}\,\tau)\tau\in S'_k(\Delta^k).$$

Then, $S'_*(X)$ is the subgroup of $S_*(X)$ spanned by

$$\big\{f_\#\tilde{\tau}\colon f\in\mathrm{Hom}(\Delta^k,X);\ \tau\in\mathcal{S}_k;\ k=0,1,\ldots\big\}.$$

Note that if $h\colon X\longrightarrow Y$ is a continuous map, the linear map

$$h_\#\colon S_*(X)\longrightarrow S_*(Y)$$

maps $S'_*(X)$ into $S'_*(Y)$.

Lemma 2.6 *The free abelian group $S'_*(X)$ is a subcomplex of $\big(S_*(X),\partial_X\big)$, i.e. $\partial_X S'_*(X)\subset S'_*(X)$.*

Proof: (1) Suppose $\tau\in\mathcal{S}_k$. For any $p=0,\ldots,k$, let $\tau_p\in\mathcal{S}_{k-1}$ be such that

$$\tau\circ\iota_{k,p}=\iota_{k,\tau(p)}\circ\tau_p\colon \Delta^{k-1}\longrightarrow\Delta^k_p\subset\Delta^k.$$

Let $\tau_{k,p}\in\mathcal{S}_k$ be defined by

$$\tau_{k,p}(q)=\begin{cases}\iota_{k,p}(q), & \text{if } q<k;\\ p, & \text{if } q=k,\end{cases}$$

Then, $\tau\circ\tau_{k,p}=\tau_{k,\tau(p)}\circ\tau_p\in\mathcal{S}_k$ for all $\tau\in\mathcal{S}_k$. Thus,

$$\operatorname{sign}\tau_p=(-1)^{(k-p)+(k-\tau(p))}\operatorname{sign}\ \tau=(-1)^{p+\tau(p)}\operatorname{sign}\tau. \tag{2.10}$$

(2) By the above, we have

$$\begin{aligned}\partial_{\Delta^k}\tau&=\sum_{p=0}^k(-1)^p\tau\circ\iota_{k,p}=\sum_{p=0}^k(-1)^p\iota_{k,\tau(p)}\circ\tau_p\\&=(\operatorname{sign}\tau)\sum_{p=0}^k(-1)^{\tau(p)}(\operatorname{sign}\tau_p)\iota_{k,\tau(p)}\circ\tau_p=(\operatorname{sign}\tau)\sum_{p=0}^k(-1)^p(\operatorname{sign}\tau_{\tau^{-1}(p)})\iota_{k,p}\circ\tau_{\tau^{-1}(p)}.\end{aligned}$$

Thus,

$$\partial_{\Delta^k}\tilde\tau=\sum_{p=0}^k(-1)^p\big(\iota_{k,p}-(\operatorname{sign}\tau_{\tau^{-1}(p)})\iota_{k,p}\circ\tau_{\tau^{-1}(p)}\big)\in S'_{k-1}(\Delta^k).$$

It follows that for any $f\in S_k(X)$,

$$\partial_X(f_\#\tilde\tau)=f_\#(\partial_{\Delta^k}\tilde\tau)\in S'_{k-1}(X).$$

Lemma 2.7 *There exists a natural transformation of functors $D_X\colon S_*\longrightarrow S_{*+1}$ such that*
(i) if $f\colon\Delta^m\longrightarrow\Delta^k$ is a linear map, D_Xf is a linear combination of linear maps $\Delta^{m+1}\longrightarrow\Delta^k$ for all $k,m=0,1,\ldots$;
(ii) $D_XS'_*(X)\subset S'_*(X)$ for all topological spaces X;
(iii) $\partial_XD_X=(-1)^{k+1}\mathrm{Id}+D_X\partial_X$ on $S'_k(X)$.

Proof: (1) Suppose $k\in\mathbb{Z}^+$. If $f\colon\Delta^m\longrightarrow\Delta^k$ is a linear map, define a new linear map

$$P_kf\colon\Delta^{m+1}\longrightarrow\Delta^k\qquad\text{by}\qquad P_kf\,(e_q)=\begin{cases}f(e_q), & \text{if } q=0,\ldots,m;\\ b_k, & \text{if } q=m+1.\end{cases} \tag{2.11}$$

The transformation P_k induces a linear map on the subchain complex of $S_*(\Delta^k)$ spanned by the linear maps. If $\tau\in\mathcal{S}_m\subset\mathcal{S}_{m+1}$ and $f\in S_m(\Delta^k)$, then

$$P_k(f\circ\tau)=P_kf\circ\tau. \tag{2.12}$$

Thus, P_k maps the subgroup of $S'_*(\Delta^k)$ spanned by the linear maps into itself. Similarly, if $\tau\in\mathcal{S}_k$,

$$\tau_\#(P_kf)\equiv\tau\circ P_kf=P_k(\tau\circ f)\equiv P_k\tau_\#f. \tag{2.13}$$

Furthermore,

$$\partial_{\Delta^k} P_k f = (-1)^{k+1} f + P_k\big(\partial_{\Delta^k} f\big). \tag{2.14}$$

(2) Let $D_X|_{S_k(X)} = 0$ if $k < 1$; then D_X satisfies (i)-(iii). Suppose $k \geq 1$ and we have defined $D_X|_{S_{k-1}(X)}$ so that the three requirements are satisfied wherever D_X is defined. Put

$$D_{\Delta^k}(\mathrm{Id}_{\Delta^k}) = P_k\big(\mathrm{Id}_{\Delta^k} + (-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big) \in S_{k+1}(\Delta^k). \tag{2.15}$$

By the inductive assumption (i) and (2.11), $D_{\Delta^k}(\mathrm{Id}_{\Delta^k})$ is a well-defined linear combination of linear maps. For any $f \in \mathrm{Hom}(\Delta^k, X)$, let

$$D_X f = f_\# D_{\Delta^k}\mathrm{Id}_{\Delta^k}. \tag{2.16}$$

This construction defines a natural transformation $S_k \longrightarrow S_{k+1}$. Since $D_{\Delta^k}(\mathrm{Id}_{\Delta^k})$ is a linear combination of linear maps, it is clear that the requirement (i) above is satisfied; it remains to check (ii) and (iii).

(3) Given $f \in \mathrm{Hom}(\Delta^k, X)$ and $\tau \in \mathcal{S}_k$, let $s = f_\#\tilde{\tau} \in S'_k(X)$. By (2.16), (2.15), (2.13), and naturality of $D_X|_{S_{k-1}}$,

$$\begin{aligned} D_X(f\circ\tau) &= f_\#\tau_\# D_{\Delta^k}\mathrm{Id}_{\Delta^k} = f_\#\tau_\# P_k\big(\mathrm{Id}_{\Delta^k} + (-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big) \\ &= f_\# P_k\big(\tau + (-1)^{k+1}\tau_\# D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big) \\ &= f_\# P_k\big(\tau + (-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\tau\big). \end{aligned} \tag{2.17}$$

Thus,

$$D_X s = f_\# P_k\big(\tilde{\tau} + (-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big). \tag{2.18}$$

By Lemma 2.6, the induction assumption (ii), and (2.12), $S'_k(\Delta^k)$ is mapped into $S'_*(\Delta^k)$ by $D_{\Delta^k}\partial_{\Delta^k}$ and by P_k. Thus, by (2.18), D_X maps $S'_k(X)$ into $S'_{k+1}(X)$. Finally, by (2.18), (2.14), and the inductive assumption (iii),

$$\begin{aligned} \partial_X D_X s &= \partial_X f_\# P_k\big(\tilde{\tau} + (-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big) = f_\#\partial_{\Delta^k} P_k\tilde{\tau} + (-1)^{k+1} f_\#\partial_{\Delta^k} P_k D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau} \\ &= f_\#\big((-1)^{k+1}\tilde{\tau} + P_k\partial_{\Delta^k}\tilde{\tau}\big) + (-1)^{k+1} f_\#\big((-1)^{k+1} D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau} + P_k\partial_{\Delta^k} D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big) \\ &= \big((-1)^{k+1} s + D_X\partial_X s\big) + f_\# P_k\partial_{\Delta^k}\tilde{\tau} + (-1)^{k+1} f_\# P_k\big((-1)^k\partial_{\Delta^k}\tilde{\tau} + D_{\Delta^k}\partial^2_{\Delta^k}\tilde{\tau}\big) \\ &= (-1)^{k+1} s + D_X\partial_X s. \end{aligned}$$

Corollary 2.8 *All homology groups of the complex* $\big(S'_*(X), \partial_X|_{S'_*(X)}\big)$ *are zero.*

Let $\bar{S}_*(X) = S_*(X)/S'_*(X)$ and denote by

$$\pi\colon S_*(X) \longrightarrow \bar{S}_*(X)$$

the projection map. Let $\bar{\partial}_X$ be boundary map on $\bar{S}_*(X)$ induced by ∂_X. We denote by $\bar{H}_*(X;\mathbb{Z})$ the homology groups of $\big(\bar{S}_*(X), \bar{\partial}_X\big)$.

Proposition 2.9 *If X is a topological space, the projection map $\pi: S_*(X) \longrightarrow \bar{S}_*(X)$ induces a natural isomorphism $H_*(X;\mathbb{Z}) \longrightarrow \bar{H}_*(X;\mathbb{Z})$. This isomorphism can be extended to relative homologies to give an isomorphism of homology theories.*

The first statement follows from the long exact sequence in homology for the short exact sequence of chain complexes

$$0 \longrightarrow S'_*(X) \longrightarrow S_*(X) \xrightarrow{\pi} \bar{S}_*(X) \longrightarrow 0$$

and Corollary 2.8. The second statement follows from the first and the Five Lemma.

For a simplicial complex K, there is a natural chain map from the *ordered* simplicial complex $C'_*(K)$ to the singular chain complex $S_*(|K|)$, which induces isomorphism in homology. If the vertices of K are ordered, there is also a chain map from $C'_*(K)$ to the *oriented* chain complex $C_*(K)$, which induces a natural isomorphism in homology. However, the chain map itself depends on the ordering of the vertices; see Section 34 in [Mu]. The advantage of the complex $\bar{S}_*(K)$ is that there is a natural chain map from $C_*(K)$ to $\bar{S}_*(K)$, which induces isomorphism in homology; this chain map is induced by the natural chain map from $C'_*(K)$ to $S_*(|K|)$ described in Section 34 of [Mu].

If $(X, \mathrm{Bd}\, X)$ is a compact oriented n-manifold, (K, K', η) a triangulation of $(X, \mathrm{Bd}\, X)$, and for each n-dimensional simplex $\sigma \in K$,

$$l_\sigma \colon \Delta^n \longrightarrow \sigma$$

is a linear map such that $\eta \circ l_\sigma$ is orientation-preserving, then the fundamental homology class $[X] \in H_n(X, \mathrm{Bd}\, X)$ is represented in $\bar{S}_k(X, \mathrm{Bd}\, X)$ by

$$\sum_{\sigma \in K, \dim \sigma = n} \{\eta \circ l_\sigma\} \equiv \sum_{\sigma \in K, \dim \sigma = n} \pi(\eta \circ l_\sigma),$$

where π is as before. Note that

$$\sum_{\sigma \in K, \dim \sigma = n} \eta \circ l_\sigma$$

may not even be a cycle in $S_k(X, \mathrm{Bd}\, X)$. It is definitely *not* a cycle if $\mathrm{Bd}\, X = \emptyset$ and n is an even positive integer, as the boundary of each term $\eta \circ l_\sigma$ contains one more term with coefficient $+1$ than -1. Similarly, if

$$h \colon (X, \mathrm{Bd}\, X) \longrightarrow (M, U)$$

is a continuous map, $h_*([X]) \in H_k(M, U)$ is represented in $\bar{S}_k(M, U)$ by

$$\sum_{\sigma \in K, \dim \sigma = n} \{h \circ \eta \circ l_\sigma\}.$$

Once again, the obvious preimage under π of the above chain in $S_k(M, U)$ may not be even a cycle.

2.4 Combinatorics of Oriented Singular Homology

In this subsection we characterize cycles and boundaries in $\bar{S}_*(X)$ in a manner suitable for converting them to pseudocycles and pseudocycle equivalences in Subsection 3.1. We will use the two lemmas proved here to glue maps from standard simplices together to construct smooth maps from smooth manifolds.

The homology groups of a smooth manifolds X can be defined with the space $\mathrm{Hom}(\Delta^k, X)$ of continuous maps from Δ^k to X replaced by the space $C^\infty(\Delta^k, X)$ of smooth maps; this is a standard fact in differential topology. Note that the operator D_X of Lemma 2.7 maps smooth maps into linear combinations of smooth maps. Thus, all of the constructions of Subsection 2.3 go through for the chain complexes based on elements in $C^\infty(\Delta^k, X)$ instead of $\mathrm{Hom}(\Delta^k, X)$. Below $\bar{S}_*(X)$ will refer to the quotient complex based on such maps.

If $s = \sum_{j=1}^{j=N} f_j$, where $f_j : \Delta^k \longrightarrow X$ is a continuous map for each j, let

$$\mathcal{C}_s = \big\{(j,p) : j=1,\ldots,N;\ p=0,\ldots,k\big\}.$$

Lemma 2.10 *If $k \geq 1$ and $s \equiv \sum_{j=1}^{j=N} f_j$ determines a cycle in $\bar{S}_k(X)$, there exist a subset $\mathcal{D}_s \subset \mathcal{C}_s \times \mathcal{C}_s$ disjoint from the diagonal and a map*

$$\tau : \mathcal{D}_s \longrightarrow \mathcal{S}_{k-1}, \qquad \big((j_1,p_1),(j_2,p_2)\big) \longrightarrow \tau_{(j_1,p_1),(j_2,p_2)},$$

such that
(i) if $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_s$, then $\big((j_2,p_2),(j_1,p_1)\big) \in \mathcal{D}_s$;
(ii) the projection $\mathcal{D}_s \longrightarrow \mathcal{C}_s$ on either coordinate is a bijection;
(iii) for all $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_s$,

$$\tau_{(j_2,p_2),(j_1,p_1)} = \tau^{-1}_{(j_1,p_1),(j_2,p_2)}, \qquad f_{j_2} \circ \iota_{k,p_2} = f_{j_1} \circ \iota_{k,p_1} \circ \tau_{(j_1,p_1),(j_2,p_2)}, \tag{2.19}$$

$$\textit{and} \qquad \mathrm{sign}\, \tau_{(j_1,p_1),(j_2,p_2)} = -(-1)^{p_1+p_2}. \tag{2.20}$$

This lemma follows from the assumption that $\bar{\partial}\{s\}=0$ and from the definition of $\bar{S}_*(X)$ in Subsection 2.3. The terms appearing in the boundary of s are indexed by the set $\mathcal{C}_s$, and the coefficient of the (j,p)th term is $(-1)^p$. Since s determines a cycle in $\bar{S}_*(X)$, these terms cancel in pairs, possibly after composition with an element $\tau \in \mathcal{S}_{k-1}$ and multiplying by $\mathrm{sign}\,\tau$. This operation does not change the equivalence class of a $(k-1)$-simplex in $\bar{S}_{k-1}(X)$.

Lemma 2.11 *Suppose $k \geq 1$ and*

$$s_0 \equiv \sum_{j=1}^{j=N_0} \{f_{0,j}\} \qquad \textit{and} \qquad s_1 \equiv \sum_{j=1}^{j=N_1} \{f_{1,j}\}$$

determine cycles in $\bar{S}_k(X)$. Let $\mathcal{D}_{s_0} \subset \mathcal{C}_{s_0} \times \mathcal{C}_{s_0}$, $\mathcal{D}_{s_1} \subset \mathcal{C}_{s_1} \times \mathcal{C}_{s_1}$,

$$\tau_0 : \mathcal{D}_{s_0} \longrightarrow \mathcal{S}_{k-1}, \qquad \textit{and} \qquad \tau_1 : \mathcal{D}_{s_1} \longrightarrow \mathcal{S}_{k-1}$$

be the subsets and maps provided by Lemma 2.10. If

$$\big[\{s_0\}\big] = \big[\{s_1\}\big] \in \bar{H}_k(X;\mathbb{Z}),$$

there exist

(a) $\tilde{s}\equiv\sum_{j=1}^{j=\tilde{N}}\tilde{f}_j\in S_{k+1}(X)$*;*

(b) $\mathcal{D}_{\tilde{s}}\subset\mathcal{C}_{\tilde{s}}\times\mathcal{C}_{\tilde{s}}$ *disjoint from the diagonal,* $\mathcal{C}_{\tilde{s}}^{(0)},\mathcal{C}_{\tilde{s}}^{(1)}\subset\mathcal{C}_{\tilde{s}}$*, and maps*

$$\tilde{\tau}\colon\mathcal{D}_{\tilde{s}}\longrightarrow\mathcal{S}_k,\quad\big((j_1,p_1),(j_2,p_2)\big)\longrightarrow\tilde{\tau}_{(j_1,p_1),(j_2,p_2)},$$
$$(\tilde{j}_i,\tilde{p}_i)\colon\big\{1,\ldots,N_i\big\}\longrightarrow\mathcal{C}_{\tilde{s}}^{(i)},\quad\textit{and}\quad\tilde{\tau}_i\colon\big\{1,\ldots,N_i\big\}\longrightarrow\mathcal{S}_k,\quad j\longrightarrow\tilde{\tau}_{(i,j)},\quad i=0,1,$$

such that

(i) if $\big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_{\tilde{s}}$*, then* $\big((j_2,p_2),(j_1,p_1)\big)\in\mathcal{D}_{\tilde{s}}$*;*

(ii) the projection $\mathcal{D}_{\tilde{s}}\longrightarrow\mathcal{C}_{\tilde{s}}$ *on either coordinate is a bijection onto the complement of* $\mathcal{C}_{\tilde{s}}^{(0)}\cup\mathcal{C}_{\tilde{s}}^{(1)}$*;*

(iii) for all $\big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_{\tilde{s}}$*,*

$$\tilde{\tau}_{(j_2,p_2),(j_1,p_1)}=\tilde{\tau}^{-1}_{(j_1,p_1),(j_2,p_2)},\qquad\tilde{f}_{j_2}\circ\iota_{k+1,p_2}=\tilde{f}_{j_1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)},\tag{2.21}$$
$$\textit{and}\qquad\operatorname{sign}\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}=-(-1)^{p_1+p_2};\tag{2.22}$$

(iv) for all $i=0,1$ *and* $j=1,\ldots,N_i$*,*

$$\tilde{f}_{\tilde{j}_i(j)}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)}=f_{i,j}\qquad\textit{and}\qquad\operatorname{sign}\tilde{\tau}_{(i,j)}=-(-1)^{i+\tilde{p}_i(j)};\tag{2.23}$$

(v) $(\tilde{j}_i,\tilde{p}_i)$ *is a bijection onto* $\mathcal{C}_{\tilde{s}}^{(i)}$ *for* $i=0,1$ *and* $\tilde{j}_0\sqcup\tilde{j}_1$ *is injective into* $\{1,\ldots,\tilde{N}\}$*;*

(vi) for all $i=0,1$ *and* $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{s_i}$*, there exist*

$$(\tilde{j}_0,\tilde{p}_0^-),(\tilde{j}_0,\tilde{p}_0^+),\ldots,(\tilde{j}_r,\tilde{p}_r^-),(\tilde{j}_r,\tilde{p}_r^+)\in\mathcal{C}_{\tilde{s}}\qquad\textit{s.t.}$$
$$(\tilde{j}_0,\tilde{p}_0^-,\tilde{p}_0^+)=\big(\tilde{j}_i(j_1),\tilde{p}_i(j_1),\iota_{k+1,\tilde{p}_0^-}\tilde{\tau}_{(i,j_1)}(p_1)\big),\tag{2.24}$$
$$(\tilde{j}_r,\tilde{p}_r^-,\tilde{p}_r^+)=\big(\tilde{j}_i(j_2),\iota_{k+1,\tilde{p}_r^+}\tilde{\tau}_{(i,j_2)}(p_2),\tilde{p}_i(j_2)\big),\tag{2.25}$$
$$\big((\tilde{j}_{r'-1},\tilde{p}_{r'-1}^+),(\tilde{j}_{r'},\tilde{p}_{r'}^-)\big)\in\mathcal{D}_{\tilde{s}}\qquad\forall\,r'=1,\ldots,r,\tag{2.26}$$
$$\iota^{-1}_{k+1,\tilde{p}^+_{r'-1}}(\tilde{p}^-_{r'-1})=\tilde{\tau}_{(\tilde{j}_{r'-1},\tilde{p}^+_{r'-1}),(\tilde{j}_{r'},\tilde{p}^-_{r'})}\iota^{-1}_{k+1,\tilde{p}^-_{r'}}(\tilde{p}^+_{r'})\qquad\forall\,r'=1,\ldots,r,\tag{2.27}$$
$$\begin{aligned}&\tilde{\tau}_{(i,j_1)}\iota_{k,p_1}\tau_{i,((j_1,p_1),(j_2,p_2))}\\&\quad=\big(\tau_{k,(\tilde{p}_0^-,\tilde{p}_0^+)}\tilde{\tau}_{(\tilde{j}_0,\tilde{p}_0^+),(\tilde{j}_1,\tilde{p}_1^-)}\big)\ldots\big(\tau_{k,(\tilde{p}_{r-1}^-,\tilde{p}_{r-1}^+)}\tilde{\tau}_{(\tilde{j}_{r-1},\tilde{p}_{r-1}^+),(\tilde{j}_r,\tilde{p}_r^-)}\big)\big(\tau_{k,(\tilde{p}_r^-,\tilde{p}_r^+)}\tilde{\tau}_{(i,j_2)}\big)\iota_{k,p_2}.\end{aligned}\tag{2.28}$$

Remark 1: By (2.24), the left-hand side of (2.28) is a linear map

$$\Delta^{k-1}\longrightarrow\Delta^k_{\tilde{\tau}_{i,j_1}(p_1)}=\Delta^k_{\iota^{-1}_{k+1,\tilde{p}_0^-}(\tilde{p}_0^+)}.$$

By (2.26), (2.27), and (2.3), for $r'=1,\ldots,r$

$$\tau_{k,(\tilde{p}^-_{r'-1},\tilde{p}^+_{r'-1})}\tilde{\tau}_{(\tilde{j}_{r'-1},\tilde{p}^+_{r'-1}),(\tilde{j}_{r'},\tilde{p}^-_{r'})}\colon\Delta^k_{\iota^{-1}_{k+1,\tilde{p}^-_{r'}}(\tilde{p}^+_{r'})}\longrightarrow\Delta^k_{\iota^{-1}_{k+1,\tilde{p}^-_{r'-1}}(\tilde{p}^+_{r'-1})}$$

is a linear map. By (2.25) and (2.3), the ending of the right-hand side of (2.28) is the linear map

$$\big(\tau_{k,(\tilde{p}_r^-,\tilde{p}_r^+)}\tilde{\tau}_{(i,j_2)}\big)\iota_{k,p_2}\colon\Delta^{k-1}\longrightarrow\Delta^k_{\iota^{-1}_{k+1,\tilde{p}_r^-}(\tilde{p}_r^+)}.$$

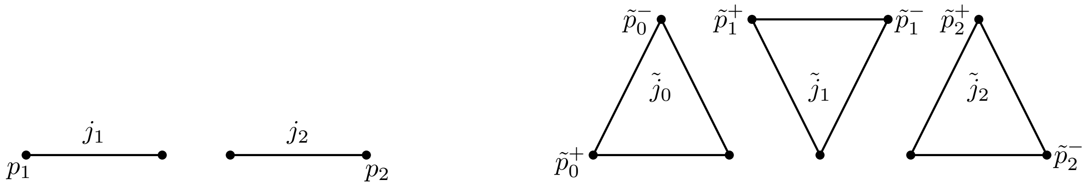

Figure 3: Illustration for Lemma 2.11 and Remark 2

Thus, just like LHS of (2.28), RHS of (2.28) is a linear map

$$\Delta^{k-1} \longrightarrow \Delta^k_{\iota^{-1}_{k+1,\tilde{p}_0^-}(\tilde{p}_0^+)}.$$

In particular, if (2.24)-(2.27) hold, the two sides of (2.28) differ by an element of $\mathcal{S}_{k-1}$.

Remark 2: In Subsection 3.1, the rather elaborate condition (vi) of Lemma 2.11 will be used to construct a manifold $\tilde{M}$ from $\tilde{s}$ such that its boundary consists of the manifolds M_0 and M_1 corresponding to the cycles s_0 and s_1. In the process of constructing $\tilde{M}$ two $(k+1)$-simplices labeled j_1 and j_2 will be identified along the k-simplices $\Delta^{k+1}_{p_1}$ and $\Delta^{k+1}_{p_2}$ after the twist determined by $\tilde{\tau}_{(j_1,p_1),(j_2,p_2)} \in \mathcal{S}_k$, whenever $((j_1,p_1),(j_2,p_2)) \in \mathcal{D}_{\tilde{s}}$. Similarly, in the process of constructing M_i, two k-simplices labeled j_1 and j_2 will be identified along the $(k-1)$-simplices $\Delta^k_{p_1}$ and $\Delta^k_{p_2}$ after the twist determined by $\tau_{i,((j_1,p_1),(j_2,p_2))} \in \mathcal{S}_{k-1}$ whenever $((j_1,p_1),(j_2,p_2)) \in \mathcal{D}_{s_i}$. The elements of $\mathcal{C}^{(0)}_{\tilde{s}}$ and $\mathcal{C}^{(1)}_{\tilde{s}}$ will index the k-simplices in the boundary of $\tilde{s}$ that correspond to simplices in s_0 and s_1. They will form the boundary of $\tilde{M}$ which will be identified with $M_1 - M_0$. In order to do this, we need to make sure that whenever $((j_1,p_1),(j_2,p_2)) \in \mathcal{D}_{s_i}$ the corresponding k simplices in $\mathcal{C}^{(i)}_{\tilde{s}}$ are identified along $(k-1)$-simplices in the same way. For example, in the case of Figure 3, the two one-simplices on LHS will be identified by joining the unlabeled vertices (0-simplices $\Delta^1_{p_1}$ and $\Delta^1_{p_2}$). These two one-simplices also correspond to the bottom boundary 1-simplices on RHS. Then, the vertices of these two bottom simplices corresponding to the unlabeled vertices on LHS must be identified as well, i.e. there should be a sequence of triangles between the two outer triangles whose faces are identified. These identifications must identify the unlabeled vertices in the outer triangles. The requirement (2.28) is automatically satisfied in this case if (2.24)-(2.27) are satisfied, since both sides are maps into

$$\Delta^1_{\tilde{\tau}_{(i,j_1)}(p_1)} \approx \Delta^0.$$

Proof: (1) Since $[\{s_0\}] = [\{s_1\}]$, there exists

$$\{\tilde{s}\} = \sum_{j=1}^{j=\tilde{N}} \{\tilde{f}_j\} \in \bar{S}_{k+1}(X) \qquad \text{s.t.} \qquad \partial\{\tilde{s}\} = \{s_1\} - \{s_0\}.$$

The terms making up $\partial\tilde{s}$ are indexed by the set $\mathcal{C}_{\tilde{s}}$. By definition of $\bar{S}_*(X)$, there exist disjoint subsets $\mathcal{C}^{(0)}_{\tilde{s}}$ and $\mathcal{C}^{(1)}_{\tilde{s}}$ of $\mathcal{C}_{\tilde{s}}$ such that for each $(j,p) \in \mathcal{C}^{(1)}_{\tilde{s}}$ the (j,p)th term of $\partial\tilde{s}$ equals one of the terms of s_i, after a composition with some $\tilde{\tau}\in\mathcal{S}_k$ and multiplying by $-(-1)^i\mathrm{sign}\,\tilde{\tau}$. The remaining terms of $\mathcal{C}_{\tilde{s}}$ must cancel in pairs, as in the case of Lemma 2.10. Thus, $\tilde{s}$ satisfies (i)-(iv) and the first condition in (v). The second condition in (v) is satisfied after subdividing each Δ^{k+1} into $k+2$ simplices with one of the vertices at b_{k+1}.

(2) A priori $\tilde{s}$ may not satisfy (vi). Given $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{s_i}$, we can construct a sequence

$$(\tilde{j}_0,\tilde{p}_0^-),(\tilde{j}_0,\tilde{p}_0^+),\ldots,(\tilde{j}_r,\tilde{p}_r^-),(\tilde{j}_r,\tilde{p}_r^+)\in\mathcal{C}_{\tilde{s}} \tag{2.29}$$

inductively, starting with (2.24) and using (2.26) and (2.27). The requirements (2.26) and (2.27) determine $(\tilde{j}_0,\tilde{p}_0^-)$ uniquely from any element of this sequence. Thus, this sequence contains no loops and must terminate at some

$$(\tilde{j}_r,\tilde{p}_r^+)=\big(\tilde{j}_{i'}(j_2'),\tilde{p}_{i'}(j_2')\big)\in\mathcal{C}_{\tilde{s}}^{(i')}. \tag{2.30}$$

Define $p_2'\in\{0,1,\ldots,k\}$ by

$$\tilde{p}_r^-=\iota_{k+1,\tilde{p}_r^+}\tilde{\tau}_{(i',j_2')}(p_2'). \tag{2.31}$$

We will call (2.29) the sequence of (vi) in Lemma 2.11 beginning with $(j_1,p_1)\in\mathcal{C}_{s_i}$ or ending with $(j_2',p_2')\in\mathcal{C}_{s_{i'}}$. Either condition determines (2.29) via (2.26) and (2.27).

We assume that (vi) fails for $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{s_i}$. We will modify $\tilde{s}$ by adding two $(k+1)$-simplices so that (j_1,p_1) and (j_2,p_2) are added to the collection of elements of $\mathcal{C}_{s_0}\cup\mathcal{C}_{s_1}$ that satisfy (vi) of Lemma 2.11 and no element is removed from this collection.

Let

$$\tau_{\mathrm{RHS}}\colon\big(\Delta^k,\iota_{k+1,\tilde{p}_r^+}^{-1}(\tilde{p}_r^-)\big)\longrightarrow\big(\Delta^k,\iota_{k+1,\tilde{p}_0^-}^{-1}(\tilde{p}_0^+)\big)$$

denote the right-hand side (2.21) without $\tilde{\tau}_{i,j_2}$ and ι_{k,p_2}. By (2.6) and (2.22),

$$\mathrm{sign}\,\tau_{\mathrm{RHS}}=-(-1)^{\tilde{p}_0^-+\tilde{p}_r^+}. \tag{2.32}$$

Define $\vartheta\in\mathcal{S}_{k-1}$ by

$$\tilde{\tau}_{(i,j_1)}\circ\iota_{k,p_1}\circ\tau_{i,((j_1,p_1),(j_2,p_2))}=\tau_{\mathrm{RHS}}\circ\tilde{\tau}_{(i',j_2')}\circ\iota_{k,p_2'}\circ\vartheta\colon\Delta^{k-1}\longrightarrow\Delta^k_{\iota^{-1}_{k+1,\tilde{p}_0^-}(\tilde{p}_0^+)}. \tag{2.33}$$

By (2.10), (2.23), (2.20), (2.32), (2.24), and (2.30),

$$\begin{aligned}\mathrm{sign}\,\vartheta&=(-1)^{p_1+p_2'}\cdot\mathrm{sign}\,\tilde{\tau}_{(i,j_1)}\cdot\mathrm{sign}\,\tilde{\tau}_{(i',j_2')}\cdot\mathrm{sign}\,\tau_{i,((j_1,p_1),(j_2,p_2))}\cdot\mathrm{sign}\,\tau_{\mathrm{RHS}}\\&=(-1)^{i+i'+p_2+p_2'}.\end{aligned} \tag{2.34}$$

Furthermore, by (2.23) used twice, (2.30), (2.4), (2.24)-(2.27), (2.21), (2.33), and (2.19),

$$\begin{aligned}f_{i',j_2'}\circ\iota_{k,p_2'}&=\tilde{f}_{\tilde{j}_r}\circ\iota_{k+1,\tilde{p}_r^+}\circ\tau_{(i',j_2')}\circ\iota_{k,p_2'}=\tilde{f}_{\tilde{j}_0}\circ\iota_{k+1,\tilde{p}_0^-}\circ\tau_{\mathrm{RHS}}\circ\tilde{\tau}_{(i',j_2')}\circ\iota_{k,p_2'}\\&=\tilde{f}_{\tilde{j}_0}\circ\iota_{k+1,\tilde{p}_0^-}\circ\tilde{\tau}_{(i,j_1)}\circ\iota_{k,p_1}\circ\tau_{i,((j_1,p_1),(j_2,p_2))}\circ\vartheta^{-1}\\&=f_{i,j_1}\circ\iota_{k,p_1}\circ\tau_{i,((j_1,p_1),(j_2,p_2))}\circ\vartheta^{-1}=f_{i,j_2}\circ\iota_{k,p_2}\circ\vartheta^{-1}.\end{aligned} \tag{2.35}$$

(3) Choose $\tilde{p}^-_{\tilde{N}+1}\in\{0,\ldots,k+1\}$ such that

$$\big|\tilde{p}^-_{\tilde{N}+1}-\tilde{p}^+_r\big|=1. \tag{2.36}$$

Let

$$\tilde{p}^+_{\tilde{N}+1}=\iota_{k+1,\tilde{p}^-_{\tilde{N}+1}}\big(\iota^{-1}_{k+1,\tilde{p}^+_r}(\tilde{p}^-_r)\big)=\begin{cases}\tilde{p}^-_r, & \text{if } \tilde{p}^-_{\tilde{N}+1}\neq\tilde{p}^-_r;\\ \tilde{p}^+_r, & \text{if } \tilde{p}^-_{\tilde{N}+1}=\tilde{p}^-_r.\end{cases} \tag{2.37}$$

Define $\tau_{(i',j_2')}\in\mathcal{S}_{k-1}$ and $\tilde{\tau}'_{(i,j_2)}\in\mathcal{S}_k$ by

$$\iota_{k,p_2'}\circ\tau^{-1}_{(i',j_2')}=\tilde{\tau}^{-1}_{(i',j_2')}\circ\iota_{k,\tilde{\tau}_{(i',j_2')}(p_2')}:\Delta^{k-1}\longrightarrow\Delta^k_{p_2'}, \tag{2.38}$$

$$\tilde{\tau}'^{\,-1}_{(i,j_2)}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}^+_{\tilde{N}+1}}(\tilde{p}^-_{\tilde{N}+1})}=\iota_{k,p_2}\circ\vartheta^{-1}\circ\tau^{-1}_{(i',j_2')}:\Delta^{k-1}\longrightarrow\Delta^k_{p_2}. \tag{2.39}$$

By (2.10), (2.23), (2.31), (2.30), (2.37), (2.34), and (2.36),

$$\begin{aligned}
\operatorname{sign}\tau_{(i',j_2')}&=(-1)^{p_2'+\tilde{\tau}_{(i',j_2')}(p_2')}\cdot\operatorname{sign}\tilde{\tau}_{(i',j_2')}=-(-1)^{i'+p_2'+\tilde{p}^+_r+\iota_{k+1,\tilde{p}^+_r}(\tilde{p}^-_r)};\\
\operatorname{sign}\tilde{\tau}'_{(i,j_2)}&=(-1)^{\iota^{-1}_{k+1,\tilde{p}^+_{\tilde{N}+1}}(\tilde{p}^-_{\tilde{N}+1})+p_2}\cdot\operatorname{sign}\tau_{(i',j_2')}\cdot\operatorname{sign}\vartheta\\
&=-(-1)^{\iota^{-1}_{k+1,\tilde{p}^+_{\tilde{N}+1}}(\tilde{p}^-_{\tilde{N}+1})+p_2}(-1)^{i'+p_2'+\tilde{p}^+_r+\iota^{-1}_{k+1,\tilde{p}^-_{\tilde{N}+1}}(\tilde{p}^+_{\tilde{N}+1})}(-1)^{i+i'+p_2+p_2'}\\
&=-(-1)^{i+\tilde{p}^+_r+\tilde{p}^+_{\tilde{N}+1}+\tilde{p}^-_{\tilde{N}+1}-1}=-(-1)^{i+\tilde{p}^+_{\tilde{N}+1}}.
\end{aligned} \tag{2.40}$$

Choose a smooth function

$$\tilde{f}_{\tilde{N}+1}:\Delta^{k+1}\longrightarrow X\qquad\text{s.t.}\qquad\begin{aligned}\tilde{f}_{\tilde{N}+1}\circ\iota_{k+1,\tilde{p}^-_{\tilde{N}+1}}&=f_{i',j_2'}\circ\tilde{\tau}^{-1}_{(i',j_2')},\\ \tilde{f}_{\tilde{N}+1}\circ\iota_{k+1,\tilde{p}^+_{\tilde{N}+1}}&=f_{i,j_2}\circ\tilde{\tau}'^{\,-1}_{(i,j_2)}.\end{aligned} \tag{2.41}$$

Such a function exists because the two requirements agree on the overlap, i.e. $\Delta^{k+1}_{\tilde{p}^-_{\tilde{N}+1},\tilde{p}^+_{\tilde{N}+1}}$:

$$\begin{aligned}
f_{i,j_2}\circ\tilde{\tau}'^{\,-1}_{(i,j_2)}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}^+_{\tilde{N}+1}}(\tilde{p}^-_{\tilde{N}+1})}&=f_{i,j_2}\circ\iota_{k,p_2}\circ\vartheta^{-1}\circ\tau^{-1}_{(i',j_2')}=f_{i',j_2'}\circ\iota_{k,p_2'}\circ\tau^{-1}_{(i',j_2')}\\
&=f_{i',j_2'}\circ\tilde{\tau}^{-1}_{(i',j_2')}\circ\iota_{k,\tilde{\tau}_{(i',j_2')}(p_2')}=f_{i',j_2'}\circ\tilde{\tau}^{-1}_{(i',j_2')}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}^+_r}(\tilde{p}^-_r)}\\
&=f_{i',j_2'}\circ\tilde{\tau}^{-1}_{(i',j_2')}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}^-_{\tilde{N}+1}}(\tilde{p}^+_{\tilde{N}+1})}
\end{aligned}$$

by (2.39), (2.35), (2.38), (2.31), and (2.37).

Let $\sigma\in\mathcal{S}_{k+1}$ be the transposition interchanging $\tilde{p}^+_r$ and $\tilde{p}^-_{\tilde{N}+1}$. For each $p=0,1,\ldots,k$, define $\sigma_p\in\mathcal{S}_k$ by

$$\sigma\circ\iota_{k+1,p}=\iota_{k+1,\sigma(p)}\circ\sigma_p. \tag{2.42}$$

Note that σ_p is the identity if $p=\tilde{p}^+_r,\tilde{p}^-_{\tilde{N}+1}$; otherwise, it interchanges $\iota^{-1}_{k+1,p}(\tilde{p}^+_r)$ and $\iota^{-1}_{k+1,p}(\tilde{p}^-_{\tilde{N}+1})$. Thus,

$$\operatorname{sign}\sigma_p=-(-1)^{p+\sigma(p)}. \tag{2.43}$$

Furthermore,

$$\tau_{k,(p,\tilde{p}_r^+)} \circ \tau_{k,(\tilde{p}_{\tilde{N}+1}^-,\sigma(p))} = \sigma_p \qquad \forall\, p \neq \tilde{p}_r^+, \tag{2.44}$$

$$\sigma_{\tilde{p}_r^-} \circ \tau_{k,(\tilde{p}_r^-,p)} \circ \sigma_p \circ \tau_{k,(\sigma(p),\tilde{p}_{\tilde{N}+1}^+)} = \mathrm{id} \qquad \forall\, p \neq \tilde{p}_r^-. \tag{2.45}$$

as can be seen from (2.3) and (2.36). Put

$$\tilde{f}_{\tilde{N}+2} = \tilde{f}_{\tilde{N}+1} \circ \sigma, \qquad \tilde{p}_{\tilde{N}+2}^+ = \sigma\big(\tilde{p}_{\tilde{N}+1}^-\big) = \tilde{p}_r^+, \qquad \tilde{p}_{\tilde{N}+2}^- = \sigma\big(\tilde{p}_{\tilde{N}+1}^+\big) = \tilde{p}_r^-. \tag{2.46}$$

The last equality follows from (2.37). We note that by (2.46), (2.30), and (2.39),

$$\iota_{k+1,\tilde{p}_{\tilde{N}+2}^+}\big(\tilde{\tau}_{(i',j_2')}(p_2')\big) = \iota_{k+1,\tilde{p}_r^+}\big(\tilde{\tau}_{(i',j_2')}(p_2')\big) = \tilde{p}_r^- = \tilde{p}_{\tilde{N}+2}^-, \tag{2.47}$$

$$\iota_{k+1,\tilde{p}_{\tilde{N}+1}^+}\big(\tilde{\tau}'_{(i,j_2)}(p_2)\big) = \tilde{p}_{\tilde{N}+1}^-. \tag{2.48}$$

We will take the replacement for $\tilde{s}$ to be

$$\tilde{s}' = \tilde{s} + \tilde{f}_{\tilde{N}+1} + \tilde{f}_{\tilde{N}+2}.$$

At this point, we need to consider two separate cases.

(4) Case 1: Suppose $(i',j_2') = (i,j_2)$, but either $p_2' \neq p_2$ or (2.28) does not hold. Let $\mathcal{C}_{\tilde{s}'}^{(0)}$ and $\mathcal{C}_{\tilde{s}'}^{(1)}$ be obtained from $\mathcal{C}_{\tilde{s}}^{(0)}$ and $\mathcal{C}_{\tilde{s}}^{(1)}$ by replacing $(\tilde{j}_i(j_2),\tilde{p}_i(j_2))$ with $(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^+)$. We modify the maps $(\tilde{j}_0,\tilde{p}_0)$, $(\tilde{j}_1,\tilde{p}_1)$, $\tilde{\tau}_0$, and $\tilde{\tau}_1$ of Lemma 2.11 by replacing $\tilde{\tau}_{(i,j_2)}$ with $\tilde{\tau}'_{(i,j_2)}$ defined in (2.39) and taking

$$\big(\tilde{j}_i'(j_2),\tilde{p}_i'(j_2)\big) = \big(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^+\big).$$

It is immediate that (v) of Lemma 2.11 is still satisfied. By (2.41),

$$\tilde{f}_{\tilde{N}+1} \circ \iota_{k+1,\tilde{p}_{\tilde{N}+1}^+} \circ \tilde{\tau}'_{(i,j_2)} = f_{i,j_2},$$

i.e. (i,j_2) still satisfies the first equation in (2.23). By (2.40), it still satisfies the second equation in (2.23).

In this case, we set

$$\begin{aligned}\mathcal{D}_{\tilde{s}'} = \mathcal{D}_{\tilde{s}} &\cup \big\{((\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)),((\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{j}_r,\tilde{p}_r^+))\big\}\\ &\cup \big\{((\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)),((\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-))\big\}\\ &\cup \bigcup_{p\le k+1,\, p\neq \tilde{p}_{\tilde{N}+1}^\pm}\big\{((\tilde{N}+1,p),(\tilde{N}+2,\sigma(p))),((\tilde{N}+2,\sigma(p)),(\tilde{N}+1,p))\big\};\end{aligned}$$

$$\begin{aligned}\tilde{\tau}_{(\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)} &= \tilde{\tau}_{(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{j}_r,\tilde{p}_r^+)} = \mathrm{id},\\ \tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)} &= \tilde{\tau}^{-1}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-)} = \sigma_{\tilde{p}_r^-} \circ \tilde{\tau}'_{(i,j_2)} \circ \tilde{\tau}^{-1}_{(i,j_2)},\\ \tilde{\tau}_{(\tilde{N}+1,p),(\tilde{N}+2,\sigma(p))} &= \tilde{\tau}_{(\tilde{N}+2,\sigma(p)),(\tilde{N}+1,p)} = \sigma_p \quad \forall\, \sigma \in \{0,1,\ldots,k+1\} - \{\tilde{p}_{\tilde{N}+1}^-,\tilde{p}_{\tilde{N}+1}^+\}.\end{aligned}$$

Since $\mathcal{D}_{\tilde{s}}$ and $\mathcal{C}_{\tilde{s}}^{(i)}$ satisfy (i) and (ii) of Lemma 2.11, so do $\mathcal{D}_{\tilde{s}'}$ and $\mathcal{C}_{\tilde{s}'}^{(i)}$. By construction, all elements of $\mathcal{D}_{\tilde{s}'}-\mathcal{D}_{\tilde{s}}$ satisfy the first condition in (2.21). By (2.41), (2.30), the first equation in (2.23), and (2.36),

$$\big((\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)\big),\big((\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{j}_r,\tilde{p}_r^+)\big)\in\mathcal{D}_{\tilde{s}'}$$

satisfy (iii) of Lemma 2.11. By (2.43), (2.46), (2.40), the second equation in (2.23), and (2.30),

$$\begin{aligned}\operatorname{sign}\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)}&=-(-1)^{\tilde{p}_r^-+\tilde{p}_{\tilde{N}+1}^+}\cdot(-1)^{i+\tilde{p}_{\tilde{N}+1}^+}\cdot(-1)^{i+\tilde{p}_r^+}\\&=-(-1)^{\tilde{p}_{\tilde{N}+2}^-+\tilde{p}_{\tilde{N}+2}^+}.\end{aligned}$$

On the other hand, by (2.46), (2.42), (2.41),

$$\begin{aligned}\tilde{f}_{\tilde{N}+2}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^+}&=\tilde{f}_{\tilde{N}+1}\circ\sigma\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^+}=\tilde{f}_{\tilde{N}+1}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+1}^-}\circ\sigma_{\tilde{p}_{\tilde{N}+1}^-}=f_{i,j_2}\circ\tilde{\tau}_{(i,j_2)}^{-1}\circ\sigma_{\tilde{p}_{\tilde{N}+1}^-}\\&=f_{i,j_2}\circ\tilde{\tau}_{(i,j_2)}'^{-1}\circ\sigma_{\tilde{p}_r^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)}\\&=\tilde{f}_{\tilde{N}+1}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+1}^+}\circ\sigma_{\tilde{p}_{\tilde{N}+2}^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)}\\&=\tilde{f}_{\tilde{N}+1}\circ\sigma\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)}\\&=\tilde{f}_{\tilde{N}+2}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)}.\end{aligned}$$

We conclude that

$$\big((\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+)\big),\big((\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^+),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-)\big)\in\mathcal{D}_{\tilde{s}'}$$

also satisfy (iii) of Lemma 2.11. For each $p\in\{0,1,\ldots,k+1\}$ different from $\tilde{p}_{N+1}^{\pm}$,

$$\big((\tilde{N}+1,p),(\tilde{N}+2,\sigma(p))\big),\big((\tilde{N}+2,\sigma(p)),(\tilde{N}+1,p)\big)\in\mathcal{D}_{\tilde{s}'}$$

satisfy (iii) of Lemma 2.11 by (2.46), (2.42), and (2.43).

The sequence of (vi) in Lemma 2.11 beginning with $(j_1,p_1)\in\mathcal{C}_{s_i}$ is modified by adding

$$(\tilde{N}+1,\tilde{p}_{N+1}^-),(\tilde{N}+1,\tilde{p}_{N+1}^+)\in\mathcal{C}_{\tilde{s}'}$$

to the right end. By (2.48), this sequence ends with $(j_2,p_2)\in\mathcal{C}_{s_i}$. The reflection of this sequence is the sequence now beginning $(j_2,p_2)\in\mathcal{C}_{s_i}$. These two sequences satisfy all of the requirements of (vi). In order to see that (2.28) holds, note that by (2.39), (2.5), (2.37), (2.31), (2.38), and (2.33), the new right-hand side of (2.28) is given by

$$\begin{aligned}\tau_{\mathrm{RHS}}\,\tilde{\tau}_{(\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)}\tau_{k,(\tilde{p}_{\tilde{N}+1}^-,\tilde{p}_{\tilde{N}+1}^+)}\tilde{\tau}_{(i,j_2)}'\iota_{k,p_2}&=\tau_{\mathrm{RHS}}\,\tau_{k,(\tilde{p}_{\tilde{N}+1}^-,\tilde{p}_{\tilde{N}+1}^+)}\iota_{k,\iota_{k+1,\tilde{p}_{\tilde{N}+1}^+}^{-1}(\tilde{p}_{\tilde{N}+1}^-)}\tau_{(i,j_2)}\vartheta\\=\tau_{\mathrm{RHS}}\,\iota_{k,\iota_{k+1,\tilde{p}_{\tilde{N}+1}^-}^{-1}(\tilde{p}_{\tilde{N}+1}^+)}\tau_{(i,j_2)}\vartheta\ &=\tau_{\mathrm{RHS}}\,\iota_{k,\iota_{k+1,\tilde{p}_r^+}^{-1}(\tilde{p}_r^-)}\tau_{(i,j_2)}\vartheta\\=\tau_{\mathrm{RHS}}\,\iota_{k,\tilde{\tau}_{(i,j_2)}(p_2')}\tau_{(i,j_2)}\vartheta=\tau_{\mathrm{RHS}}\,\tilde{\tau}_{(i,j_2)}\iota_{k,p_2'}\vartheta&=\tilde{\tau}_{(i,j_1)}\circ\iota_{k,p_1}\circ\tau_{i,(j_1,p_1),(j_2,p_2)}.\end{aligned}$$

On the other hand, by (2.25)-(2.27), (2.37), (2.31), (2.42), (2.46), and (2.48), for each $(j_2,p)\!\in\!\mathcal{C}_{s_i}$ with $p\!\neq\!p_2,p_2'$ the sequence of (vi) in Lemma 2.11 ending with (j_2,p) is modified by adding

$$\begin{gathered}\big(\tilde N\!+\!1,\tilde p^-_{\tilde N+1}\big),\big(\tilde N\!+\!1,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p)\big),\big(\tilde N\!+\!2,\sigma\,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p)\big),\big(\tilde N\!+\!2,\tilde p^+_{\tilde N+2}\big),\\ \big(\tilde N\!+\!2,\tilde p^-_{\tilde N+2}\big),\big(\tilde N\!+\!2,\sigma\,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{(i,j_2)}(p)\big),\big(\tilde N\!+\!1,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p)\big),\big(\tilde N\!+\!1,\tilde p^+_{\tilde N+1}\big)\in\mathcal{C}_{\tilde s'}\end{gathered}$$

as the last four pairs. Thus, the new sequence still ends with (j_2,p). Furthermore, the corresponding sequence in (2.28) is modified by replacing $\tilde\tau_{i,j_2}$ with

$$\begin{aligned}&\tilde{\mathcal{T}}_{(j_r,\tilde p_r^+),(\tilde N+1,\tilde p^-_{\tilde N+1})}\mathcal{T}_{k,(\tilde p^-_{\tilde N+1},\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p))}\tilde{\mathcal{T}}_{(\tilde N+1,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p)),(\tilde N+2,\sigma\,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p))}\\ &\qquad\times\mathcal{T}_{k,(\sigma\,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p),\tilde p^+_{\tilde N+2})}\tilde{\mathcal{T}}_{(\tilde N+2,\tilde p^+_{\tilde N+2}),(\tilde N+2,\tilde p^-_{\tilde N+2})}\mathcal{T}_{k,(\tilde p^-_{\tilde N+2},\sigma\,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{(i,j_2)}(p))}\\ &\qquad\times\tilde{\mathcal{T}}_{(\tilde N+2,\sigma\,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{(i,j_2)}(p)),(\tilde N+1,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p))}\mathcal{T}_{k,(\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p),\tilde p^+_{\tilde N+1})}\tilde\tau'_{i,j_2}\\ &=\big(\mathcal{T}_{k,(\tilde p^-_{\tilde N+1},\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p))}\sigma_{\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p)}\mathcal{T}_{k,(\sigma\,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{(i,j_2)}(p),\tilde p^+_{\tilde N+2})}\big)\\ &\qquad\times\tilde\tau_{i,j_2}\tilde\tau'^{\,-1}_{i,j_2}\big(\sigma_{\tilde p_r^-}\mathcal{T}_{k,(\tilde p^-_{\tilde N+2},\sigma\,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{(i,j_2)}(p))}\sigma_{\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p)}\mathcal{T}_{k,(\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p),\tilde p^+_{\tilde N+1})}\big)\tilde\tau'_{i,j_2}\\ &=\mathrm{id}\cdot\tilde\tau_{i,j_2}\tilde\tau'^{\,-1}_{i,j_2}\cdot\mathrm{id}\cdot\tilde\tau'_{i,j_2}=\tilde\tau_{i,j_2}\end{aligned}$$

by (2.46), (2.44), and (2.45). Thus, the above procedure does not change the ends of the sequence of (vi) or the difference between LHS and RHS in (2.28) when one of the ends of the original sequence is (j_2,p) with $p\!\neq\!p_2,p_2'$. The only other elements of $\mathcal{C}_{s_0}\!\cup\!\mathcal{C}_{s_1}$ for which the corresponding sequences of (vi) in Lemma 2.11 are modified are (j_2,p_2') and the endpoint of the sequence previously corresponding to (j_2,p_2). We conclude that this construction adds a new element of $\mathcal{C}_{s_0}\!\cup\!\mathcal{C}_{s_1}$ to the set of elements satisfying (vi) without removing any of the elements from this set.

(5) Case 2: Suppose $(i',j_2')\!\neq\!(i,j_2)$. Let $\mathcal{C}^{(0)}_{\tilde s'}$ and $\mathcal{C}^{(1)}_{\tilde s'}$ be obtained from $\mathcal{C}^{(0)}_{\tilde s}$ and $\mathcal{C}^{(1)}_{\tilde s}$ by replacing $(\tilde j_i(j_2),\tilde p_i(j_2))$ and $(\tilde j_{i'}(j_2'),\tilde p_{i'}(j_2'))$ with $(\tilde N\!+\!1,\tilde p^+_{\tilde N+1})$ and $(\tilde N\!+\!2,\tilde p^+_{\tilde N+2})$, respectively. We modify the maps $(\tilde j_0,\tilde p_0)$, $(\tilde j_1,\tilde p_1)$, $\tilde\tau_0$, and $\tilde\tau_1$ of Lemma 2.11 by replacing $\tilde\tau_{(i,j_2)}$ with $\tilde\tau'_{(i,j_2)}$ defined in (2.39) and taking

$$\big(\tilde j_i'(j_2),\tilde p_i'(j_2)\big)=\big(\tilde N\!+\!1,\tilde p^+_{\tilde N+1}\big)\qquad\text{and}\qquad\big(\tilde j_{i'}'(j_2'),\tilde p_{i'}'(j_2')\big)=\big(\tilde N\!+\!2,\tilde p^+_{\tilde N+2}\big).$$

It is immediate that (v) of Lemma 2.11 is still satisfied. By (2.41), (2.46), and (2.42), (i',j_2') and (i',j_2') still satisfy the first equation in (2.23). By (2.40) and (2.46), they still satisfy the second equation as well.

In this case, we take

$$\begin{aligned}\mathcal{D}_{\tilde s'}=\mathcal{D}_{\tilde s}&\cup\big\{((\tilde j_r,\tilde p_r^+),(\tilde N\!+\!1,\tilde p^-_{\tilde N+1})),((\tilde N\!+\!1,\tilde p^-_{\tilde N+1}),(\tilde j_r,\tilde p_r^+))\big\}\\ &\cup\big\{((\tilde N\!+\!2,\tilde p^-_{\tilde N+2}),(\tilde j_i(j_2),\tilde p_i(j_2))),((\tilde j_i(j_2),\tilde p_i(j_2)),(\tilde N\!+\!2,\tilde p^-_{\tilde N+2}))\big\}\\ &\cup\bigcup_{p\le k+1,p\neq\tilde p^{\pm}_{\tilde N+1}}\big\{((\tilde N\!+\!1,p),(\tilde N\!+\!2,\sigma(p))),((\tilde N\!+\!2,\sigma(p)),(\tilde N\!+\!1,p))\big\};\end{aligned}$$

$$\begin{aligned}
\tilde{\tau}_{(\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)} &= \tilde{\tau}_{(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{j}_r,\tilde{p}_r^+)} = \mathrm{id},\\
\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))} &= \tilde{\tau}^{-1}_{(\tilde{j}_i(j_2),\tilde{p}_i(j_2)),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-)} = \sigma_{\tilde{p}_r^-}\circ\tilde{\tau}'_{(i,j_2)}\circ\tilde{\tau}^{-1}_{(i,j_2)},\\
\tilde{\tau}_{(\tilde{N}+1,p),(\tilde{N}+2,\sigma(p))} &= \tilde{\tau}_{(\tilde{N}+2,\sigma(p)),(\tilde{N}+1,p)} = \sigma_p \quad \forall\, \sigma\in\big\{0,1,\ldots,k+1\big\}-\big\{\tilde{p}_{\tilde{N}+1}^-,\tilde{p}_{\tilde{N}+1}^+\big\}.
\end{aligned}$$

Since $\mathcal{D}_{\tilde{s}}$ and $\mathcal{C}_{\tilde{s}}^{(i)}$ satisfy (i) and (ii) of Lemma 2.11, so do $\mathcal{D}_{\tilde{s}'}$ and $\mathcal{C}_{\tilde{s}'}^{(i)}$. By construction, all elements of $\mathcal{D}_{\tilde{s}'}-\mathcal{D}_{\tilde{s}}$ satisfy the first condition in (2.21). The elements

$$\begin{gathered}
\big((\tilde{j}_r,\tilde{p}_r^+),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-)\big),\big((\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{j}_r,\tilde{p}_r^+)\big),\\
\big((\tilde{N}+1,p),(\tilde{N}+2,\sigma(p))\big),\big((\tilde{N}+2,\sigma(p)),(\tilde{N}+1,p)\big), \quad \text{with} \quad p\neq\tilde{p}_{\tilde{N}+1}^{\pm},
\end{gathered}$$

of $\mathcal{D}_{\tilde{s}'}$ satisfy (iii) of Lemma 2.11 for the same reasons as in Case 1. On the other hand, by (2.43), (2.46), (2.40), the second equation in (2.23), and (2.30),

$$\begin{aligned}
\operatorname{sign}\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))} &= -(-1)^{\tilde{p}_r^-+\tilde{p}_{\tilde{N}+1}^+}\cdot(-1)^{i+\tilde{p}_{\tilde{N}+1}^+}\cdot(-1)^{i+\tilde{p}_r^+}\\
&= -(-1)^{\tilde{p}_{\tilde{N}+2}^-+\tilde{p}_i(j_2)}.
\end{aligned}$$

By the first equation in (2.23), (2.41), (2.42), and (2.46),

$$\begin{aligned}
\tilde{f}_{\tilde{j}_i(j_2)}\circ\iota_{k+1,\tilde{p}_i(j_2)} &= f_{i,j_2}\circ\tilde{\tau}^{-1}_{(i,j_2)} = f_{i,j_2}\circ\tilde{\tau}'^{-1}_{(i,j_2)}\circ\sigma_{\tilde{p}_r^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))}\\
&= \tilde{f}_{\tilde{N}+1}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+1}^+}\circ\sigma_{\tilde{p}_r^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))}\\
&= \tilde{f}_{\tilde{N}+1}\circ\sigma\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))}\\
&= \tilde{f}_{\tilde{N}+2}\circ\iota_{k+1,\tilde{p}_{\tilde{N}+2}^-}\circ\tilde{\tau}_{(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))}.
\end{aligned}$$

We conclude that

$$\big((\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-),(\tilde{j}_i(j_2),\tilde{p}_i(j_2))\big),\big((\tilde{j}_i(j_2),\tilde{p}_i(j_2)),(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-)\big)\in\mathcal{D}_{\tilde{s}'}$$

also satisfy (iii) of Lemma 2.11.

Finally, the sequence of (vi) in Lemma 2.11 beginning with $(j_1,p_1)\in\mathcal{C}_{s_i}$ is modified by adding

$$(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^-),(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^+)\in\mathcal{C}_{\tilde{s}'}$$

to the right end. By (2.48), this sequence ends with $(j_2,p_2)\in\mathcal{C}_{s_i}$. The reflection of this sequence is now the sequence beginning with $(j_2,p_2)\in\mathcal{C}_{s_i}$. By exactly the same computation as in Case 1, these two sequences now satisfy all of the requirements of (vi) in Lemma 2.11. By (2.25)-(2.27), (2.42), (2.46), and (2.48), for each $(j_2,p)\in\mathcal{C}_{s_i}$ with $p\neq p_2$ the sequence of (vi) in Lemma 2.11 ending with (j_2,p) is modified by adding

$$\big(\tilde{N}+2,\tilde{p}_{\tilde{N}+2}^-\big),\big(\tilde{N}+2,\iota_{k+1,\tilde{p}_{\tilde{N}+2}^-}\sigma_{\tilde{p}_r^-}\tilde{\tau}'_{i,j_2}(p)\big),\big(\tilde{N}+1,\iota_{k+1,\tilde{p}_{\tilde{N}+1}^+}\tilde{\tau}'_{i,j_2}(p)\big),\big(\tilde{N}+1,\tilde{p}_{\tilde{N}+1}^+\big)\in\mathcal{C}_{\tilde{s}'}$$

as the last two pairs. Thus, the new sequence still ends with (j_2,p). Furthermore, the corresponding sequence in (2.28) is modified by replacing $\tilde\tau_{i,j_2}$ with

$$\begin{split}
&\tilde{\mathcal{T}}_{(j_i(j_2),\tilde p_i(j_2)),(\tilde N+2,\tilde p^-_{\tilde N+2})}\mathcal{T}_{k,(\tilde p^-_{\tilde N+2},\iota_{k+1,\tilde p^-_{\tilde N+2}}\sigma_{\tilde p^-_r}\tilde\tau'_{i,j_2}(p))}\tilde{\mathcal{T}}_{(\tilde N+2,\iota_{k+1,\tilde p^-_{\tilde N+2}}\sigma_{\tilde p^-_r}\tilde\tau'_{i,j_2}(p)),(\tilde N+1,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p))}\\
&\qquad\quad\times\mathcal{T}_{k,(\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p),\tilde p^+_{\tilde N+1})}\tilde{\mathcal{T}}'_{i,j_2}\\
&\quad=\tilde\tau_{(i,j_2)}\tilde{\mathcal{T}}'^{-1}_{(i,j_2)}\big(\sigma_{\tilde p^-_r}\mathcal{T}_{k,(\tilde p^-_{\tilde N+2},\sigma\,\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p))}\sigma_{\iota_{k+1,\tilde p^+_{\tilde N+1}}}\tilde\tau'_{i,j_2}(p))\mathcal{T}_{k,(\iota_{k+1,\tilde p^+_{\tilde N+1}}\tilde\tau'_{i,j_2}(p),\tilde p^+_{\tilde N+1})}\big)\tilde{\mathcal{T}}'_{i,j_2}\\
&\quad=\tilde\tau_{(i,j_2)}\tilde{\mathcal{T}}'^{-1}_{(i,j_2)}\,\mathrm{id}\,\tilde\tau'_{i,j_2}=\tilde\tau_{i,j_2},
\end{split}$$

by (2.42), (2.46), and (2.45). Thus, the above procedure does not change the ends of the sequence in (vi) or the difference between LHS and RHS in (2.28) when one of the ends of the original sequence is (j_2,p) with $p\neq p_2$.

Similarly, for each $(j_2',p)\in\mathcal{C}_{s_{i'}}$ with $p\neq p_2'$ the sequence of (vi) in Lemma 2.11 that ends with (j_2',p) is modified by adding

$$(\tilde N+1,\tilde p^-_{\tilde N+1}),\,\big(\tilde N+1,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{i',j_2'}(p)\big),\,\big(\tilde N+2,\iota_{k+1,\tilde p^+_{\tilde N+2}}\tilde\tau_{i',j_2'}(p)\big),\,\big(\tilde N+2,\tilde p^+_{\tilde N+2}\big)\in\mathcal{C}_{\tilde s'}$$

as the last two pairs. Note that $\sigma_{\tilde p^+_r}=\mathrm{id}$. Thus, the new sequence still ends with (j_2',p). The corresponding sequence in (2.28) is modified by replacing $\tilde\tau_{i',j_2'}$ with

$$\begin{split}
&\tilde{\mathcal{T}}_{(\tilde j_r,\tilde p^+_r),(\tilde N+1,\tilde p^-_{\tilde N+1})}\mathcal{T}_{k,(\tilde p^-_{\tilde N+1},\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{i',j_2'}(p))}\tilde{\mathcal{T}}_{(\tilde N+1,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{i',j_2'}(p)),(\tilde N+2,\iota_{k+1,\tilde p^+_{\tilde N+2}}\tilde\tau_{i',j_2'}(p))}\\
&\qquad\quad\times\mathcal{T}_{k,(\iota_{k+1,\tilde p^+_{\tilde N+2}}\tilde\tau_{i',j_2'}(p),\tilde p^+_{\tilde N+2})}\tilde{\mathcal{T}}_{i',j_2'}\\
&\quad=\big(\mathcal{T}_{k,(\tilde p^-_{\tilde N+1},\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{i',j_2'}(p))}\sigma_{\iota_{k+1,\tilde p^-_{\tilde N+1}}}\tilde\tau_{i',j_2'}(p)\mathcal{T}_{k,(\sigma\,\iota_{k+1,\tilde p^-_{\tilde N+1}}\tilde\tau_{i',j_2'}(p),\tilde p^+_{\tilde N+2})}\big)\tilde\tau_{i',j_2'}=\tilde\tau_{i',j_2'},
\end{split}$$

by (2.42), (2.46), and (2.44). Thus, the above procedure does not change the ends of the sequence in (vi) or the difference between LHS and RHS in (2.28) when one of the ends of the original sequence is (j_2',p) with $p\neq p_2'$. The only other elements of $\mathcal{C}_{s_0}\cup\mathcal{C}_{s_1}$ for which the corresponding sequences in (vi) of Lemma 2.11 are modified are $(j_2',p_2')\in\mathcal{C}_{s_{i'}}$ and the endpoint of sequence previously corresponding to (j_2,p_2). We conclude that this construction adds a new element of $\mathcal{C}_{s_0}\cup\mathcal{C}_{s_1}$ to the set of elements satisfying (vi) without removing any of the elements from this set.

3 Integral Homology and Pseudocycles

3.1 From Integral Cycles to Pseudocycles

In this subsection, we prove

Proposition 3.1 *If X is a smooth manifold, there exists a homomorphism*

$$\Psi_*\colon H_*(X;\mathbb{Z})\longrightarrow\mathcal{H}_*(X),$$

which is natural with respect to smooth maps.

In the proof of Lemma 3.2, we construct a homomorphism from the subgroup of cycles in $\bar{S}_*(X)$ to $\mathcal{H}_*(X)$. Starting with a cycle $\{s\}$ as in Lemma 2.10, we will glue the functions $f_j\circ\varphi_k$ together, where φ_k is the self-map of Δ^k provided by Lemma 2.1. These functions continue to satisfy the second equation in (2.19), i.e.

$$f_{j_2}\circ\varphi_k\circ\iota_{k,p_2} = f_{j_1}\circ\varphi_k\circ\iota_{k,p_1}\circ\tau_{(j_1,p_1),(j_2,p_2)} \qquad \forall\,\big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_s, \tag{3.1}$$

because $\varphi_k=\mathrm{id}$ on $\Delta^k-\mathrm{Int}\,\Delta^k$ by the first equation (2.7). The proof of Lemma 3.2 implements a construction suggested in Section 7.1 of [McSa].

Lemma 3.3 shows that the map of Lemma 3.2 descends to the homology groups. Starting with a chain $\{\tilde{s}\}$ as in Lemma 2.11, we will glue the functions $\tilde{f}_j\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}$ together, where $\tilde{\varphi}_{k+1}$ and φ_{k+1} are the self-maps of Δ^{k+1} provided by Lemma 2.1. If $i=0,1$ and $j=1,\ldots,N_i$, by the third equation in (2.8), the second equation in (2.7), and the first equation in (2.23)

$$\begin{aligned}\tilde{f}_{\tilde{j}_i(j)}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)} = \tilde{f}_{\tilde{j}_i(j)}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\varphi_k\circ\tilde{\tau}_{(i,j)} &= \tilde{f}_{\tilde{j}_i(j)}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)}\circ\varphi_k \\ &= f_{i,j}\circ\varphi_k.\end{aligned}$$

Since $\varphi_{k+1}=\mathrm{id}$ on $\Delta^{k+1}-\mathrm{Int}\,\Delta^{k+1}$, it follows that

$$\tilde{f}_{\tilde{j}_i(j)}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)} = f_{i,j}\circ\varphi_k \qquad \forall\; j=1,\ldots,N_i,\;\; i=0,1. \tag{3.2}$$

Similarly, if $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{\tilde{s}}$, by the third equation in (2.8) used twice, the second equation in (2.21), and the second equation in (2.7),

$$\begin{aligned}\tilde{f}_{j_2}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,p_2} &= \tilde{f}_{j_2}\circ\iota_{k+1,p_2}\circ\varphi_k = \tilde{f}_{j_1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}\circ\varphi_k \\ &= \tilde{f}_{j_1}\circ\iota_{k+1,p_1}\circ\varphi_k\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)} = \tilde{f}_{j_1}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}.\end{aligned}$$

Since $\varphi_{k+1}=\mathrm{id}$ on $\Delta^{k+1}-\mathrm{Int}\,\Delta^{k+1}$, it follows that

$$\tilde{f}_{j_2}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,p_2} = \tilde{f}_{j_1}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)} \quad \forall\;((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{\tilde{s}}. \tag{3.3}$$

Thus, the functions $\tilde{f}_j\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}$ are the analogues (in the sense of Lemma 2.11) of the functions $\tilde{f}_j$ for the maps $f_{0,j}\circ\varphi_k$ and $f_{1,j}\circ\varphi_k$.

Lemma 3.2 *If X is a smooth manifold, every integral k-cycle in X, based on $C^\infty(\Delta^k;X)$, determines an element of $\mathcal{H}_k(X)$.*

Proof: (1) If $k=0$, this is obvious. Suppose $k\geq 1$ and

$$s\equiv\sum_{j=1}^{j=N} f_j$$

determines a cycle in $\bar{S}_k(X)$. Let $\mathcal{D}_s$ be the set provided by Lemma 2.10 and let $\tau\colon\mathcal{D}_s\longrightarrow\mathcal{S}_{k-1}$ be the corresponding map. Let

$$M' = \Big(\bigsqcup_{j=1}^{j=N}\{j\}\times\Delta^k\Big)/\sim, \qquad \text{where}$$

$$\big(j_1,\iota_{k,p_1}(\tau_{(j_1,p_1),(j_2,p_2)}(t))\big)\sim\big(j_2,\iota_{k,p_2}(t)\big) \quad \forall\;\big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_s,\;\; t\in\Delta^{k-1}.$$

Let π be the quotient map. Define

$$F\colon M' \longrightarrow X \qquad \text{by} \quad F\big([j,t]\big) = f_j\big(\varphi_k(t)\big).$$

This map is well-defined by (3.1) and continuous by the Pasting Lemma. Let M be the complement in M' of the set

$$\pi\Big(\bigsqcup_{j=1}^{j=N}\{j\}\times Y\Big),$$

where Y is the $(k\!-\!2)$-skeleton of Δ^k. By continuity of F, compactness of M', and the first equation in (2.7),

$$\mathrm{Bd}\, F|_M = F(M'\!-\!M) = \bigcup_{j=1}^{j=N} f_j\big(\varphi_k(Y)\big) = \bigcup_{j=1}^{j=N} f_j(Y). \tag{3.4}$$

Since $f_j|_{\mathrm{Int}\,\sigma}$ is smooth for all $j=1,\ldots,N$ and all simplices $\sigma\subset\Delta^k$, $\mathrm{Bd}\,F|_M$ has dimension at most $k-2$ by (3.4). Thus, $F|_M$ is a k-pseudocycle, provided M is a smooth oriented manifold and $F|_M$ is a smooth map. This is shown below.

(2) Let $[j,t]\in M$ be any point. If $t\in\mathrm{Int}\,\Delta^k$, then $\pi(\{j\}\times\mathrm{Int}\,\Delta^k)$ is an open set about $[j,t]$, which is naturally homeomorphic to $\mathrm{Int}\,\Delta^k$. If

$$[j,t] = \big[j_1, \iota_{k,p_1}(t_1)\big] = \big[j_2, \iota_{k,p_2}(t_2)\big]$$

with $(j_1,p_1)\neq(j_2,p_2)$ and $t_1\in\mathrm{Int}\ \Delta^{k-1}$, let

$$U = \pi\big(\{j_1\}\times U^k_{p_1}\big)\cup\pi\big(\{j_2\}\times U^k_{p_2}\big).$$

This is an open neighborhood of $[j,t]$ in M. It is homeomorphic in a canonical way to the disjoint union of $U^k_{p_1}$ and $U^k_{p_2}$ with $\mathrm{Int}\,\Delta^k_{p_1}\subset U^k_{p_1}$ and $\mathrm{Int}\,\Delta^k_{p_2}\subset U^k_{p_2}$ identified by the linear map

$$\iota_{k,p_1}\circ\tau_{(j_1,p_1),(j_2,p_2)}\circ\iota^{-1}_{k,p_2}\colon \mathrm{Int}\,\Delta^k_{p_2} \longrightarrow \mathrm{Int}\,\Delta^k_{p_1} \tag{3.5}$$

and thus to an open subset of $\mathbb{R}^k$. By (2.20), the transition map (3.5) is orientation-reversing if the open simplices $\mathrm{Int}\,\Delta^k_{p_1}$ and $\mathrm{Int}\,\Delta^k_{p_2}$ are oriented as boundaries of the k-manifolds $U^k_{p_1}$ and $U^k_{p_2}$ with their natural orientations. This means that the induced orientations of T_pU coming from the two k-manifolds with boundary agree. On any nonempty overlap of this coordinate chart with any other coordinate chart, the transition map is the identity map on an open subset of $\mathrm{Int}\,\Delta^k$. Thus, M is a smooth oriented manifold. The map F is smooth on $\{j\}\times\mathrm{Int}\,\Delta^k$ for all j by our assumptions on F. If

$$[j,t] = \big[j_1, \iota_{k,p_1}(t_1)\big] = \big[j_2, \iota_{k,p_2}(t_2)\big],$$

then F is smooth on the open set U, defined as above, because it is smooth on

$$\pi\big(\{j_1\}\times U^k_{p_1}\big) \quad\text{and}\quad \pi\big(\{j_2\}\times U^k_{p_2}\big),$$

and all derivatives in the direction normal to $\pi(\{j_1\}\times\mathrm{Int}\,\Delta^k_{p_1})$ vanish by the first equation in (2.7).

Remark: The pseudocycle $F|_M$ constructed above depends on the choice of $\mathcal{D}_s$ and τ. However, as the next lemma shows, the image of $F|_M$ in $\mathcal{H}_k(X)$ depends only on $[\{s\}]$.

Given $\tilde{s}$ as in Lemma 2.11 and $j=1,\ldots,\tilde{N}$, denote by

$$\mathcal{E}_j(\tilde{s}) \subset \big\{0,1,\ldots,k+1\big\} \times \big\{0,1,\ldots,k+1\big\}$$

the set of pairs (p^-,p^+) such that

$$\big(j,p^-,p^+\big) = \big(\tilde{j}_{r'},\tilde{p}^-_{r'},\tilde{p}^+_{r'}\big)$$

for the sequence of (vi) in Lemma 2.11 beginning with some $(j_1,p_1)\in\mathcal{C}_{s_i}$, $i=0,1$, and for some r'. The set $\mathcal{E}_j(\tilde{s})$ depends on the choice of $\mathcal{D}_{\tilde{s}}$. Let

$$\Delta^{k+1}_{\mathcal{E}_j(\tilde{s})} = \Delta^{k+1}$$

be the union of the $(k-2)$-skeleton of Δ^{k+1} with all $(k-1)$-simplices in Δ^{k+1} that are *not* of the form $\Delta^{k+1}_{p^-,p^+}$ for some $(p^-,p^+)\in\mathcal{E}_j(\tilde{s})$.

Lemma 3.3 *Under the construction of Lemma 3.2, homologous k-cycles determine the same equivalence class of pseudocycles in $\mathcal{H}_k(X)$.*

Proof: (1) If $k=0$, this is obvious. Suppose $k>0$ and

$$s_0 \equiv \sum_{j=1}^{j=N_0} f_{0,j} \qquad \text{and} \qquad s_1 \equiv \sum_{j=1}^{j=N_1} f_{1,j}$$

determine two homologous k-cycles in $\bar{S}_k(X)$. Let $\mathcal{D}_{s_0}$ and $\mathcal{D}_{s_1}$ be the sets provided by Lemma 2.10 and let τ_0 and τ_1 be the corresponding maps into $\mathcal{S}_{k-1}$. Denote by (M_0',M_0,F_0) and (M_1',M_1,F_1) the triples constructed in the proof of Lemma 3.2 corresponding to s_0 and s_1. Let

$$\tilde{s} = \sum_{j=1}^{j=\tilde{N}} \tilde{f}_j$$

be a chain in $S_{k+1}(X)$ provided by Lemma 2.11 for the homologous cycles s_0 and s_1. Denote by $\mathcal{C}^{(0)}_{\tilde{s}}$, $\mathcal{C}^{(1)}_{\tilde{s}}$, $\mathcal{D}_{\tilde{s}}$, $(\tilde{j}_i,\tilde{p}_i,\tilde{\tau}_i)$, and $\tilde{\tau}$ the corresponding objects of Lemma 2.11.

(2) Put

$$\tilde{M}' = \Big(\bigsqcup_{j=1}^{j=\tilde{N}} \{j\}\times\Delta^{k+1}\Big), \qquad \text{where}$$

$$\big(j_1,\iota_{k+1,p_1}(\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}(t))\big) \sim \big(j_2,\iota_{k+1,p_2}(t)\big) \quad \forall\ \big((j_1,p_1),(j_2,p_2)\big)\in\tilde{\mathcal{D}}_{\tilde{s}},\ t\in\Delta^k.$$

Let

$$\tilde{\pi}\colon \bigsqcup_{j=1}^{j=\tilde{N}} \{j\}\times\Delta^{k+1} \longrightarrow \tilde{M}'$$

be the quotient map. Define

$$\tilde{F}\colon \tilde{M}' \longrightarrow X \qquad \text{by} \qquad \tilde{F}\big([j,t]\big) = \tilde{f}_j\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(t))\big).$$

This map is well-defined by (3.3) and continuous by the Pasting Lemma. Let $\tilde{M}$ be the complement in $\tilde{M}'$ of the set

$$\tilde{\pi}\Big(\bigsqcup_{j=1}^{j=\tilde{N}}\{j\}\times\Delta^{k+1}_{\mathcal{E}_j(\tilde{s})}\Big).$$

By continuity of $\tilde{F}$, compactness of $\tilde{M}'$, and the first equation in (2.8),

$$\mathrm{Bd}\,\tilde{F}|_{\tilde{M}} = \tilde{F}(\tilde{M}'-\tilde{M}) = \bigcup_{j=1}^{j=\tilde{N}} \tilde{f}_j\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(\Delta^{k+1}_{\mathcal{E}_j(\tilde{s})}))\big) = \bigcup_{j=1}^{j=\tilde{N}} \tilde{f}_j\big(\Delta^{k+1}_{\mathcal{E}_j(\tilde{s})}\big). \tag{3.6}$$

Since $\tilde{f}_j|_{\mathrm{Int}\,\sigma}$ is smooth for all $j=1,\ldots,\tilde{N}$ and all simplices $\sigma\subset\Delta^{k+1}$, $\mathrm{Bd}\,\tilde{F}|_{\tilde{M}}$ has dimension at most $k-1$ by (3.6). Thus, $\tilde{F}|_{\tilde{M}}$ is a pseudocycle equivalence between $F_0|_{M_0}$ and $F_1|_{M_1}$, provided $\tilde{M}$ is a smooth oriented manifold, $\tilde{F}|_{\tilde{M}}$ is a smooth map, and

$$\partial\big(\tilde{F}|_{\tilde{M}}\big) = F_1|_{M_1} - F_0|_{M_0}.$$

This is shown below.

(3) Let $[j,t]\in\tilde{M}$ be any point. If $t\in\mathrm{Int}\,\Delta^{k+1}$, then

$$U_j \equiv \tilde{\pi}(\{j\}\times\mathrm{Int}\,\Delta^{k+1})$$

is an open set about $[j,t]$, which is naturally homeomorphic to $\mathrm{Int}\,\Delta^{k+1}$. If

$$[j,t] = \big[j,\iota_{k+1,p}(t)\big]$$

for some $(j,t)\in\mathcal{C}^{(0)}_{\tilde{s}}\cup\mathcal{C}^{(1)}_{\tilde{s}}$, then

$$U_{(j,p)} \equiv \tilde{\pi}\big(\{j\}\times(\mathrm{Int}\,\Delta^{k+1}\cup\mathrm{Int}\,\Delta^{k+1}_p)\big)$$

is an open neighborhood of $[j,t]$ in $\tilde{M}$ naturally homeomorphic to an open subset of $\mathbb{R}^{k+1}\times\bar{\mathbb{R}}^+$. If

$$[j,t] = \big[j_1,\iota_{k+1,p_1}(t_1)\big] = \big[j_2,\iota_{k+1,p_2}(t_2)\big]$$

with $(j_1,p_1)\neq(j_2,p_2)$ and $t_1\in\mathrm{Int}\,\Delta^k$, let

$$U_{(j_1,p_1),(j_2,p_2)} \equiv \tilde{\pi}\big(\{j_1\}\times(\mathrm{Int}\,\Delta^{k+1}\cup\mathrm{Int}\,\Delta^{k+1}_{p_1})\big)\cup\tilde{\pi}\big(\{j_2\}\times(\mathrm{Int}\,\Delta^{k+1}\cup\mathrm{Int}\,\Delta^{k+1}_{p_2})\big).$$

Similarly to the case of Lemma 3.2, this is an open neighborhood of $[j,t]$ in $\tilde{M}$ which is naturally homeomorphic to an open subset of $\mathbb{R}^{k+1}$ by an orientation-preserving map. It overlaps smoothly with the charts U_j and $U_{(j,p)}$ above.

Finally, suppose

$$t\in\Delta^{k+1}_{p^-,p^+} \qquad \text{for some} \qquad (p^-,p^+)\in\mathcal{E}_j(\tilde{s}).$$

By definition, there exist $i\!=\!0,1$, $((j_1,p_1),(j_2,p_2))\!\in\!\mathcal{D}_{s_i}$, and a sequence

$$(\tilde{j}_0,\tilde{p}_0^-),(\tilde{j}_0,\tilde{p}_0^+),\ldots,(\tilde{j}_r,\tilde{p}_r^-),(\tilde{j}_r,\tilde{p}_r^+)\in\mathcal{C}_{\tilde{s}}$$

satisfying (vi) of Lemma 2.11 such that

$$(j,p^-,p^+)=\big(\tilde{j}_{r'},\tilde{p}_{r'}^-,\tilde{p}_{r'}^+\big)\qquad\text{for some}\qquad r'=0,1,\ldots,r.$$

By (2.24)-(2.27), this sequence (up to reflection) depends only on j and $\{p^-,p^+\}$. Let

$$U_{(j_1,p_1),(j_2,p_2)}=\bigcup_{l=0}^{l=r}\tilde{\pi}\big(\tilde{j}_l\!\times\!\tilde{U}^{k+1}_{p_l^-,p_l^+}\big).$$

This is a neighborhood of $[j,t]$ in $\tilde{M}$. For each $l\!=\!1,\ldots,r$, define $\tau_l\!\in\!\mathcal{S}_{k-1}$ by

$$\begin{aligned}\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)}&\equiv\iota_{k,\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)}\circ\tau_l\\&=\iota_{k,\iota^{-1}_{k+1,\tilde{p}_{l-1}^+}(\tilde{p}_{l-1}^-)}\circ\tau_l;\end{aligned}\tag{3.7}$$

the last equality holds by (2.27). We define a linear map

$$\begin{gathered}\psi_l\colon\mathrm{CH}\big(b_{k,\tilde{p}_l^-},b_{k,\tilde{p}_l^+},\{e_q\colon q\!\neq\!p_l^{\pm}\}\big)\longrightarrow\mathbb{R}^{k+1}\!=\!\mathbb{R}^{k-1}\!\times\!\mathbb{C}\qquad\text{by}\\ \psi_l(b_{k,\tilde{p}_l^-})=(0,e^{\pi\mathrm{i}(r-l)/(r+1)}),\qquad\psi_l(b_{k,\tilde{p}_l^+})=(0,e^{\pi\mathrm{i}(r+1-l)/(r+1)}),\\ \psi_l\circ\iota_{k+1,(\tilde{p}_l^-,\tilde{p}_l^+)}=\mathrm{id}_{\Delta^{k-1}}\circ\tau_1\circ\ldots\circ\tau_l.\end{gathered}$$

Then, by (2.27),

$$\begin{aligned}\big\{\psi_l\!\circ\!\iota_{k+1,\tilde{p}_l^-}\big\}\big(\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)\big)&=\big\{\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\big\}\big(\iota^{-1}_{k+1,\tilde{p}_{l-1}^+}(\tilde{p}_{l-1}^-)\big)\\&=\big\{\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\big\}\big(\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)\big)\\&=\big\{\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\!\circ\!\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\big\}\big(\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)\big).\end{aligned}\tag{3.8}$$

On the other hand, by (2.1) used twice and (3.7),

$$\begin{aligned}\psi_l\!\circ\!\iota_{k+1,\tilde{p}_l^-}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)}&=\psi_l\!\circ\!\iota_{k+1,(\tilde{p}_l^-,\tilde{p}_l^+)}=\psi_{l-1}\!\circ\!\iota_{k+1,(\tilde{p}_{l-1}^-,\tilde{p}_{l-1}^+)}\circ\tau_l\\&=\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}_{l-1}^+}(\tilde{p}_{l-1}^-)}\circ\tau_l\\&=\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\!\circ\!\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\circ\iota_{k,\iota^{-1}_{k+1,\tilde{p}_l^-}(\tilde{p}_l^+)}.\end{aligned}\tag{3.9}$$

By (3.8), (3.9), and linearity of ψ_{l-1} and ψ_l,

$$\psi_l\circ\iota_{k+1,\tilde{p}_l^-}=\psi_{l-1}\!\circ\!\iota_{k+1,\tilde{p}_{l-1}^+}\circ\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\qquad\forall\ l\!=\!1,\ldots,r.$$

Thus, by (2.2), the maps ψ_l induce a well-defined homeomorphism

$$\psi\colon U_{(j_1,p_1),(j_2,p_2)}\longrightarrow P,\qquad\text{where}$$

$$P=\Big\{\sum_{p=0}^{k-1}t_pe_p+\sum_{l=0}^{r+1}\tilde{t}_l(0,e^{\pi\mathrm{i}l/(r+1)})\colon t_p\!>\!0\ \forall\,p,\ \tilde{t}_l\!\geq\!0\ \forall l;\ \sum_{p=0}^{k-1}t_p\!+\!\sum_{l=0}^{r+1}\tilde{t}_l\!=\!1\Big\}\subset\mathbb{R}^{k-1}\!\times\!\mathbb{C}=\mathbb{R}^{k+1}.$$

Note that P is an open subset of $\mathbb{R}^k \times \bar{\mathbb{R}}^+$.

The chart $(U_{(j_1,p_1),(j_2,p_2)}, \psi)$ on $\tilde{M}$ intersects

$$U_{\tilde{j}_l}, \qquad U_{(\tilde{j}_0,\tilde{p}_0^-)}, \quad \text{and} \quad U_{(\tilde{j}_r,\tilde{p}_r^+)},$$

with the overlap map equal to the restriction of the diffeomorphism ψ_l, with $l=0,r$ in the last two cases, to

$$\mathrm{Int}\,\Delta^{k+1}\cap U^{k+1}_{\tilde{p}_l^-,\tilde{p}_l^+}, \qquad (\mathrm{Int}\,\Delta^{k+1}\cup \mathrm{Int}\,\Delta^{k+1}_{\tilde{p}_0^-})\cap U^{k+1}_{\tilde{p}_0^-,\tilde{p}_0^+}, \quad \text{and} \quad (\mathrm{Int}\,\Delta^{k+1}\cup \mathrm{Int}\,\Delta^{k+1}_{\tilde{p}_r^+})\cap U^{k+1}_{\tilde{p}_r^-,\tilde{p}_r^+}$$

respectively. It also intersects the open sets $U_{(\tilde{j}_{l-1},\tilde{p}_{l-1}),(\tilde{j}_l,\tilde{p}_l)}$, with $l=1,\ldots,r$. The overlap map in this case is the diffeomorphism

$$\begin{gathered}\tilde{\pi}\big(\{\tilde{j}_{l-1}\}\times((\mathrm{Int}\,\Delta^{k+1}\cup \mathrm{Int}\,\Delta^{k+1}_{\tilde{p}_{l-1}^+})\cap U^{k+1}_{\tilde{p}_{l-1}^-,\tilde{p}_{l-1}^+})\big)\cup\pi\big(\{\tilde{j}_l\}\times((\mathrm{Int}\,\Delta^{k+1}\cup \mathrm{Int}\,\Delta^{k+1}_{\tilde{p}_l^-})\cap U^{k+1}_{\tilde{p}_l^-,\tilde{p}_l^+})\big)\\ \longrightarrow\Big\{\sum_{p=0}^{k-1}t_pe_p+\sum_{l'=l-1}^{l+1}\tilde{t}_{l'}(0,e^{\pi\mathrm{i}(r+1-l')/(r+1)})\colon t_p>0\;\forall p;\; t_l>0,\; t_{l\pm1}\ge0;\; \sum_{p=0}^{k-1}t_p+\sum_{l'=l-1}^{l+1}\tilde{t}_{l'}=1\Big\}\end{gathered}$$

induced by ψ_{l-1} and ψ_l. The open set $U_{(j_1,p_1),(j_2,p_2)}$ does not intersect any of the other charts described above. Thus, $\tilde{M}$ is a smooth oriented manifold with boundary.

(4) For the same reasons as in the proof of Lemma 3.2, the function $\tilde{F}$ is smooth on the open sets

$$U_j, \quad U_{(j',p)}, \quad \text{and} \quad U_{(j_1,p_1),(j_2,p_2)} \qquad j=1,\ldots,\tilde{N}, \;\; (j',p)\in\mathcal{C}^{(0)}_{\tilde{s}}\cup\mathcal{C}^{(1)}_{\tilde{s}}, \;\; \big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_{\tilde{s}},$$

defined above. If $i=0,1$ and $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{s_i}$, $\tilde{F}$ is also smooth on $U_{(j_1,p_1),(j_2,p_2)}$ because it is smooth on $\tilde{\pi}(\tilde{j}_l\times U^{k+1}_{\tilde{p}_l^\pm})$ and $\tilde{\pi}(\tilde{j}_l\times U^{k+1}_{\tilde{p}_l^-,\tilde{p}_l^+})$, with $\tilde{j}_l,\tilde{p}_l^-,\tilde{p}_l^+$ as in (3) above, and all its derivatives in the directions normal to

$$\tilde{\pi}\big(\tilde{j}_l\times\Delta^{k+1}_{\tilde{p}_l^\pm}\big)\subset\tilde{\pi}\big(\tilde{j}_l\times U^{k+1}_{\tilde{p}_l^\pm}\big) \qquad \text{and} \qquad \tilde{\pi}\big(\tilde{j}_l\times\Delta^{k+1}_{\tilde{p}_l^-,\tilde{p}_l^+}\big)\subset\tilde{\pi}\big(\tilde{j}_l\times U^{k+1}_{\tilde{p}_l^-,\tilde{p}_l^+}\big)$$

vanish by Lemma 2.1. Thus, the restriction of $\tilde{F}$ to $\tilde{M}$ is smooth.

(5) For $i=0,1$, define

$$\begin{gathered}\kappa_i\colon M_i\equiv M_i'-\pi\Big(\bigsqcup_{j=1}^{j=N_i}\{j\}\times Y\Big)\longrightarrow\tilde{M}\equiv\tilde{M}'-\tilde{\pi}\Big(\bigsqcup_{j=1}^{j=\tilde{N}}\{j\}\times\Delta^{k+1}_{\mathcal{E}_j(\tilde{s})}\Big) \qquad \text{by}\\ \kappa_i\big([j,t]\big)=\big[\tilde{j}_i(j),\iota_{k+1,\tilde{p}_i(j)}(\tilde{\tau}_{(i,j)}(t))\big].\end{gathered}$$

To see that this map is well-defined, suppose

$$[j,t]=\big[j_1,\iota_{k,p_1}(t_1)\big]=\big[j_2,\iota_{k,p_2}(t_2)\big] \qquad \text{for some} \quad \big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_{s_i},\; t_1,t_2\in\mathrm{Int}\;\Delta^{k-1}.$$

By definition of the equivalence relation in the proof of Lemma 3.2,

$$t_1=\tau_{i,((j_1,p_1),(j_2,p_2))}(t_2). \tag{3.10}$$

On the other hand, by definition of the equivalence relation in (2) above and (2.4),

$$\begin{aligned}\big[\tilde{j}_{l-1},\iota_{k+1,\tilde{p}_{l-1}^+}(\tilde{\tau}_{(\tilde{j}_{l-1},\tilde{p}_{l-1}^+),(\tilde{j}_l,\tilde{p}_l^-)}\tau_{k,(\tilde{p}_l^-,\tilde{p}_l^+)}(t))\big] &= \big[\tilde{j}_l,\iota_{k+1,\tilde{p}_l^-}(\tau_{k,(\tilde{p}_l^-,\tilde{p}_l^+)}(t))\big]\\ &= \big[\tilde{j}_l,\iota_{k+1,\tilde{p}_l^+}(t)\big] \qquad\qquad \forall\, t\in\Delta^k_{\iota^{-1}_{k+1,\tilde{p}_l^+}(\tilde{p}_l^-)},\end{aligned} \tag{3.11}$$

where $(\tilde{j}_l,\tilde{p}_l^-,\tilde{p}_l^+)$ are as in (3) above. By (3.11) and (2.25),

$$\begin{aligned}\big[\tilde{j}_0,\iota_{k+1,\tilde{p}_0^+}(\tau_{k,(\tilde{p}_0^+,\tilde{p}_0^-)}\iota_{\mathrm{RHS}}(t_2))\big] &= \big[\tilde{j}_r,\iota_{k+1,\tilde{p}_r^+}(\tilde{\tau}_{(i,j_2)}\iota_{k,p_2}(t_2))\big]\\ &= \big[\tilde{j}_i(j_2),\iota_{k+1,\tilde{p}_i(j_2)}(\tilde{\tau}_{(i,j_2)}\iota_{k,p_2}(t_2))\big],\end{aligned} \tag{3.12}$$

where ι_{RHS} denotes the right-hand side of (2.28). Finally, by (2.24), (2.4), (3.10), (2.28), and (3.12),

$$\begin{aligned}\big[\tilde{j}_i(j_1),\iota_{k+1,\tilde{p}_i(j_1)}(\tilde{\tau}_{(i,j_1)}\iota_{k,p_1}(t_1))\big] &= \big[\tilde{j}_0,\iota_{k+1,\tilde{p}_0^-}(\tilde{\tau}_{(i,j_1)}\iota_{k,p_1}(t_1))\big]\\ &= \big[\tilde{j}_0,\iota_{k+1,\tilde{p}_0^+}(\tau_{k,(\tilde{p}_0^+,\tilde{p}_0^-)}\tilde{\tau}_{(i,j_1)}\iota_{k,p_1}\tau_{i,((j_1,p_1),(j_2,p_2))}(t_2))\big]\\ &= \big[\tilde{j}_i(j_2),\iota_{k+1,\tilde{p}_i(j_2)}(\tilde{\tau}_{(i,j_2)}\iota_{k,p_2}(t_2))\big].\end{aligned}$$

Thus, the map κ_i is well-defined.

As can be seen from its restrictions to the individual simplices, the map κ_i is a diffeomorphism onto the union of components of the boundary of $\tilde{M}$ given

$$\bigcup_{(j,p)\in\mathcal{C}_{\tilde{s}}^{(i)}}\tilde{\pi}\big(j\times(\Delta_p^{k+1}-\Delta_{\mathcal{E}_j(\tilde{s})}^{k+1})\big).$$

By the second equation in (2.23), κ_0 is orientation-reversing, while κ_1 is orientation-preserving. Thus,

$$\partial\tilde{M}\approx M_1-M_0,$$

with the isomorphism given by $\kappa_0\sqcup\kappa_1$. Furthermore, by (3.2), $F_i|_{M_i}=\tilde{F}\circ\kappa_i$ for $i=0,1$. Thus,

$$\partial\big(\tilde{F}|_{\tilde{M}}\big)=F_1|_{M_1}-F_0|_{M_0},$$

as claimed.

3.2 From Pseudocycles to Integral Cycles

In this subsection, we prove

Proposition 3.4 *If X is a smooth manifold, there exists a homomorphism*

$$\Phi_*:\mathcal{H}_*(X)\longrightarrow H_*(X;\mathbb{Z}),$$

which is natural with respect to smooth maps.

Lemma 3.5 *Every k-pseudocycle determines a class in $H_k(X;\mathbb{Z})$.*

Proof: (1) Suppose $h\colon M \longrightarrow X$ is a k-pseudocycle and $f\colon N \longrightarrow X$ a smooth map such that

$$\dim N = k-2 \qquad \text{and} \qquad \mathrm{Bd}\, h \subset \mathrm{Im}\, f.$$

By Proposition 2.2, there exists an open neighborhood U of $\mathrm{Bd}\, h$ in X such that

$$H_l(U;\mathbb{Z}) = 0 \qquad \forall l > k-2.$$

Let $K = M - h^{-1}(U)$. Since the closure of $h(M)$ is compact in X, K is a compact subset of M by definition of $\mathrm{Bd}\, h$. Let V be an open neighborhood of K in M such that $\bar{V}$ is a compact manifold with boundary. It inherits an orientation from the orientation of M and thus defines a homology

$$[\bar{V}] \in H_k(\bar{V}, \mathrm{Bd}\, \bar{V};\mathbb{Z}).$$

Put

$$[h] = h_*([\bar{V}]) \in H_k(X,U;\mathbb{Z}) \approx H_k(X;\mathbb{Z}), \tag{3.13}$$

where

$$h_*\colon H_k(\bar{V}, \mathrm{Bd}\, \bar{V};\mathbb{Z}) \longrightarrow H_k(X,U;\mathbb{Z}) \tag{3.14}$$

is the homology homomorphism induced by h. The isomorphism in (3.13) is induced by inclusion. It is an isomorphism by the assumption on the homology of U as follows from the long exact sequence in homology for the pair (X,U).

(2) The homology class $[h]$ is independent of the choice of V. Suppose V' is another choice such that $\bar{V} \subset V'$. Choose a triangulation of $\bar{V}'$ extending some triangulation of $(\mathrm{Bd}\, \bar{V}) \bigcup (\mathrm{Bd}\, \bar{V}')$; such a triangulation exists by Section 16 in [Mu]. The cycles

$$h_*([\bar{V}]), h_*([\bar{V}']) \in H_k(X,U;\mathbb{Z})$$

then differ by singular simplices lying in U; see discussion at the end of Subsection 2.3. Thus,

$$h_*([\bar{V}']) = h_*([\bar{V}]) \in H_k(X,U;\mathbb{Z}).$$

(3) The cycle $[h]$ is also independent of the choice of U. Suppose $U' \subset U$ is another choice. By (2), it can be assumed that V and V' chosen as in (1) are the same. Since the isomorphism in (3.13) is the composite of isomorphisms

$$H_k(X;\mathbb{Z}) \longrightarrow H_k(X,U';\mathbb{Z}) \longrightarrow H_k(X,U;\mathbb{Z})$$

induced by inclusions and the homomorphism (3.14) is the composition

$$H_k(\bar{V}, \mathrm{Bd}\, \bar{V};\mathbb{Z}) \longrightarrow H_k(X,U';\mathbb{Z}) \longrightarrow H_k(X,U;\mathbb{Z}),$$

the homology classes obtained in $H_k(X;\mathbb{Z})$ from U and U' are equal. Finally, if U and U' are two arbitrary choices of open sets in (1), by Proposition 2.2 there exists a third choice $U'' \subset U \cap U'$.

Lemma 3.6 *Equivalent k-pseudocycles determine the same class in $H_k(X,\mathbb{Z})$.*

Proof: Suppose $h_i : M_i \longrightarrow X$, $i=0,1$, are two equivalent k-pseudocycles and $\tilde{h} : \tilde{M} \longrightarrow X$ is an equivalence between them. In particular, $\tilde{M}$ is oriented,

$$\partial \tilde{M} = M_1 - M_0, \qquad \text{and} \qquad \tilde{h}|_{M_i} = h_i.$$

Let $\tilde{U}$ be an open neighborhood of $\operatorname{Bd} \tilde{h}$ in X such that

$$H_l(\tilde{U}; \mathbb{Z}) = 0 \qquad \forall\, l > k-1.$$

Let U_i be an open neighborhood of $\operatorname{Bd} h_i \subset \operatorname{Bd} \tilde{h}$ in $\tilde{U}$ such that

$$H_l(U_i; \mathbb{Z}) = 0 \qquad \forall\, l > k-2,$$

as provided by Proposition 2.2. Let $V_i \subset M_i$ be a choice of an open set as in (1) of the proof of Lemma 3.5. For $i=0,1$, choose a triangulation of M_i that extends a triangulation of $\operatorname{Bd} \bar{V}_i$. Extend these two triangulations to a triangulation $\tilde{T} = (\tilde{K}, \tilde{\eta})$ of $\tilde{M}$. Let K be a finite sub-complex of $\tilde{K}$ such that

$$V_0, V_1 \subset \tilde{\eta}(|K|) \qquad \text{and} \qquad \tilde{M} - \tilde{h}^{-1}(\tilde{U}) \subset \tilde{\eta}(\operatorname{Int} |K|).$$

Such a subcomplex exists because $\tilde{h}(\tilde{M})$ is a pre-compact subset of X and thus $\tilde{M} - \tilde{h}^{-1}(\tilde{U})$ is a compact subset of $\tilde{M}$. Put

$$K_i = \left\{ \sigma \in K : \eta(\sigma) \subset \bar{V}_i \right\} \qquad \text{for} \;\; i = 0, 1.$$

By the proof of Lemma 3.5, $(K_i, \tilde{h} \circ \tilde{\eta}|_{|K_i|})$ determines the homology class $[h_i] \in H_k(X, U_i; \mathbb{Z})$. Let $[h_i']$ denote its image in $H_k(X, \tilde{U}; \mathbb{Z})$ under the homomorphism induced by inclusion. The above assumptions on K imply that

$$\partial (K, \tilde{h} \circ \tilde{\eta}|_K) = (K_1, \tilde{h} \circ \tilde{\eta}|_{K_1}) - (K_0, \tilde{h} \circ \tilde{\eta}|_{K_0})$$

in $\bar{S}(M, \tilde{U})$. Thus,

$$[h_0'] = [h_1'] \in H_k(X, \tilde{U}; \mathbb{Z}),$$

and this class lies in the image of the homomorphism

$$H_k(X; \mathbb{Z}) \longrightarrow H_k(X, \tilde{U}; \mathbb{Z}) \tag{3.15}$$

induced by inclusion. This map is equal to the composites

$$H_k(X; \mathbb{Z}) \longrightarrow H_k(X, U_0; \mathbb{Z}) \longrightarrow H_k(X, \tilde{U}; \mathbb{Z}) \qquad \text{and}$$
$$H_k(X; \mathbb{Z}) \longrightarrow H_k(X, U_1; \mathbb{Z}) \longrightarrow H_k(X, \tilde{U}; \mathbb{Z}).$$

Since $H_k(\tilde{U}; \mathbb{Z}) = 0$, the homomorphism (3.15) is injective. Thus, $[h_0]$ and $[h_1]$ come from the same element of $H_k(X; \mathbb{Z})$.

3.3 Isomorphism of Homology Theories

In this subsection we conclude the proof of Theorem 1.1.

Lemma 3.7 *If X is a smooth manifold, the composition*

$$\Phi_* \circ \Psi_* \colon H_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})$$

is the identity map on $H_*(X;\mathbb{Z})$.

Proof: Suppose

$$\{s\} = \sum_{j=1}^{N} \{f_j\} \in \bar{S}_k(X)$$

is a cycle and $F\colon M \longrightarrow X$ is a pseudocycle corresponding to s via the construction of Lemma 3.2. Recall that M is the complement of the $(k-2)$-simplices in a compact space M' and F is the restriction of a continuous map $F'\colon M' \longrightarrow X$ induced by the maps

$$f_j \circ \varphi_k \colon \Delta^k \longrightarrow X, \qquad j = 1, \ldots, N.$$

Since φ_k is homotopic to the identity on Δ^k, with boundary fixed,

$$f_j \circ \varphi_k - f_j \in \partial S_{k+1}(X) \qquad \forall\, j = 1, \ldots, N. \tag{3.16}$$

Let U be a neighborhood of $\mathrm{Bd}\, F$ such that

$$H_l(U;\mathbb{Z}) = 0 \qquad \forall\, l > k-2.$$

Put $K = M - f^{-1}(U)$. Let V be a pre-compact neighborhood of K such that $(\bar{V}, \partial\bar{V})$ is a smooth manifold with boundary. Choose a triangulation $T = (K, \eta)$ of $(\bar{V}, \partial\bar{V})$ such that every k-simplex of T is contained in a set of the form $\pi(\{j\} \times \Delta^k)$, where π is as in the proof of Lemma 3.2. Put

$$K_j = \big\{\sigma \in K \colon \eta(\sigma) \subset \pi(\{j\} \times \Delta^k)\big\}, \qquad K_j^{\mathrm{top}} = \big\{\sigma \in K_j \colon \dim\sigma = k\big\}.$$

Let $\tilde{T}_j = (\tilde{K}_j, \eta_j)$ be a triangulation of a subset of Δ^k that along with K_j gives a triangulation of Δ^k. Put

$$\tilde{K}_j^{\mathrm{top}} = \big\{\sigma \in \tilde{K}_j \colon \dim\sigma = k\big\}.$$

By definition of T,

$$f_j \circ \varphi_k\big(\eta_j(\sigma)\big) \subset U \qquad \forall\, \sigma \in \tilde{K}_j^{\mathrm{top}}. \tag{3.17}$$

Furthermore, by (3.16)

$$\begin{aligned} \{s\} &= \sum_{\sigma \in K^{\mathrm{top}}} \{f_j \circ \varphi_k \circ \eta \circ l_\sigma\} \\ &= \sum_{j=1}^{N} \sum_{\sigma \in K_j^{\mathrm{top}}} \{f_j \circ \varphi_k \circ \eta \circ l_\sigma\} + \sum_{j=1}^{N} \sum_{\sigma \in \tilde{K}_j^{\mathrm{top}}} \{f_j \circ \varphi_k \circ \tilde{\eta}_j \circ l_\sigma\} \qquad \mathrm{mod}\ \partial\bar{S}_{k+1}(X), \end{aligned} \tag{3.18}$$

since subdivisions of cycles do not change the homology class. By the proof of Lemma 3.5, the first sum on the right-hand side of (3.18) represents $[F]$ in $\bar{S}_k(X, U)$. By (3.17), the second sum lies in $\bar{S}_k(U)$. Since the sum of the two terms is a cycle in $\bar{S}_k(X)$, it must represent $[F]$ in $\bar{S}_k(X)$. Thus,

$$\{F\} = \{s\} \in H_k(X;\mathbb{Z}),$$

and the claim follows.

Lemma 3.8 *If X is a smooth manifold, the homomorphism $\Phi_*\colon \mathcal{H}_*(X)\longrightarrow H_*(X;\mathbb{Z})$ is injective.*

Proof: (1) Suppose a k-pseudocycle $h: M'\longrightarrow X$ determines the zero homology class. It can be assumed that $k\ge 1$; otherwise, there is nothing to prove. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of open pre-compact neighborhoods of $\mathrm{Bd}\, h$ in X such that

$$U_{i+1}\subset U_i, \qquad \bigcap_{i=1}^{\infty} U_i=\mathrm{Bd}\, h, \quad \text{and} \quad H_l(U_i;\mathbb{Z})=0 \quad \forall\, l>k-2.$$

Existence of such a collection follows from Proposition 2.2 and metrizability of any manifold. Let $\{V_i\}_{i=1}^{\infty}$ be a corresponding collection of open sets in M' as in (1) of the proof of Lemma 3.5. It can be assumed that $\bar{V}_i\subset V_{i+1}$. Choose a triangulation $T=(K,\eta)$ of M' that extends a triangulation of $\bigcup\limits_{i=1}^{\infty}\mathrm{Bd}\,\bar{V}_i$. Let

$$K^{\mathrm{top}}=\big\{\sigma\in K\colon \dim\sigma=k\big\}, \qquad \mathcal{C}_\eta=\big\{(\sigma,p)\colon \sigma\in K^{\mathrm{top}},\ p=0,1,\ldots,k\big\}.$$

For each $\sigma\in K^{\mathrm{top}}$, let

$$l_\sigma\colon \Delta^k\longrightarrow \sigma\subset |K|\subset \mathbb{R}^\infty$$

be a linear map such that $\eta\circ l_\sigma$ is orientation-preserving. Put

$$f_\sigma=h\circ\eta\circ l_\sigma \qquad \forall\, \sigma\in K^{\mathrm{top}} \qquad \text{and}$$
$$\mathcal{D}_\eta=\big\{((\sigma_1,p_1),(\sigma_2,p_2))\in\mathcal{C}_\eta\times\mathcal{C}_\eta : (\sigma_1,p_1)\neq(\sigma_2,p_2),\ l_{\sigma_1}(\Delta^k_{p_1})=l_{\sigma_2}(\Delta^k_{p_2})\big\}.$$

For each $((\sigma_1,p_1),(\sigma_2,p_2))\in\mathcal{D}_\eta$, define

$$\tau_{(\sigma_1,p_1),(\sigma_2,p_2)}\in\mathcal{S}_{k-1} \qquad \text{by} \qquad l_{\sigma_2}\circ\iota_{k,p_2}=l_{\sigma_1}\circ\iota_{k,p_1}\circ\tau_{(\sigma_1,p_1),(\sigma_2,p_2)}.$$

Since K is an oriented simplicial complex,

$$\mathcal{D}_\eta\subset\mathcal{C}_\eta\times\mathcal{C}_\eta \qquad \text{and} \qquad \tau\colon \mathcal{D}_\eta\longrightarrow\mathcal{S}_{k-1}$$

satisfy (i)-(iii) of Lemma 2.10. Furthermore, M' is the topological space corresponding to $(\mathcal{C}_\eta,\mathcal{D}_\eta,\tau)$ via the construction of Lemma 3.2 and h is the continuous map described by

$$h|_{\pi(\sigma\times\Delta^k)}=f_\sigma.$$

As in the proof of Lemma 3.2, let M be the complement of the $(k-2)$-simplices in M'; the pseudocycles h and $h|_M$ are equivalent. Since φ_k is homotopic to the identity on Δ^k with boundary fixed, the pseudocycle $h|_M$ is in turn equivalent to the pseudocycle $F|_M$, where as in the proof of Lemma 3.2

$$F\colon M'\longrightarrow X, \qquad F\circ\eta\circ l_\sigma=f_\sigma\circ\varphi_k.$$

(2) For each $i\ge 1$, let

$$K_i^{\mathrm{top}}=\big\{\sigma\in K^{\mathrm{top}}\colon \eta(\sigma)\subset\bar{V}_i\big\}, \qquad \mathcal{C}_{\eta;i}=\big\{(\sigma,p)\in\mathcal{C}_\eta : \sigma\in K_i^{\mathrm{top}}\big\}, \quad \text{and} \quad \mathcal{D}_{\eta;i}=\mathcal{D}_\eta\cap(\mathcal{C}_{\eta;i}\times\mathcal{C}_{\eta;i}).$$

By construction of $[h]$, for every $i\geq 1$, there exists a singular chain

$$s_i \equiv \sum_{j=1}^{N_i} f_{i,j} \in S_k(U_i)$$

such that

$$\sum_{\sigma\in K_i^{\rm top}} \{h\circ\eta\circ l_\sigma\} + \{s_i\}$$

is a cycle in $\bar{S}_k(X)$ representing $[h]$. Similarly to Lemma 2.10, there exist a symmetric subset

$$\mathcal{D}_i \subset (\mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i})\times(\mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i})$$

disjoint from the diagonal and a map

$$\tau_i\colon \mathcal{D}_i \longrightarrow \mathcal{S}_{k-1}$$

such that

(i) $\mathcal{D}_{\eta;i}\subset\mathcal{D}_i$ and $\tau_i|_{\mathcal{D}_{\eta;i}}=\tau|_{\mathcal{D}_{\eta;i}}$;

(ii) the projection map $\mathcal{D}_i \longrightarrow \mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i}$ on either coordinate is a bijection;

(iii) for all $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_i$,

$$\tau_{(j_2,p_2),(j_1,p_1)} = \tau^{-1}_{(j_1,p_1),(j_2,p_2)}, \qquad f_{i,j_2}\circ\iota_{k,p_2} = f_{i,j_1}\circ\iota_{k,p_1}\circ\tau_{(j_1,p_1),(j_2,p_2)},$$
$$\text{and} \qquad \operatorname{sign}\tau_{(j_1,p_1),(j_2,p_2)} = -(-1)^{p_1+p_2},$$

where $f_{i,\sigma}\equiv f_\sigma$ for all $\sigma\in K_i^{\rm top}$.

(3) By (2), for each $i\geq 2$

$$\sum_{\sigma\in K_i^{\rm top}-K_{i-1}^{\rm top}} \{h\circ\eta\circ l_\sigma\} + \{s_i\} - \{s_{i-1}\} \in \bar{S}_k(U_{i-1})$$

is a cycle. Since $H_k(U_{i-1};\mathbb{Z})=0$, it must be a boundary. If $i=1$, this conclusion is still true with $U_0=X$, $K_0^{\rm top}=\emptyset$, and $s_0=0$, since $[h]=0$ by assumption. Therefore, similarly to Lemma 2.11, there exist

$$\tilde{s}_i \equiv \sum_{j=1}^{\tilde{N}_i} \tilde{f}_{i,j} \in S_{k+1}(U_{i-1}), \qquad \tilde{\mathcal{C}}_i^{(0)} \subset \tilde{\mathcal{C}}_i \equiv \bigsqcup_{i'=1}^{i'=i} \mathcal{C}_{\tilde{s}_{i'}},$$

a symmetric subset $\tilde{\mathcal{D}}_i\subset\tilde{\mathcal{C}}_i\times\tilde{\mathcal{C}}_i$ disjoint from the diagonal, and maps

$$\tilde{\tau}_i\colon \tilde{\mathcal{D}}_i \longrightarrow \mathcal{S}_k, \quad \big((j_1,p_1),(j_2,p_2)\big) \longrightarrow \tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))},$$
$$(\tilde{j}_i,\tilde{p}_i)\colon K_i^{\rm top}\sqcup\{1,\ldots,N_i\} \longrightarrow \tilde{\mathcal{C}}_i^{(0)}, \quad \text{and} \quad \tilde{\tau}_i\colon K_i^{\rm top}\sqcup\{1,\ldots,N_i\} \longrightarrow \mathcal{S}_k, \quad j \longrightarrow \tilde{\tau}_{(i,j)},$$

such that

(i) $\tilde{\mathcal{D}}_i\subset\tilde{\mathcal{D}}_{i+1}$, $\tilde{\tau}_{i+1}|_{\tilde{\mathcal{D}}_i}=\tilde{\tau}_i$, and $(\tilde{j}_{i+1},\tilde{p}_{i+1},\tilde{\tau}_{i+1})|_{K_i^{\rm top}} = (\tilde{j}_i,\tilde{p}_i,\tilde{\tau}_i)|_{K_i^{\rm top}}$;

(ii) the projection $\tilde{\mathcal{D}}_i \longrightarrow \tilde{\mathcal{C}}_i$ on either coordinate is a bijection onto the complement of $\tilde{\mathcal{C}}_i^{(0)}$;

(iii) for all $((j_1,p_1),(j_2,p_2)) \in \tilde{\mathcal{D}}_i \cap (\mathcal{C}_{\tilde{s}_{i_1}} \times \mathcal{C}_{\tilde{s}_{i_2}})$,

$$\tilde{\tau}_{i,((j_2,p_2),(j_1,p_1))} = \tilde{\tau}^{-1}_{i,((j_1,p_1),(j_2,p_2))}, \qquad \tilde{f}_{i_2,j_2} \circ \iota_{k+1,p_2} = \tilde{f}_{i_1,j_1} \circ \iota_{k+1,p_1} \circ \tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))},$$
$$\text{and} \qquad \operatorname{sign} \tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))} = -(-1)^{p_1+p_2};$$

(iv) for all $\sigma \in K_i^{\mathrm{top}} - K_{i-1}^{\mathrm{top}}$,

$$\tilde{f}_{i,\tilde{j}_i(j)} \circ \iota_{k+1,\tilde{p}_i(j)} \circ \tilde{\tau}_{(i,j)} = f_\sigma \qquad \text{and} \qquad \operatorname{sign} \tilde{\tau}_{(i,j)} = -(-1)^{\tilde{p}_i(j)};$$

(v) $(\tilde{j}_i, \tilde{p}_i)$ is a bijection onto $\tilde{\mathcal{C}}_i^{(0)}$ and $\tilde{j}_i|_{K_i^{\mathrm{top}} - K_{i-1}^{\mathrm{top}}}$ is injective into $\{1,\ldots,\tilde{N}_i\}$;

(vi) for all $((j_1,p_1),(j_2,p_2)) \in \mathcal{D}_{\eta;i}$, there exist

$$\begin{gathered}
(\tilde{j}_0, \tilde{p}_0^-), (\tilde{j}_0, \tilde{p}_0^+), \ldots, (\tilde{j}_r, \tilde{p}_r^-), (\tilde{j}_r, \tilde{p}_r^+) \in \tilde{\mathcal{C}}_i \qquad \text{s.t.} \\
(\tilde{j}_0, \tilde{p}_0^-, \tilde{p}_0^+) = \big(\tilde{j}_i(j_1), \tilde{p}_i(j_1), \iota_{k+1,\tilde{p}_0^-} \tilde{\tau}_{(i,j_1)}(p_1)\big), \\
(\tilde{j}_r, \tilde{p}_r^-, \tilde{p}_r^+) = \big(\tilde{j}_i(j_2), \iota_{k+1,\tilde{p}_r^+} \tilde{\tau}_{(i,j_2)}(p_2), \tilde{p}_i(j_2)\big), \\
\big((\tilde{j}_{r'-1}, \tilde{p}_{r'-1}^+), (\tilde{j}_{r'}, \tilde{p}_{r'}^-)\big) \in \tilde{\mathcal{D}}_i \qquad \forall\, r' = 1,\ldots,r, \\
\iota^{-1}_{k+1,\tilde{p}_{r'-1}^+}(\tilde{p}_{r'-1}^-) = \tilde{\tau}_{i,((\tilde{j}_{r'-1},\tilde{p}_{r'-1}^+),(\tilde{j}_{r'},\tilde{p}_{r'}^-))} \iota^{-1}_{k+1,\tilde{p}_{r'}^-}(\tilde{p}_{r'}^+) \qquad \forall\, r' = 1,\ldots,r,
\end{gathered}$$

$$\begin{aligned}
&\tilde{\tau}_{(i,j_1)} \iota_{k,p_1} \tau_{i,((j_1,p_1),(j_2,p_2))} \\
&\quad = \big(\tau_{k,(\tilde{p}_0^-,\tilde{p}_0^+)} \tilde{\tau}_{i,((\tilde{j}_0,\tilde{p}_0^+),(\tilde{j}_1,\tilde{p}_1^-))}\big) \ldots \big(\tau_{k,(\tilde{p}_{r-1}^-,\tilde{p}_{r-1}^+)} \tilde{\tau}_{i,((\tilde{j}_{r-1},\tilde{p}_{r-1}^+),(\tilde{j}_r,\tilde{p}_r^-))}\big) \big(\tau_{k,(\tilde{p}_r^-,\tilde{p}_r^+)} \tilde{\tau}_{(i,j_2)}\big) \iota_{k,p_2}.
\end{aligned}$$

For each $i \geq 1$ and $j = 1,\ldots,\tilde{N}_i$, denote by

$$\mathcal{E}_{i,j} \subset \big\{0,1,\ldots,k+1\big\} \times \big\{0,1,\ldots,k+1\big\}$$

the set of pairs (p^-,p^+) such that

$$\big(j,p^-,p^+\big) = \big(\tilde{j}_{r'}, \tilde{p}_{r'}^-, \tilde{p}_{r'}^+\big)$$

for the sequence of (vi) above beginning with some $(j_1,p_1) \in \mathcal{C}_\eta$ and some r'. Let

$$\Delta^{k+1}_{\mathcal{E}_{i,j}} \subset \Delta^{k+1}$$

be the union of the $(k-2)$-skeleton of Δ^{k+1} with all $(k-1)$-simplices in Δ^{k+1} that are *not* of the form $\Delta^{k+1}_{p^-,p^+}$ for some $(p^-,p^+) \in \mathcal{E}_{i,j}$.

(4) Put

$$\tilde{M}' = \Big(\bigsqcup_{i=1}^{\infty} \bigsqcup_{j=1}^{\tilde{N}_i} \{i\} \times \{j\} \times \Delta^{k+1}\Big) \Big/ \sim, \qquad \text{where}$$

$$\big(i_1, j_1, \iota_{k,p_1}(\tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))}(t))\big) \sim \big(i_2, j_2, \iota_{k,p_2}(t)\big) \quad \forall\ ((j_1,p_1),(j_2,p_2)) \in \tilde{\mathcal{D}}_i \cap (\mathcal{C}_{\tilde{s}_{i_1}} \times \mathcal{C}_{\tilde{s}_{i_2}}),\ t \in \Delta^k.$$

Let

$$\tilde{\pi}\colon \bigsqcup_{i=1}^{\infty} \bigsqcup_{j=1}^{\tilde{N}_i} \{i\} \times \{j\} \times \Delta^{k+1} \longrightarrow \tilde{M}'$$

be the quotient map. Define

$$\tilde{F}\colon \tilde{M}' \longrightarrow X \qquad \text{by} \qquad \tilde{F}\big([i,j,t]\big) = \tilde{f}_{i,j}\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(t))\big),$$

where $\tilde{\varphi}_{k+1}$ and φ_{k+1} are the self-maps of Δ^{k+1} provided by Lemma 2.1. Similarly to the proof of Lemma 3.3, this map is well-defined and continuous. Since the image of

$$\bigsqcup_{i=2}^{\infty}\bigsqcup_{j=1}^{\tilde{N}_i}\{i\}\times\{j\}\times\Delta^{k+1}$$

under $\tilde{F}$ is contained in the pre-compact subset U_1 of X, $\tilde{F}(\tilde{M}')$ is a pre-compact subset of X as well.

Let $\tilde{M}$ be the complement in $\tilde{M}'$ of the set

$$\tilde{\pi}\Big(\bigsqcup_{i=1}^{\infty}\bigsqcup_{j=1}^{j=\tilde{N}_i}\{j\}\times\Delta^{k+1}_{\mathcal{E}_{i,j}}\Big).$$

Similarly to the proof of Lemma 3.3, $\tilde{M}$ is a smooth manifold, $\tilde{F}|_{\tilde{M}}$ is a smooth map, and $\operatorname{Bd}\tilde{F}|_{\tilde{M}}$ is of dimension at most $k-1$. Furthermore, there is a well-defined map

$$\kappa_0\colon M \longrightarrow \tilde{M}$$

which is an orientation-reversing diffeomorphism onto $\partial\tilde{M}$ such that

$$F|_M = \tilde{F}\circ\kappa_0.$$

Thus,

$$\partial\big(\tilde{F}|_{\tilde{M}}\big) = -F|_M,$$

i.e. $F|_M$ and h represent the zero element in $\mathcal{H}_k(M)$.

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