On the Geometry of Genus 1 Gromov-Witten Invariants

Aleksey Zinger

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From String Theory to Enumerative Geometry

A-Model partition function for Calabi-Yau 3-fold $X$ \[ \overset{\text{MIRROR}}{\text{principle}} \] B-Model partition function for mirror (family) of $X$

Generating function for GWs
"counts of complex curves in" $X$

Something about geometry of moduli spaces of CYs
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“Simplest” Calabi-Yau 3-fold

- quintic 3-fold $X_5 = \text{degree 5 hypersurface in } \mathbb{P}^4$
- expected # of genus $g$ degree $d$ curves is finite: $n_{g,d}$
- genus $g$ degree $d$ GW-invariant $N_{g,d}$ is made up of $n_{h,d}$
- A-model partition function:
  \[ F^A_g(q) = \sum_{d=1}^{\infty} N_{g,d} q^d. \]
- B-model partition function $F^B_g$ “measures” geometry of moduli spaces of CYs
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B-Side Computations

- Candelas-de la Ossa-Green-Parkes’91 construct mirror family, compute $F^B_0$
- Bershadsky-Cecotti-Ooguri-Vafa’93 (BCOV) compute $F^B_1$ using physics arguments
- Fang-Z. Lu-Yoshikawa’03 compute $F^B_1$ mathematically
- Huang-Klemm-Quackenbush’06 compute $F^B_g$, $g \leq 52$ using physics
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Mirror Symmetry Predictions and Verifications

Predictions

\[ F_g^A(q) \equiv \sum_{d=1}^{\infty} N_{g,d} q^d \Rightarrow F_g^B(q). \]

Theorem (Givental’96, Lian-Liu-Yau’97,...........~’00)
\( g = 0 \) predict. of Candelas-de la Ossa-Green-Parkes’91 holds

Theorem (Z.’07)
\( g = 1 \) predict. of Bershadsky-Cecotti-Ooguri-Vafa’93 holds
## Mirror Symmetry Predictions and Verifications

### Predictions

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General Approach to Verifying $F^A_g = F^B_g$
(works for $g = 0, 1$)

Need to compute each $N_{g,d}$ and all of them (for fixed $g$):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{P}^4 \supset X_5$

Step 2: use $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$ to compute each $N_{g,d}$ by localization

Step 3: find some recursive feature(s) to compute $N_{g,d} \forall d$

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$\iff F^A_g$
GW-Invariants of $X_5 \subset \mathbb{P}^4$

$\overline{M}_g(X_5, d) = \{ [u: \Sigma \to X_5] | \ g(\Sigma) = g, \deg u = d, \bar{\partial}u = 0 \}$

$N_{g,d} \equiv \deg [\overline{M}_g(X_5, d)]^{\text{vir}}$

$\equiv \# \{ [u: \Sigma \to X_5] | \ g(\Sigma) = g, \deg u = d, \bar{\partial}u = \nu(u) \}$

$\nu = \text{small generic deformation of } \bar{\partial}\text{-equation}$
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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4$

$\mathcal{L}$  

$\overline{M}_g(\mathcal{L}, d)$  

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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4$

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\[ L \equiv \mathcal{O}(5) \]

\[ V_{g,d} \equiv \overline{M}_g(L, d) \]

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$V_{g,d} \equiv \overline{M}_g(L, d)$

$\tilde{s} \uparrow \downarrow \tilde{\pi}$

$\tilde{\pi}([\xi: \Sigma \to L]) = [\pi \circ \xi: \Sigma \to \mathbb{P}^4]$  

$\tilde{s}([u: \Sigma \to \mathbb{P}^4]) = [s \circ u: \Sigma \to L]$
From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

\[
\begin{align*}
\mathcal{L} & \equiv \mathcal{O}(5) \\
\mathcal{V}_{g,d} & \equiv \overline{M}_g(\mathcal{L}, d) \\
X_5 \equiv s^{-1}(0) & \subset \mathbb{P}^4 \\
\overline{M}_g(X_5, d) & \equiv \tilde{s}^{-1}(0) \subset \overline{M}_g(\mathbb{P}^4, d)
\end{align*}
\]

\[
\pi \downarrow \\
\begin{array}{c}
X_5 \\
\end{array} \\
\downarrow \\
\begin{array}{c}
\overline{M}_g(X_5, d) \\
\end{array}
\]

\[
\tilde{\pi} ([\xi : \Sigma \to \mathcal{L}]) = [\pi \circ \xi : \Sigma \to \mathbb{P}^4] \\
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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

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\begin{align*}
\mathcal{L} & \equiv \mathcal{O}(5) \\
\nu_{g,d} & \equiv \overline{M}_g(\mathcal{L}, d) \\
X_5 & \equiv s^{-1}(0) \subset \mathbb{P}^4 \\
\overline{M}_g(X_5, d) & \equiv \bar{s}^{-1}(0) \subset \overline{M}_g(\mathbb{P}^4, d)
\end{align*}
\]

This suggests: **Hyperplane Property**

\[
N_{g,d} \equiv \deg \left[ \overline{M}_g(X_5, d) \right]^{vir} \equiv \pm |\bar{s}^{-1}(0)|
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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

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\[ N_{g,d} \equiv \deg [\overline{M}_g(X_5, d)]^{\text{vir}} \equiv \pm |\tilde{s}^{-1}(0)| \]

\[ ? \equiv \langle e(\mathcal{V}_{g,d}), \overline{M}_g(\mathbb{P}^4, d) \rangle \]
Genus 0 vs. Positive Genus

$g = 0$ everything is as expected:

- $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ is smooth
- $[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)]^{vir} = [\overline{\mathcal{M}}_g(\mathbb{P}^4, d)]$
- $\mathcal{N}_{0,d} \rightarrow \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ is vector bundle
- hyperplane prop. makes sense and holds

$g \geq 1$ none of these holds
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Genus 1 Analogue

Thm. A (J. Li–Z.'04): HP holds for reduced genus 1 GWs

\[ [\overline{M}_1(X_5, d)]^{vir} = e(\nu_{1,d}) \cap \overline{M}_1^0(\mathbb{P}^4, d). \]

This generalizes to complete intersections \( X \subset \mathbb{P}^n \).

- \( \overline{M}_1^0(\mathbb{P}^4, d) \subset \overline{M}_1(\mathbb{P}^4, d) \) main irred. component
  closure of \( \{ [u: \Sigma \rightarrow \mathbb{P}^4] \in \overline{M}_1(\mathbb{P}^4, d): \Sigma \text{ is smooth} \} \)

- \( \nu_{1,d} \rightarrow \overline{M}_1^0(\mathbb{P}^4, d) \) not vector bundle, but
  \( e(\nu_{1,d}) \) well-defined (0-set of generic section)
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\overline{M}_1(X_5, d) \overset{\text{vir}}{=} e(V_1, d) \cap \overline{M}_1^0(\mathbb{P}^4, d).
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Standard vs. Reduced GWs

Thm. A \implies N^0_{1,d} \equiv \deg \overline{\mathcal{M}}^0_1(X,d)^{\text{vir}} = \int_{\overline{\mathcal{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})

\overline{\mathcal{M}}^0_1(X,d) \equiv \overline{\mathcal{M}}^0_1(\mathbb{P}^4,d) \cap \overline{\mathcal{M}}^0_1(X,d)

Thm. B (Z.'04,'07): N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}

This generalizes to all symplectic manifolds:

[standard] − [reduced genus 1 GW] = f(\text{genus 0 GW})

\therefore to check BCOV, enough to compute \int_{\overline{\mathcal{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})
Standard vs. Reduced GWs

Thm. A \implies N_{1,d}^0 \equiv \deg \left[ \mathcal{M}_1^0(X,d) \right]^{\text{vir}} = \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4,d)} e(\mathcal{N}_1,d)

\overline{\mathcal{M}}_1^0(X,d) \equiv \overline{\mathcal{M}}_1^0(\mathbb{P}^4,d) \cap \overline{\mathcal{M}}_1(X,d)

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Standard vs. Reduced GWs

Thm. A \[\Rightarrow \quad N_{1,d}^0 \equiv \text{deg} [\overline{\mathcal{M}}_{1}(X, d)]^{vir} = \int_{\overline{\mathcal{M}}_{1}(\mathbb{P}^4, d)} e(\nu_{1,d})\]

\[\overline{\mathcal{M}}_{1}(X, d) \equiv \overline{\mathcal{M}}_{1}(\mathbb{P}^4, d) \cap \overline{\mathcal{M}}_{1}(X, d)\]

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∴ to check BCOV, enough to compute \[\int_{\overline{\mathcal{M}}_{1}(\mathbb{P}^4, d)} e(\nu_{1,d})\]
Standard vs. Reduced GWs

Thm. A $\Rightarrow N_{1,d}^0 \equiv \deg \overline{\mathcal{M}}_1(X, d)^{\text{vir}} = \int_{\overline{\mathcal{M}}_1(\mathbb{P}^4, d)} e(\mathcal{V}_{1,d})$

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Thm. B (Z.’04,’07): $N_{1,d} - N_{1,d}^0 = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

[standard] $-$ [reduced genus 1 GW] = $f(\text{genus 0 GW})$

$\therefore$ to check BCOV, enough to compute $\int_{\overline{\mathcal{M}}_1(\mathbb{P}^4, d)} e(\mathcal{V}_{1,d})$
Standard vs. Reduced GWs

Thm. A \[\implies\] \( N_{1,d}^0 \equiv \deg [\overline{\mathcal{M}}_1^0(X,d)]^{\text{vir}} = \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4,d)} e(\mathcal{V}_1,d) \)

\[\overline{\mathcal{M}}_1^0(X,d) \equiv \overline{\mathcal{M}}_1^0(\mathbb{P}^4,d) \cap \overline{\mathcal{M}}_1(X,d)\]

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\( \therefore \) to check BCOV, enough to compute \( \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4,d)} e(\mathcal{V}_1,d) \)
Standard vs. Reduced GWs

Thm. A \[ \implies N^0_{1,d} \equiv \deg \left[ \overline{\mathcal{M}}^0_1 (X, d) \right]^{vir} = \int_{\overline{\mathcal{M}}^0_1 (\mathbb{P}^4, d)} e(\mathcal{V}_1, d) \]

\[ \overline{\mathcal{M}}^0_1 (X, d) \equiv \overline{\mathcal{M}}^0_1 (\mathbb{P}^4, d) \cap \overline{\mathcal{M}}_1 (X, d) \]

Thm. B (Z.’04,’07): \[ N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d} \]

This generalizes to all symplectic manifolds:

[standard] – [reduced genus 1 GW] = f(genus 0 GW)

\[ \therefore \text{to check BCOV, enough to compute} \int_{\overline{\mathcal{M}}^0_1 (\mathbb{P}^4, d)} e(\mathcal{V}_1, d) \]
Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\implies$ on $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ (with simple fixed loci) and on $\mathcal{V}_{g,d} \to \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$
- $\int_{\overline{\mathcal{M}}_g^0(\mathbb{P}^4, d)} \mathcal{E}(\mathcal{V}_{g,d})$ localizes to fixed loci

$g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{\text{graphs}}$

$g = 1$: $\overline{\mathcal{M}}_g^0(\mathbb{P}^4, d), \mathcal{V}_{g,d}$ singular $\implies$ AB does not apply
Torus Actions

- \((\mathbb{C}^*)^5\) acts on \(\mathbb{P}^4\) (with 5 fixed pts)
- \(\longrightarrow\) on \(\overline{M}_g(\mathbb{P}^4, d)\) (with simple fixed loci)
  
  and on \(V_{g,d} \longrightarrow M_g(\mathbb{P}^4, d)\)

\[\int_{\overline{M}_g(\mathbb{P}^4, d)} e(V_{g,d})\] localizes to fixed loci

- \(g = 0: \) Atiyah-Bott Localization Thm reduces \(\int\) to \(\sum_{\text{graphs}}\)
- \(g = 1: \) \(\overline{M}_g(\mathbb{P}^4, d), V_{g,d}\) singular \(\longrightarrow\) AB does not apply
Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\longrightarrow$ on $\overline{M}_g(\mathbb{P}^4, d)$ (with simple fixed loci) and on $\mathcal{V}_{g,d} \longrightarrow \overline{M}_g(\mathbb{P}^4, d)$
- $\int_{\overline{M}_0^g(\mathbb{P}^4, d)} e(\mathcal{V}_{g,d})$ localizes to fixed loci

$g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{\text{graphs}}$

$g = 1$: $\overline{M}_0^g(\mathbb{P}^4, d)$, $\mathcal{V}_{g,d}$ singular $\longrightarrow$ AB does not apply
Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\Rightarrow$ on $\overline{M}_g(\mathbb{P}^4, d)$ (with simple fixed loci) and on $\mathcal{N}_{g,d} \rightarrow \overline{M}_g(\mathbb{P}^4, d)$
- $\int_{\overline{M}_g(\mathbb{P}^4, d)} e(\mathcal{N}_{g,d})$ localizes to fixed loci

- $g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{\text{graphs}}$
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Torus Actions

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- on $\overline{M}_g(\mathbb{P}^4, d)$ (with simple fixed loci)
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Torus Actions

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- $g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{\text{graphs}}$ graphs
- $g = 1$: $\overline{M}_g(\mathbb{P}^4, d), \mathcal{V}_{g,d}$ singular $\Rightarrow$ AB does not apply
Thm. C (Vakil–Z.’05): $\mathcal{V}_{1,d} \to \overline{M}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

$$\tilde{\mathcal{V}}_{1,d} \to \mathcal{V}_{1,d}$$

$$\overline{\mathcal{M}}_1^0(\mathbb{P}^4, d') \to \overline{\mathcal{M}}_1^0(\mathbb{P}^4, d)$$

$$\int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4, d)} e(\mathcal{V}_{1,d}) = \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4, d')} e(\tilde{\mathcal{V}}_{1,d})$$
Thm. C (Vakil–Z.’05): $\mathcal{V}_{1,d} \rightarrow \overline{\mathcal{M}}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

$$\tilde{\mathcal{V}}_{1,d} \rightarrow \mathcal{V}_{1,d}$$

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$$\Rightarrow \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4, d)} e(\mathcal{V}_{1,d}) = \int_{\overline{\mathcal{M}}_1^0(\mathbb{P}^4, d)} e(\tilde{\mathcal{V}}_{1,d})$$
Genus 1 Bypass

Thm. C (Vakil–Z.’05): $\mathcal{V}_{1,d} \rightarrow \overline{M}_{1}^{0}(\mathbb{P}^{4}, d)$ admit natural desingularizations:

$$
\tilde{\mathcal{V}}_{1,d} \rightarrow \mathcal{V}_{1,d} \rightarrow \overline{M}_{1}^{0}(\mathbb{P}^{4}, d) \rightarrow \overline{M}_{1}^{0}(\mathbb{P}^{4}, d)
$$

$$
\Rightarrow \int_{\overline{M}_{1}^{0}(\mathbb{P}^{4}, d)} e(\mathcal{V}_{1,d}) = \int_{\overline{M}_{1}^{0}(\mathbb{P}^{4}, d)} e(\tilde{\mathcal{V}}_{1,d})
$$
Computation of Genus 1 GWs of CIs

Thm. C generalizes to all \( \mathcal{V}_{1,d} \rightarrow \overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d) \):

\[
\mathcal{L} \equiv \mathcal{O}(a) \quad \pi \\
\mathbb{P}^n
\]

\[
\overline{\mathcal{M}}_{1,k}(\mathcal{L}, d) \quad \overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)
\]

\[\therefore \text{Thms A,B,C provide an algorithm for computing genus 1 GWs of complete intersections } X \subset \mathbb{P}^n\]
Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\nu_1,d \rightarrow \overline{M}_{1,k}^0(\mathbb{P}^n, d)$:

$$L \equiv \mathcal{O}(a)$$

$$\pi$$

$$\mathbb{P}^n$$

$\therefore$ Thms A, B, C provide an algorithm for computing genus 1 GWs of complete intersections $X \subset \mathbb{P}^n$
Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{O}(\nu_{1,d} \rightarrow \overline{M}_{1,k}^0(\mathbb{P}^n, d))$:

$$L \equiv \mathcal{O}(a)$$

$$\overline{M}_{1,k}(\mathcal{L}, d)$$

$\pi$ $\overline{M}_{1,k}(\mathbb{P}^n, d)$

$\tilde{\pi}$

$\mathbb{P}^n$

$\mathbb{P}^n$

∴ Thms A,B,C provide an algorithm for computing genus 1 GWs of complete intersections $X \subset \mathbb{P}^n$
Computation of Genus 1 GWs of CIs

Thm. C generalizes to all \( V_{1,d} \rightarrow \overline{M}_{1,k}(\mathbb{P}^n, d) \):

\[
\begin{align*}
\mathcal{L} & \equiv \mathcal{O}(a) \\
\nu_{1,d} & \equiv \overline{M}_{1,k}(\mathcal{L}, d) \\
\mathbb{P}^n & \xrightarrow{\pi} \overline{M}_{1,k}(\mathbb{P}^n, d)
\end{align*}
\]

\( \therefore \) Thms A, B, C provide an algorithm for computing genus 1 GWs of complete intersections \( X \subset \mathbb{P}^n \)
Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \rightarrow \overline{\mathcal{M}}_{1,k}^{0}(\mathbb{P}^n, d)$:

$$\mathcal{L} \equiv \mathcal{O}(a) \quad \pi$$

$$\mathbb{P}^n$$

$$\overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)$$

\[\vdash \text{Thms A, B, C provide an algorithm for computing genus 1 GWs of complete intersections } X \subset \mathbb{P}^n\]
Computation of $N_{1,d}$ for all $d$

- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
- extract non-equivariant part of elements in $H_T^*(\mathbb{P}^4)$
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Key Geometric Foundation

A Sharp Gromov’s Compactness Thm in Genus 1 (Z.’04)

- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a \(\bar{\partial}\)-equation with limited perturbation
- allows use of topological techniques to study genus 1 GWs
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Main Tool

Analysis of Local Obstructions

- study obstructions to smoothing pseudo-holomorphic maps from smooth domains
- not just potential existence of obstructions
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