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COMPLETION OF KATZ-QIN-RUAN'S ENUMERATION OF GENUS-TWO PLANE CURVES

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Abstract

We give a formula for the number of genus-two fixed-complex-structure degree-d plane curves passing through $3d-2$ points in general position. This is achieved by completing Katz-Qin-Ruan's approach. This paper's formula agrees with the one obtained by the author in a completely different way.

1. Introduction

In the past decade, significant progress has been in enumerative algebraic geometry based on ideas of Gromov's compactness and quantum cohomology. In particular, [KM] and [RT] derived a recursive formula for the number N_d of rational degree-d plane curves passing through $(3d-1)$ points in general position. In [I] and [P], a simple relation between the number $N_{1,d}$ of fixed-j-invariant elliptic degree-d plane curves passing through $(3d-1)$ points and the number N_d is obtained. The approaches in the two papers are drastically different. In [P], the number $N_{1,d}$ is computed by a beautiful degeneration argument. In [I], the number $N_{1,d}$ is compared to the corresponding symplectic invariant as defined in [RT]. Like the methods of [KM] and [RT] in the genus-zero case, the approach of [I] applies to all projective spaces.

The subject of this paper is the number $N_{2,d}$ of genus-two degree-d plane curves that have a fixed complex structure on the normalization and pass through $(3d-2)$ points in general position. Using a degeneration argument similar to [P], [KQR] express $N_{2,d}$ in terms of the numbers $N_{d'}$ with $d' \le d$. Recently the author extended the approach of [I] to obtain formulas for the genus-two numbers in $\mathbb{P}^2$ and $\mathbb{P}^3$. However, the formulas for $N_{2,d}$ in [KQR] and [Z] are not equivalent. The relation between the two is

$$N^Z_{2,d} = 6\big(N^{KQR}_{2,d} + T_d\big),$$

where T_d is the number of degree-d tacnodal rational plane curves passing through $(3d-2)$ points. The formulas in [Z] satisfy all the required classical checks that the author is aware of. In particular, $N^Z_{2,4}$ is the same as the corresponding number for three points and seven lines in $\mathbb{P}^3$. The author then explored the details of the argument [KQR] and found three errors, one of which is significant. They are described briefly in the paragraph following the table and in more detail in Section 3. Once these errors are corrected, the formula of [Z] is recovered:

Theorem 1.1.

$$N_{2,d} = 3(d^2-1)N_d + \frac{1}{2}\sum_{d_1+d_2=d}\Big(d_1^2d_2^2 + 28 - 16\frac{9d_1d_2-1}{3d-2}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

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The table below gives the numbers $N_{2,d}$ for small values of d, computed directly from Theorem 1.1. The first three values have long been known to be zero. We use $N_1=N_2=1$, $N_3=12$, $N_4=620$, $N_5=87,304$, $N_6=26,312,976$, and $N_7=14,616,808,192$.

d	1	2	3	4	5	6	7
$N_{2,d}$	0	0	0	14,400	6,350,400	3,931,128,000	3,718,909,209,600

The first step in the proof of Theorem 1.1 via the recipe of [KQR] is Lemma 2.1, which allows one to reduce the computation to a very degenerate genus-two curve. The relevant intersection number is then computed by Propositions 3.1-3.4. Propositions 3.1 and 3.3 are proved in [KQR]. Proposition 3.4 is implied by Remark 3.12 in [KQR]. However, this remark is stated without a proof and contradicts Proposition 3.2. This is the significant error in [KQR]. A minor error is the statement about boundary relations at the beginning of the proof of Lemma 2.18. A posteriori, it turns out that this statement is in fact correct, at least in the relevant cases, but it does not follow from the argument given. The remaining error is dividing by an extra factor of six when computing contributions to the intersection number.

Since our goal is to correct the computation in [KQR], we attempt to follow their notation as closely as possible. The outline of this paper is as follows. We first review the notation and setup in [KQR]. In Section 3, four propositions that imply Theorem 1.1 are stated. The last two sections prove the two propositions not proved in [KQR].

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2. Review of Notation and Setup

Denote by $\overline{\mathfrak{M}}_2$ the Deligne-Mumford moduli space of stable genus-two curves. If $d\geq 3$, let

$$\overline{\mathfrak{M}}_2(d)\equiv\overline{\mathfrak{M}}_{2,3d-2}\big(\mathbb{P}^2,d\ell\big)$$

be Kontsevich's moduli space of stable maps from $(3d-2)$-pointed genus-two curves to $\mathbb{P}^2$ of degree d, where $\ell\in H_2(\mathbb{P}^2;\mathbb{Z})$ is the homology class of a line. Let $\pi\colon\overline{\mathfrak{M}}_2(d)\longrightarrow\overline{\mathfrak{M}}_2$ be the forgetful map. Denote by $W_2(d)\subset\overline{\mathfrak{M}}_2(d)$ the subset of stable maps with irreducible domains and by $\overline{W}_2(d)$ the closure of $W_2(d)$ in $\overline{\mathfrak{M}}_2(d)$.

Every element of $\overline{\mathfrak{M}}_2(d)$ can be written as $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]$, where D is a prestable genus-two curve, $\mu\colon D\longrightarrow\mathbb{P}^2$ is a (holomorphic) map, and $p_1,\ldots,p_{3d-2}\in D$ are the marked points. There are natural evaluation maps

$$e_i\colon\overline{\mathfrak{M}}_2(d)\longrightarrow\mathbb{P}^2,\quad e_i\big(\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\big)=\mu(p_i),\qquad i=1,\ldots,3d-2.$$

Let $\mathcal{L}_i=e_i^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)$ and

$$Z=\big[\overline{W}_2(d)\big]\cap c_1^2(\mathcal{L}_1)\cap\ldots\cap c_1^2(\mathcal{L}_{3d-2})\in H_6\big(\overline{W}_2(d)\big).$$

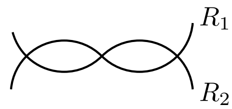
If $q_1,\ldots,q_{3d-2}$ are points in $\mathbb{P}^2$ in general position, then $\big\{e_1\times\ldots\times e_{3d-2}\big\}^{-1}(q_1\times\ldots\times q_{3d-2})$ is a representative for Z; see [KQR] for details.

Lemma 2.1. *For every* $[C]\in\overline{\mathfrak{M}}_2$,

$$N_{2,d}=[\pi^{-1}(C)]\cdot Z,$$

where $[\pi^{-1}(C)]\cdot Z$ *is the intersection pairing of* $\pi^{-1}([C])$ *and* Z *in* $\overline{W}_2(d)$.

This is a special case of Lemma 2.5 in [KQR]. In particular, if C_0 consists of two rational components identified at 3 pairs of points, i.e.

$C_0=$ 

then $N_{2,d}=[\pi^{-1}(C_0)]\cdot Z$. The space $\pi^{-1}(C_0)\subset\overline{\mathfrak{M}}_2(d)$ can be written as the disjoint union $\bigsqcup W_T$, where W_T is the space of stable maps $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]$, such that the domain D is the union of R_1, R_2, and trees $T_1,\ldots,T_s$ of $\mathbb{P}^1$ in a way encoded by T. The stable reduction of D must be C_0. See Figure 1 below for some examples.

In order to compute $[\pi^{-1}(C_0)]\cdot Z$, [KQR] consider the intersection of Z with every nonempty space W_T. It is fairly easy to show that $Z\cap W_T$ is empty for all but a small number of trees T, independent of d. If $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in Z\cap W_T$, the map $\mu\colon D\longrightarrow\mathbb{P}^2$ has degree d and passes through $3d-2$ points in $\mathbb{P}^2$ in general position. Thus, if $D_1,\ldots,D_m$ are the irreducible components of D to which μ restricts non-trivially, $m=1$ or $m=2$. Then D can have at most two components, other than R_1, R_2, on which the map μ is constant.

The complete list of possibilities for D, up to equivalence, is given in Figure 1. Denote by C_{ij} the curve as in the ith row and jth column of Figure 1. Similarly, denote by W_{ij} be the space of stable maps with domain C_{ij} and a distribution of the degree d between the components of C_{ij} such that the image of some stable map in W_{ij} passes through $(3d-2)$ points. We clarify this statement in the relevant cases:
(1) if $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]$ lies in W_{13}, W_{32}, W_{41}, W_{43}, or W_{5j}, the degree of $\mu|D_i$ is $d_i\neq0$, and the restriction of μ to all other components is constant;
(2) if $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]$ lies in W_{24}, W_{31}, or W_{42}, the degree of $\mu|D_1$ is $d_1\neq0$, $\mu|R_i$ is constant, and in the case of W_{42} the restriction of μ to the vertical component (in the diagram) is constant.
Furthermore, for stability reasons, every component of C_{ij}, on which μ is constant and which does not contain three singular point of C_{ij}, must contain one of the marked points p_i.

3. Computation of the Intersection Number

Proposition 3.1. *The contribution to* $[\pi^{-1}(C_0)]\cdot Z$ *from* W_{11} *is*

$$\frac{3(d-1)(d-2)(d-3)}{d}N_d+\frac{1}{2}\sum_{d_1+d_2=d}\Big(d_1^2d_2^2-6d_1d_2-4+18\frac{d_1d_2}{d}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

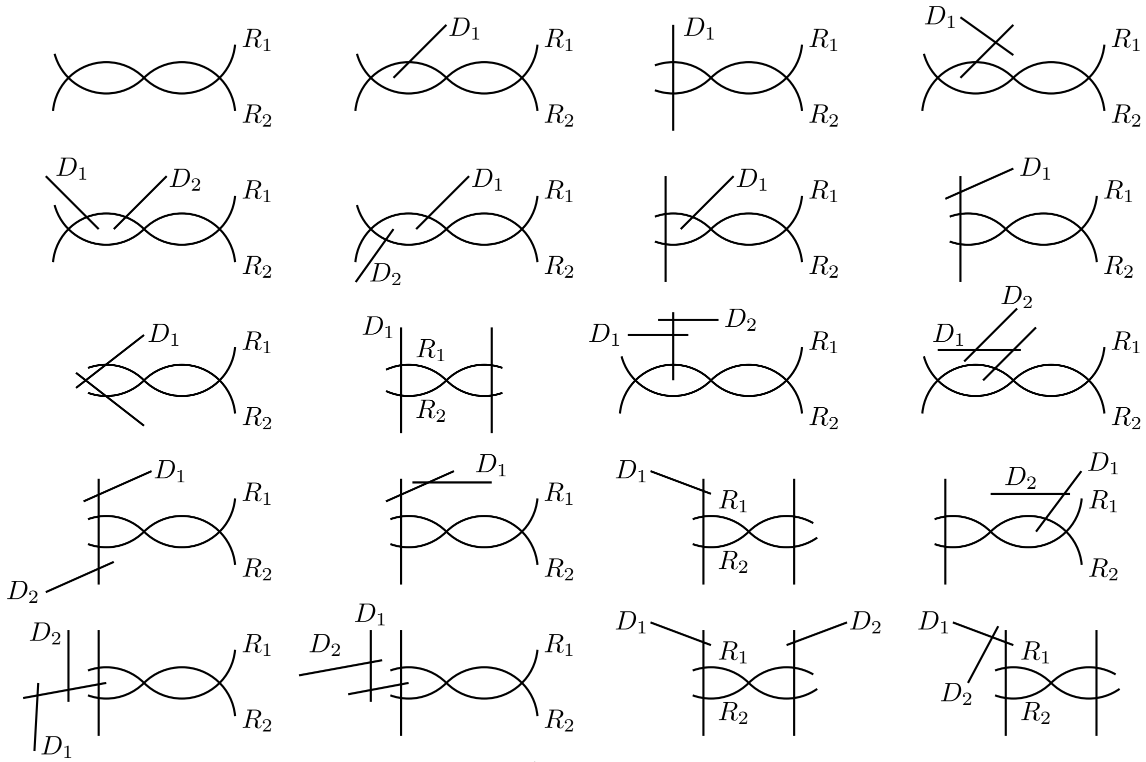


Figure 1

This proposition is essentially proved in [KQR]; see equation (2.9) and Lemmas 2.12, 2.16, and 3.2 in [KQR]. The above number is six times the number given by Theorem 1.1 of [KQR]. It is easy to see that the authors divide by six an extra time. For example, in Lemma 2.12, one should take *ordered* triplets of nodes, i.e. $\binom{d_1d_2}{3}$ should be replaced by

$$d_1d_2(d_1d_2-1)(d_1d_2-2),$$

since they are dividing by the order of $\mathrm{Aut}(C_0)$. Similarly, the number in Lemma 2.16 should be replaced by six times itself.

Proposition 3.2. *The contribution to* $[\pi^{-1}(C_0)]\cdot Z$ *from* W_{13} *is*

$$\frac{6(3d^2-12d+9)n_d}{d}+3\sum_{d_1+d_2=d}\Big(d_1d_2+4-9\frac{d_1d_2}{d}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

We prove this proposition in Section 5. What we show is that $\overline{W}_2(d)\cap W_{13}$ is the space of all stable maps $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]$ such that $\mu(D)$ is a tacnodal curve in $\mathbb{P}^2$, and μ maps the two nodes of D to the same tacnode of $\mu(D)$. The number of Proposition 3.2 is $6T_d$. Note that the number T_d is well-known; see equation (1.2) in [DH] and Subsection 3.2 in [V1].

Proposition 3.3. *If* $(i,j)\in\big\{(1,2),(1,4),(2,1),(2,2),(2,3),(3,3),(3,4),(4,4)\big\}$, $Z\cap W_{ij}=\emptyset$. *Thus,* W_{ij} *does not contribute to* $[\pi^{-1}(C_0)]\cdot Z$.

Most of this proposition is proved by Lemmas 2.18 and 3.7 of [KQR]. The cases $(i,j)=(3,3)$ and $(i,j)=(3,4)$ can be deduced from the proofs of these two lemmas. The modification required is similar to the extension of the main part of the proof of Lemma 1 in [P] to cases of multiple blowups; see also the proof of Lemma 4.4 below. Note that since Lemma 3.7 of [KQR] does not apply to the remaining possibilities for (i,j), neither does Lemma 2.18 of [KQR].

Proposition 3.4. *If* $(i,j)\in\{(2,4),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4)\}$, $Z\cap W_{ij}=\emptyset$. *Thus,* W_{ij} *does not contribute to* $[\pi^{-1}(C_0)]\cdot Z$.

We prove this proposition in the next section. The number in Theorem 1.1 is the sum of the numbers in Propositions 3.1 and 3.2. However, one has to make use of Kontsevich's recursion to obtain the formula in Theorem 1.1:

$$N_d=\frac{1}{6(d-1)}\sum_{d_1+d_2=d}\Big(d_1d_2-2\frac{(d_1-d_2)^2}{3d-2}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

4. Proof of Proposition 3.4

4.1. The Semi-Standard Cases. We prove Proposition 3.4 by exhibiting conditions that stable maps in $\overline{W}_2(d)\cap W_{ij}$ must satisfy. This approach is analogous to methods in [P] and [KQR], but we make no use of the spaces X and Y of these two papers. It should be possible to describe the space $\overline{W}_2(d)\cap\pi^{-1}(C_0)\subset\overline{\mathfrak{M}}_2(d)$ explicitly by using arguments as in this section to obtain necessary conditions for an element of $\pi^{-1}(C_0)$ to be in $\overline{W}_2(d)$ and by applying methods similar to the next section to show that these conditions are sufficient. However, much less is needed to prove Theorem 1.1.

Suppose $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(g)\cap W_{ij}$. Then by definition of stable-map convergence, there exist
(T1) a one-parameter family of curves $\tilde{\eta}\colon\tilde{\mathcal{F}}\longrightarrow\Delta$ such that Δ is a neighborhood of 0 in $\mathbb{C}$, $\tilde{\mathcal{F}}$ is a smooth space, $\tilde{\eta}^{-1}(0)=D$, and $C_t\equiv\tilde{\eta}^{-1}(t)$ is a smooth genus-two curve for all $t\in\Delta^*\equiv\Delta-\{0\}$;
(T2) a map $\tilde{\mu}\colon\tilde{\mathcal{F}}\longrightarrow\mathbb{P}^2$ such that $\tilde{\mu}|\eta^{-1}(0)=\mu$.
In many cases, $\tilde{\mathcal{F}}$ can be obtained by a sequence of blowups from another smooth bundle $\eta\colon\mathcal{F}\longrightarrow\Delta$ of curves. This observation is used often in the proofs of the lemmas that follow.

Lemma 4.1. *If* $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{24}$ *and the degree of* $\mu|D_1$ *is* d, $\mu\big(D\big)$ *has a cusp at the image of the node of* D_1.

Proof. (1) Let $\tilde{\eta}\colon\tilde{\mathcal{F}}\longrightarrow\Delta$ be a family as in (T1) above with central fiber $\tilde{C}_0=D$, and $\tilde{\mu}\colon\tilde{\mathcal{F}}\longrightarrow\mathbb{P}^2$ a map as in (T2). Then there exists another family $\eta\colon\mathcal{F}\longrightarrow\Delta$ as in (T1) such that the central fiber is C_{13} and $\tilde{\mathcal{F}}$ is the blowup of $\mathcal{F}$ at a smooth point $p\in D_1\subset C_{13}$.
(2) Let $\psi\in H^0\big(C_{13};\omega_{C_{13}}\big)$ be an element such that $\psi|D_1\neq 0$. From the point of view of complex geometry, $H^0\big(C_{13};\omega_{C_{13}}\big)$ is the space harmonic $(1,0)$-forms on the three components of C_{13}, which have simple poles at the singular points with residues that add up to zero at each node. Thus, such an element exists. Let (t,w) be coordinates near $p\in\mathcal{F}$ such that w is the vertical coordinate, i.e. $d\eta\big|\frac{\partial}{\partial w}=0$. Then ψ extends to a family of elements $\psi_t\in H^0\big(C_t;\omega_{C_t}\big)$ such that

$$\psi_t\big|_w=a\big(1+o(1_{(t,w)})\big)dw, \tag{4.1}$$

for some $a\in\mathbb{C}^*$.
(3) On a neighborhood of $D_1^*\subset D=C_{24}$, the complement of the node in D_1, we have local coordinates

$(t,z)\longrightarrow\big(t,w\!=\!tz,[1,z]\big)$. Note that in these coordinates, (4.1) becomes

$$\psi_t\big|_z = at\big(1+o(1_t)\big)dz. \tag{4.2}$$

Let L_1 and L_2 be any two lines in general position in $\mathbb{P}^2$. In particular, we assume that they miss the image under μ of the node of D_1. Then for all $t\!\in\!\Delta$, sufficiently small,

$$\mu_t^{-1}(L_i)=\big\{z_1^{(i)}(t),\ldots,z_d^{(i)}(t)\big\}\subset C_t \quad\text{and}\quad z_j^{(i)}(t)=z_j^{(i)}(0)+o(1_t), \tag{4.3}$$

where $\mu_t\!=\!\tilde{\mu}|C_t$. Since $\sum z_j^{(1)}(t)$ and $\sum z_j^{(2)}(t)$ are linearly equivalent divisors in C_t,

$$\sum_{j=1}^{j=d}\int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)}\psi_t=0 \qquad \forall t\in\Delta^*, \tag{4.4}$$

where the line integrals are taken inside of the coordinate chart. Plugging (4.2) and (4.3) into (4.4) gives

$$at\sum_{j=1}^{j=d}\big(z_j^{(2)}(0)-z_j^{(1)}(0)+o(1_t)\big)=0 \qquad \forall t\!\in\!\Delta^*. \tag{4.5}$$

Dividing this equation by at and then taking the limit as $t\!\longrightarrow\!0$, we conclude that

$$\sum_{j=1}^{j=d}z_j^{(1)}(0)=\sum_{j=1}^{j=d}z_j^{(2)}(0). \tag{4.6}$$

Condition (4.6) can be explicitly interpreted as follows. Let $[u,v]$ be homogeneous coordinates on D_1 such that $z\!=\!\frac{v}{u}$. Then a map $D_1\!\longrightarrow\!\mathbb{P}^2$ of degree-d corresponds to three homogeneous polynomials

$$p_i=\sum_{j=0}^{j=d}p_{ij}u^jv^{d-j}.$$

Since equality (4.6) holds for a dense subset of lines in $\mathbb{P}^2$, there exists $K\!=\!K(\mu)\!\in\!\mathbb{P}^1$ such that

$$\frac{c_0p_{0,d-1}+c_1p_{1,d-1}+c_2p_{2,d-1}}{c_0p_{0,d}+c_1p_{1,d}+c_2p_{2,d}}=K \quad \forall(c_0,c_1,c_2)\!\in\!\mathbb{C}^3\!-\!\{0\}\Longrightarrow$$
$$(p_{0,d-1},p_{1,d-1},p_{2,d-1})=K(p_{0,d},p_{1,d},p_{2,d}). \tag{4.7}$$

Equation (4.7) imposes two linearly independent conditions on the map $\mu|D_1$ if $\mu\in\overline{W}_2(d)\cap W_{24}$. Geometrically, they mean that $\mu\big(D\big)$ has a cusp at the image of the node of D_1. □

Corollary 4.2. *If $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\!\in\!Z\cap W_{24}$, the degree of $\mu|D_1$ is less than d.*

Proof. Suppose the degree of $\mu|D_1$ is d. Then by Lemma 4.1, $\mu(D_1)$ has a cusp at the image of the node of D_1. Since the points $q_1,\ldots,q_{3d-2}$ are in general position, $\mu(D_1)$ has one simple cusp and $\binom{d-1}{2}\!-\!1$ simple nodes. Let $\tilde{\mathcal{F}}$ and $\tilde{\mu}$ be as in the proof of Lemma 4.1. Then $\tilde{\mu}(C_t)$ converges to $\mu(D_1)$. By Lemma 2.4.1 or Example 3.2.2 in [V2], D_1 must have an elliptic tail, i.e. the map $\tilde{\mu}:\tilde{\mathcal{F}}\longrightarrow\mathbb{P}^2$ cannot exist. In the given case, this can also be seen directly as follows. The image under μ_t of the intersection of C_t with the coordinate chart described in (3) of the proof of Lemma 4.1 has $\binom{d-1}{2}\!-\!1$ simple nodes, close to the simple nodes of $\mu(D_1)$. The complement of the coordinate chart in C_t is a genus two curve with a small coordinate neighborhood removed. Thus, it contributes at least 2 to the arithmetic genus of $\mu(C_t)$. This means that the arithmetic genus of $\mu(C_t)$ is at least $\binom{d-1}{2}\!+\!1$, instead of $\binom{d-1}{2}$. □

Lemma 4.3. *The image of every element* $\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{43}$ *has a cusp at* $\mu(p_i)$ *for some* $i=1,\ldots,3d-2$. *The same is true for every element of* $\overline{W}_2(d)\cap W_{42}$ *such that the degree of* $\mu|D_1$ *is* d. *Thus,* $Z\cap W_{43}=\emptyset$, *while for every element* $\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in Z\cap W_{42}$, *the degree of* $\mu|D_1$ *is less than* d.

Proof. (1) The proof of the first statement is nearly the same as the proof of Lemma 4.1. The only difference is that the central fiber of $\mathcal{F}$ will be C_{32}.
(2) The family $\tilde{\mathcal{F}}$ of the second claim of this lemma is obtained from $\tilde{\mathcal{F}}$ of Lemma 4.1 by blowing up a smooth point of the exceptional divisor $D_1\subset C_{24}$. Thus, nearly the same argument as in Lemma 4.1 applies if the degree of $\mu|D_1$ is d; see [P] for an extension in an analogous situation. □

Lemma 4.4. *If* $(i,j)\in\big\{(5,2),(5,4)\big\}$, *the image of every element of* $\overline{W}_2(d)\cap W_{ij}$ *is a two-component rational cuspidal curve. The same is true for all* $\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{42}$, *such that the degree of* $\mu|D_1$ *is less than* d. *Thus,* $Z\cap W_{ij}=\emptyset$ *in all three cases.*

Proof. (1) We first consider the case $\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{42}$ and the degree of $\mu|D_1$ is $d_1<d$. The case $d_1=d$ is considered in Lemma 4.3. The family $\tilde{\mathcal{F}}\longrightarrow\Delta$ corresponding to this case can be obtained as follows. We start with a family $\mathcal{F}\longrightarrow\Delta$ as in (2) of the proof of Lemma 4.1, blow it up at a smooth point $p\in D_1\subset C_{13}$, and then blow up the resulting space at a smooth point p_1 of the new exceptional divisor $E\equiv D_1\subset C_{24}$. Denote the last exceptional divisor by E_1. We use coordinates (t,z) near E^* as before and coordinates $(t,z_1)\longrightarrow\big(t,z=p_1+tz_1,[1,z_1]\big)$ near E_1^*. Then,

$$\psi_t\big|_z=at\big(1+o(1_t)\big)dz,\qquad \psi_t\big|_{z_1}=at^2\big(1+o(1_t)\big)dz_1;$$
$$\mu_t^{-1}(L_i)=\big\{z_{1,1}^{(i)}(t),\ldots,z_{1,d_1}^{(i)}(t),z_{d_1+1}^{(i)}(t),\ldots,z_d^{(i)}(t)\big\}\subset C_t,\qquad\text{with}$$
$$z_{1,j}^{(i)}(t)=z_{1,j}^{(i)}(0)+o(1_t),\quad z_j^{(i)}(t)=z_j^{(i)}(0)+o(1_t);$$
$$\sum_{j=1}^{j=d_1}\int_{z_{1,j}^{(1)}(t)}^{z_{1,j}^{(2)}(t)}\psi_t+\sum_{j=d_1+1}^{j=d}\int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)}\psi_t=0\qquad\forall t\in\Delta^*.$$

Each line integral is taken inside the corresponding coordinate chart. Proceeding as in the proof of Lemma 4.1, we obtain

$$at^2\sum_{j=1}^{j=d_1}\big(z_{1,j}^{(1)}(0)-z_{1,j}^{(2)}(0)+o(1_t)\big)+at\sum_{j=d_1+1}^{j=d}\big(z_j^{(2)}(0)-z_j^{(1)}(0)+o(1_t)\big)=0\qquad\forall t\in\Delta^*$$
$$\Longrightarrow\sum_{j=d_1+1}^{j=d}z_j^{(1)}(0)=\sum_{j=d_1+1}^{j=d}z_j^{(2)}(0).$$

As before, the last identity implies that $\mu|E$ maps $z=\infty\in E$ to a cusp of $\mu(E)$.
(2) The argument in the case of W_{54} is the same, except we replace the family $\mathcal{F}$ of Lemma 4.1 with the family $\mathcal{F}$ of (1) of Lemma 4.3. Finally, the case of W_{52} simply involves an extra blowup at a smooth point as compared to the case of W_{42}. □

Lemma 4.5. *If* $(i,j)\in\big\{(4,1),(5,1)\big\}$, *the image of every element of* $\overline{W}_2(d)\cap W_{ij}$ *is a two-component rational curve that has a tacnode. Thus,* $Z\cap W_{ij}=\emptyset$.

Proof. (1) The family $\tilde{\mathcal{F}}$ corresponding to the case of W_{41} is obtained by blowing up the family $\mathcal{F}$ of Lemma 4.1 at two smooth points, p_1 and p_2, of $D_1\subset C_{13}$. On a neighborhood of $D_i^*\subset C_{41}$, we use local coordinates $(t,z_i)\longrightarrow\big(t,p_i+tz_i,[1,z_i]\big)$. Then,

$$\psi_t\big|_{z_i}=a_it\big(1+o(1_t)\big)dz_i;$$
$$\mu_t^{-1}(L_i)=\big\{z_{1,1}^{(i)}(t),\ldots,z_{1,d_1}^{(i)}(t),z_{2,d_1+1}^{(i)}(t),\ldots,z_{2,d}^{(i)}(t)\big\}\subset C_t,\quad z_{\iota,j}^{(i)}(t)=z_{\iota,j}^{(i)}(0)+o(1_t);$$
$$\sum_{j=1}^{j=d_1}\int_{z_{1,j}^{(1)}(t)}^{z_{1,j}^{(2)}(t)}\psi_t+\sum_{j=d_1+1}^{j=d}\int_{z_{2,j}^{(1)}(t)}^{z_{2,j}^{(2)}(t)}\psi_t=0\qquad\forall t\in\Delta^*.$$

for some $a_1,a_2\in\mathbb{C}^*$, which depend on D, but not on $\mu|D_i$. Proceeding as before, we obtain

$$a_1t\sum_{j=1}^{j=d_1}\big(z_{1,j}^{(1)}(0)-z_{1,j}^{(2)}(0)+o(1_t)\big)+a_2t\sum_{j=d_1+1}^{j=d}\big(z_{2,j}^{(2)}(0)-z_{2,j}^{(1)}(0)+o(1_t)\big)=0\qquad\forall t\in\Delta^*\Longrightarrow$$
$$a_1\sum_{j=1}^{j=d_1}z_{1,j}^{(1)}(0)+a_2\sum_{j=d_1+1}^{j=d}z_{2,j}^{(1)}(0)=a_1\sum_{j=1}^{j=d_1}z_{1,j}^{(2)}(0)+a_2\sum_{j=d_1+1}^{j=d}z_{2,j}^{(2)}(0).\tag{4.8}$$

Let $p_i^{(1)}$ and $p_i^{(2)}$ be the homogeneous polynomials corresponding to $\mu|D_1$ and $\mu|D_2$, respectively. Since (4.8) holds for a dense subset of lines, there exist $K=K(\mu)\in\mathbb{C}$ such that

$$a_1\frac{c_0p_{0,d_1-1}^{(1)}+c_1p_{1,d_1-1}^{(1)}+c_2p_{2,d_1-1}^{(1)}}{c_0p_{0,d_1}^{(1)}+c_1p_{1,d_1}^{(1)}+c_2p_{2,d_1}^{(1)}}+a_2\frac{c_0p_{0,d_2-1}^{(2)}+c_1p_{1,d_2-1}^{(2)}+c_2p_{2,d_2-1}^{(2)}}{c_0p_{0,d_2}^{(2)}+c_1p_{1,d_2}^{(2)}+c_2p_{2,d_2}^{(2)}}=K,\tag{4.9}$$

for all $(c_0,c_1,c_2)\in\mathbb{C}^3-\{0\}$. Since μ maps the nodes of D_1 and D_2 to the same point,

$$\big(p_{0,d_1}^{(1)},p_{1,d_1}^{(1)},p_{2,d_1}^{(1)}\big)=\kappa\big(p_{0,d_2}^{(2)},p_{1,d_2}^{(2)},p_{2,d_2}^{(2)}\big)$$

for some $\kappa\in\mathbb{C}$. Using this equation, it is easy to see that condition (4.9) is equivalent to saying that μ maps the singular points of D_1 and D_2 into a tacnode of its image. Thus, the image of very element of $\overline{W}_2(d)\cap W_{41}$ is a two-component curve with a tacnode.
(2) Nearly the same argument applies to W_{51}. In this case, an extra blowup is required, and we will have $a_1=a_2=a$. □

Lemma 4.6. *The image of every element of $\overline{W}_2(d)\cap W_{53}$ is a two-component rational curve such that both components have a cusp at one of the nodes of the image curve. Thus, $Z\cap W_{53}=\emptyset$.*

Proof. The proof is a minor modification of the proof of Lemma 4.1. The central fiber of $\mathcal{F}$ in this case is C_{32}. We can then choose $\psi\in H^0\big(C_{32};\omega_{C_{32}}\big)$ such that the restriction of ψ to the right vertical component (in the diagram) is zero. In terms of coordinates (t,w_1) and (t,w_2) near the smooth points p_1 and p_2 of the two vertical components, we will have

$$\psi_t\big|_{w_1}=a\big(1+o(1_{(t,w)})\big)dw_1\quad\text{and}\quad\psi_t\big|_{w_2}=o(1_t)dw_2,$$

for some $a\in\mathbb{C}^*$. Proceeding as above, we then conclude that μ maps the node of $D_1\subset C_{53}$ to a cusp of $\mu(D_1)$. The same argument applies to $\mu|D_2$. □

4.2. The Remaining Cases. The arguments in the previous subsection look very much like the arguments in [P] and [KQR]. However, some differences appear in this subsection.

Lemma 4.7. *If $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{13}$, the image of μ is a tacnodal rational curve and μ maps the nodes of D to a tacnode of $\mu(D)$.*

Proof. (1) We use coordinates (t,w) near $D_1^* \subset C_{13}$ such that the two nodes of D_1 correspond to $w=0$ and $w=\infty$. Let $\psi_t \in H^0\big(C_t;\omega_{C_t}\big)$ be such that

$$\psi_t\big|_w = \big(1+o(1_t)\big)\frac{dw}{w}.$$

Proceeding as above, we obtain

$$\mu_t^{-1}(L_i) = \big\{w_1^{(i)}(t),\ldots,w_d^{(i)}(t)\big\} \subset C_t, \quad w_j^{(i)}(t) = w_j^{(i)}(0)+o(1_t);$$

$$\sum_{j=1}^{j=d}\int_{w_j^{(1)}(t)}^{w_j^{(2)}(t)}\psi_t = 0 \in \mathbb{C}/2\pi i\mathbb{Z} \qquad \forall t\in\Delta^*;$$

$$\prod_{j=1}^{j=d} w_j^{(1)}(0) = \prod_{j=1}^{j=d} w_j^{(2)}(0) \equiv K; \tag{4.10}$$

$$\big(p_{0,0},p_{1,0},p_{2,0}\big) = K\big(p_{0,d},p_{1,d},p_{2,d}\big). \tag{4.11}$$

for some $K=K(\mu)\in\mathbb{C}$. Condition (4.11) on the coefficients of the homogeneous polynomials corresponding to $\mu|D_1$ follows from the fact that (4.10) holds for a dense subset of lines in $\mathbb{P}^2$. However, (4.11) by itself tells us nothing new about $\mu|D_1$, since we already know that μ maps the nodes of D_1 to the same point.

(2) We instead consider the limit of the left-hand side of (4.10) as L_1 approaches the line tangent to the branch $w=0$ of $\mu(D)$. If the node $\mu(0)$ of $\mu(D)$ is simple, two of the numbers $w_j^{(1)}(0)$ tend to 0 and one to ∞, all at comparable rates. Thus, we must have $K=0$. By the same argument, $K=\infty$. This means

$$p_{0,0}=p_{1,0}=p_{2,0}=p_{0,d}=p_{1,d}=p_{2,d}=0.$$

If $[u,v]$ are homogeneous coordinates on $E^{(1)}$ with $w=\frac{v}{u}$, it follows that uv divides all three homogeneous polynomials p_0,p_1,p_2, i.e. $\mu|D$ has degree at most $d-2$, not d, contrary to the assumption. Thus, $\mu(0)=\mu(\infty)$ has to be a tacnode of $\mu(D)$ if $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{13}$. □

Lemma 4.8. *The image of every element* $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{32}$ *has a tacnode at* $\mu(p_i)$ *for some* $i=1,\ldots,3d-2$. *If* $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{24}$ *and the degree of* $\mu|D_1$ *is less than d, then* $\mu(D)$ *is a two-component rational tacnodal curve. Thus,* $Z\cap W_{ij}=\emptyset$ *in both cases.*

Proof. Since the proof of Lemma 4.7 carries over to the case of W_{32} with no change, the first claim is clear. For the second claim, we use coordinates (t,w) and (t,z) as in the proofs of Lemmas 4.1 and 4.7. Then,

$$\psi_t\big|_w = \big(1+o(1_t)\big)\frac{dw}{w}, \qquad \psi_t\big|_z = o(1_t);$$

$$\mu_t^{-1}(L_i) = \big\{z_1^{(i)}(t),\ldots,z_{d_1}^{(i)}(t),w_{d_1+1}^{(i)}(t),\ldots,w_d^{(i)}(t)\big\} \subset C_t, \qquad \text{with}$$

$$z_j^{(i)}(t) = z_j^{(i)}(0)+o(1_t), \quad w_j^{(i)}(t) = w_j^{(i)}(0)+o(1_t);$$

$$\sum_{j=1}^{j=d_1}\int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)}\psi_t + \sum_{j=d_1+1}^{j=d}\int_{w_j^{(1)}(t)}^{w_j^{(2)}(t)}\psi_t = 0 \in \mathbb{C}/2\pi i\mathbb{Z} \qquad \forall t\in\Delta^*;$$

$$\prod_{j=d_1+1}^{j=d} w_j^{(1)}(0) = \prod_{j=d_1+1}^{j=d} w_j^{(2)}(0).$$

The last identity implies that $\mu|D_2$ has a tacnode. The remaining claim of the lemma follows from the first two and Corollary 4.2. □

Lemma 4.9. *If $\big[\mu:(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{31}$ and the degree of $\mu|D_1$ is d, $\mu(D)$ has a tacnode at $\mu(p_i)$ for some $i=1,\ldots,3d$–2. If the degree of $\mu|D_1$ is less than d, $\mu(D)$ is a two-component tacnodal rational curve. Thus, $Z\cap W_{31}=\emptyset$.*

Proof. The proof of Lemma 4.7 applies to the first case with no change. For the second case, we use coordinate $(t,w_1)=(t,w)$ and (t,w_2) analogous to (t,w), such that $w_1=\infty$ and $w_2=\infty$ are identified in C_{31}. Since the residues of $\psi\in H^0\big(\tilde{C}_0;\omega_{\tilde{C}_0}\big)$ at $w_1=\infty$ and $w_2=\infty$ add up to zero, $\psi|D_2=-\frac{dw_2}{w_2}$. Thus, proceeding as in the proof of Lemma 4.7, we obtain

$$\prod_{j=1}^{j=d_1} w_{1,j}^{(1)}(0)\cdot\Big(\prod_{j=d_1+1}^{j=d} w_{2,j}^{(1)}(0)\Big)^{-1}=\prod_{j=1}^{j=d_1} w_{1,j}^{(2)}(0)\cdot\Big(\prod_{j=d_1+1}^{j=d} w_{2,j}^{(2)}(0)\Big)^{-1}\equiv K;$$

$$\frac{c_0p_{0,0}^{(1)}+c_1p_{1,0}^{(1)}+c_2p_{2,0}^{(1)}}{c_0p_{0,d_1}^{(1)}+c_1p_{1,d_1}^{(1)}+c_2p_{2,d_1}^{(1)}}\cdot\frac{c_0p_{0,d_2}^{(2)}+c_1p_{1,d_2}^{(2)}+c_2p_{2,d_2}^{(2)}}{c_0p_{0,0}^{(2)}+c_1p_{1,0}^{(2)}+c_2p_{2,0}^{(2)}}=K\quad\forall(c_0,c_1,c_2)\in\mathbb{C}^3-\{0\},\tag{4.12}$$

for some $K\in\mathbb{C}$. Since $\mu\big(w_2=\infty\big)=\mu\big(w_1=\infty\big)$,

$$\big(p_{0,d_1}^{(1)},p_{1,d_1}^{(1)},p_{2,d_1}^{(1)}\big)=\kappa\big(p_{0,d_2}^{(2)},p_{1,d_2}^{(2)},p_{2,d_2}^{(2)}\big)$$

for some $\kappa\in\mathbb{C}^*$. Thus, as a condition on μ, (4.12) is equivalent to

$$\big(p_{0,0}^{(1)},p_{1,0}^{(1)},p_{2,0}^{(1)}\big)=K\big(p_{0,0}^{(2)},p_{1,0}^{(2)},p_{2,0}^{(2)}\big)$$

for some $K\in\mathbb{C}$. Suppose $\mu\big(w_2=\infty\big)=\mu\big(w_1=0\big)$ is not a tacnode of $\mu(D)$. Then as in (2) of the proof of Lemma 4.7, we conclude that

$$p_{0,0}^{(1)}=p_{1,0}^{(1)}=p_{2,0}^{(1)}\;=p_{0,0}^{(2)}=p_{1,0}^{(2)}=p_{2,0}^{(2)}.$$

This means $\mu|D_1$ and $\mu|D_2$ have degrees at most d_1-1 and d_2-1, respectively, contrary to the assumption. □

5. Proof of Proposition 3.2

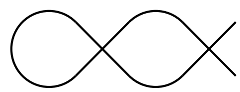
By Lemma 4.7, if $\big[\mu:(D,p_1,\ldots,p_{3d-2})\big]\in Z\cap W_{13}$, μ maps the nodes of D into the tacnode of $\mu(D)$. We now prove the converse and determine the multiplicity with which the number T_d enters into $[\pi^{-1}(C_0)]\cdot Z$.

Lemma 5.1. *Suppose C_0' is a tacnodal rational curve and $\eta:\mathcal{W}\longrightarrow\mathcal{B}$ is a local deformation space for C_0. Let $q_1,\ldots,q_{3d-2}$ be points in general position in $\mathbb{P}^2$ and $f:C_0'\longrightarrow\mathbb{P}^2$ be a map of degree d passing through the $(3d-2)$ points. Then there exists a map $\tilde{f}:\mathcal{W}\longrightarrow\mathbb{P}^2$, perhaps after shrinking $\mathcal{B}$, such that $\tilde{f}|C_0'=f$ and $\tilde{f}\big|\eta^{-1}(t)$ passes through the $(3d-2)$ points.*

Proof. Since $T_d=0$ for $d\leq 3$, we can assume $d\geq 3$. Then $H^1\big(C_0';f^*\mathcal{O}_{\mathbb{P}^2}(1)\big)=0$. Thus, there is no obstruction to extending f to a neighborhood of C_0' in $\mathcal{W}$. □

Corollary 5.2. *Suppose* $\big[\mu:(D,p_1,\ldots,p_{3d-2})\big]\in W_{13}$*,* $\mu(p_i)=q_i$ *for all* $i=1,\ldots,3d-2$*, and* μ *maps the nodes of* D_1 *to the tacnode of* $\mu(D)$*. Then* $\big[\mu:(D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)$*.*

Proof. We apply Lemma 5.1 to the normalization $f\colon C_0'\longrightarrow\mu(D)$ of $\mu(D)$ at the simple nodes. Let C_t be a family of rational curves identified at two pairs of points, i.e.

$C_t =$ 

As the nodes of C_t come together, C_t approaches C_0' in $\mathcal{B}$. For all $t\neq 0$ sufficiently small, let $f_t\colon C_t\longrightarrow\mathbb{P}^2$ be the maps provided by Lemma 5.1. Then $f_t(C_t)$ converges to $f(C_0')$. Furthermore, C_t converges to C_0 in $\overline{\mathfrak{M}}_2$. Thus, if

$$\lim_{t\longrightarrow 0}\big[f_t\colon(C_t,f_t^{-1}(q_1),\ldots,f_t^{-1}(q_{3d-2}))\big]=\big[\mu'\colon(D',p_1',\ldots,p_{3d-2}')\big]\in\overline{\mathfrak{M}}_2(d),$$

D' must be one of the curves C_{ij} of Figure 1, and $\mu'(D')$ is a tacnodal rational curve. By Propositions 3.3 and 3.4, we conclude that

$$\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]=\big[\mu'\colon(D',p_1',\ldots,p_{3d-2}')\big]\in\overline{\mathfrak{M}}_2(d).$$ □

Lemma 5.3. *The contribution of* W_{13} *to* $[\pi^{-1}(C_0)]\cdot Z$ *is* $6T_d$*.*

Proof. Suppose $\big[\mu\colon(D,p_1,\ldots,p_{3d-2})\big]\in Z\cap W_{13}$. Given a fixed complex structure j on Σ such that (Σ,j) is very close to $[C_0]$ in $\overline{\mathfrak{M}}_2$, we need to determine the number of maps $\mu_j\colon\Sigma\longrightarrow\mathbb{P}^2$ close to μ. By Corollary 5.2, there exists a family of curves $\tilde{\eta}\colon\tilde{\mathcal{F}}\longrightarrow\Delta$ and of maps $\tilde{\mu}\colon\tilde{\mathcal{F}}\longrightarrow\mathbb{P}^2$ restricting to μ on the central fiber D. There are six automorphisms of C_0 that preserve its components. Corresponding to these automorphisms and $(\tilde{\mathcal{F}},\tilde{\eta})$, we obtain six maps $\mu_j\colon\Sigma\longrightarrow\mathbb{P}^2$. None of these maps are equivalent, since we did not switch the two components of C_0. □

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