On the Geometry of Genus 1 Gromov-Witten Invariants

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From String Theory to Enumerative Geometry

A-Model partition function for Calabi-Yau 3-fold X	$\overset{\text{MIRROR}}{\underset{\text{principle}}{=}}$	B-Model partition function for mirror (family) of X

generating function for GWs "counts of complex curves in" X

something about geometry of moduli spaces of CYs

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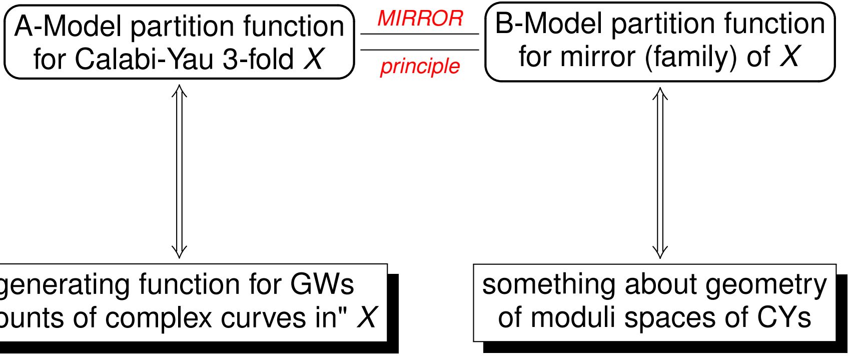

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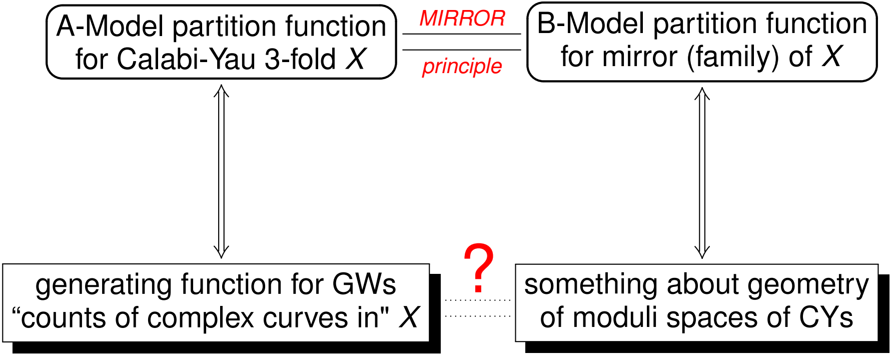

"Simplest" Calabi-Yau 3-fold

- quintic 3-fold $X_5 =$ degree 5 hypersurface in $\mathbb{P}^4$
- expected # of genus g degree d curves is finite: $n_{g,d}$
- genus g degree d GW-invariant $N_{g,d}$ is made up of $n_{h,d}$
- A-model partition function:

$$F_g^A(q) = \sum_{d=1}^{\infty} N_{g,d} q^d.$$

- B-model partition function F_g^B "measures" geometry of moduli spaces of CYs

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B-Side Computations

- Candelas-de la Ossa-Green-Parkes'91
  construct mirror family, compute F_0^B
- Bershadsky-Cecotti-Ooguri-Vafa'93 (BCOV)
  compute F_1^B using physics arguments
- Fang-Z. Lu-Yoshikawa'03 compute F_1^B mathematically
- Huang-Klemm-Quackenbush'06
  compute F_g^B, $g \leq 52$ using physics

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Mirror Symmetry Predictions and Verifications

Predictions

$$F_g^A(q) \equiv \sum_{d=1}^{\infty} N_{g,d} q^d \stackrel{?}{=} F_g^B(q).$$

Theorem (Givental'96, Lian-Liu-Yau'97,.........∼'00)

$g=0$ predict. of Candelas-de la Ossa-Green-Parkes'91 holds

Theorem (Z.'07)

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General Approach to Verifying $F_g^A = F_g^B$ (works for $g = 0, 1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{P}^4 \supset X_5$

Step 2: use $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$ to compute each $N_{g,d}$ by localization

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GW-Invariants of $X_5 \subset \mathbb{P}^4$

$$\overline{\mathfrak{M}}_g(X_5, d) = \big\{[u\colon \Sigma \longrightarrow X_5] \big|\ g(\Sigma)=g, \deg u = d,\ \bar\partial u = 0\big\}$$

$$N_{g,d} \equiv \deg\big[\overline{\mathfrak{M}}_g(X_5,d)\big]^{vir} \equiv \#\big\{[u\colon \Sigma \longrightarrow X_5] \big|\ g(\Sigma)=g, \deg u = d,\ \bar\partial u = \nu(u)\big\}$$

ν = small generic deformation of $\bar\partial$-equation

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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$\mathcal{L}$ $\overline{\mathfrak{M}}_g(\mathcal{L}, d)$

$X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4$ $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$

From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(5) & & \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ s \uparrow \downarrow \pi & & \\ X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4 & & \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) \end{array}$$

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This suggests: *Hyperplane Property*

$$N_{g,d} \equiv \deg \left[\overline{\mathfrak{M}}_g(X_5,d)\right]^{vir} \equiv \; ^{\pm}\left|\tilde{s}^{-1}(0)\right|$$

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Genus 0 vs. Positive Genus

$g = 0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]$
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Thm. A (J. Li–Z.'04): HP holds for reduced genus 1 GWs

$$\left[\overline{\mathfrak{M}}_1(X_5, d)\right]^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d).$$

This generalizes to complete intersections $X \subset \mathbb{P}^n$.

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Standard vs. Reduced GWs

Thm. A $\Longrightarrow$ $N^0_{1,d} \equiv \deg\,[\overline{\mathfrak{M}}^0_1(X,d)]^{vir} = \int_{\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$

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Thm. B (Z.'04,'07): $N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

[standard] − [reduced genus 1 GW] = f(genus 0 GW)

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Standard vs. Reduced GWs

$$\text{Thm. A} \implies N^0_{1,d} \equiv \deg\,[\overline{\mathfrak{M}}^0_1(X,d)]^{vir} = \int_{\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$$

$$\overline{\mathfrak{M}}^0_1(X,d) \equiv \overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d) \cap \overline{\mathfrak{M}}_1(X,d)$$

Thm. B (Z.'04,'07): $N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}$

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- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ (with simple fixed loci)
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  $g=0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{graphs}$

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Genus 1 Bypass

Thm. C (Vakil–Z.'05): $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

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- split genus 1 graphs into many genus 0 graphs at special vertex
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- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a $\bar{\partial}$-equation with limited perturbation
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