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ENUMERATION OF GENUS-THREE PLANE CURVES WITH A FIXED COMPLEX STRUCTURE

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Abstract

We give a practical formula for counting irreducible nodal genus-three plane curves with a fixed general complex structure on the normalization. As an intermediate step, we enumerate rational plane curves that have a $(3,4)$-cusp.

Contents

1. Introduction

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1.1. Background and Results. Let (Σ, j_Σ) be a nonsingular Riemann surface of genus-3 and let d be a positive integer. Denote by $\mathcal{H}_{\Sigma,d}(\mathbb{P}^2)$ the set of simple holomorphic maps from Σ to $\mathbb{P}^2$ of degree d. We call a holomorphic map $u : \Sigma \longrightarrow \mathbb{P}^2$ *simple* if $u : \Sigma \longrightarrow u(\Sigma)$ is a normalization map. Let $\mu = (\mu_1, \ldots, \mu_{3d-4})$ be a tuple of points in $\mathbb{P}^2$ in general position. Then the set

$$\mathcal{H}_{\Sigma,d}(\mu) = \big\{(y_1, \ldots, y_{3d-4}; u) \colon u \in \mathcal{H}_{\Sigma,d}(\mathbb{P}^2); \; y_l \in \Sigma,\; u(y_l) = \mu_l \;\forall l = 1, \ldots, 3d-4\big\} \tag{1.1}$$

is finite. For a dense open subset of complex structures j_Σ on Σ, the cardinality of this set is the same number $n_{3,d}$. More intrinsically, $n_{3,d}$ is the number of genus-3 degree-d plane curves that pass through $3d-4$ points in general position and have a pre-specified general complex structure on the normalization.

Enumerative numbers, such as $n_{3,d}$, have been of interest in algebraic geometry for a long time. The low-genus numbers, $n_d \equiv n_{0,d}$, $n_{1,d}$, and $n_{2,d}$ are computed in [KM], [RT], [I], [P1], [Z2], and [KQR] with completion in [Z1]. In this paper, we apply the machinery developed in [Z2] and [Z3] to compute the numbers $n_{3,d}$.

It is shown in Section 10 of [RT] that

$$n_d = \mathrm{RT}_{0,d}(\mu_1, \mu_2, \mu_3; \mu_4, \ldots, \mu_{3d-1}),$$

where $\mathrm{RT}_{0,d}(\cdot;\cdot)$ denotes the symplectic invariant of $\mathbb{P}^2$ as defined in [RT]. In [I], the difference

$$\mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_{3d-1}) - 2n_{1,d}$$

is shown to be a certain multiple of n_d. Extending the general approach of [I], in [Z2], the difference

$$\mathrm{RT}_{2,d}(; \mu_1, \ldots, \mu_{3d-2}) - 2n_{2,d}$$

is expressed in terms of the numbers $n_{d'}$ with $d' \le d$. Due to the two composition laws of [RT], the symplectic invariants $\mathrm{RT}_{g,d}(\cdot;\cdot)$ of $\mathbb{P}^2$ are easily computable. Thus, comparing enumerative invariants of $\mathbb{P}^2$ to the symplectic ones as above is sufficient for computing the enumerative invariants. In this paper, we prove

Theorem 1.1. *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_{3,d} = RT_{3,d}(;\mu) - CR_3(\mu), \qquad \textit{where}$$

$$\begin{aligned}\frac{1}{12}CR_3(\mu) =& \big\langle 413a^2c_1^2(\mathcal{L}^*) + 210ac_1^3(\mathcal{L}^*) + 44c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle + 18\big|\mathcal{V}_3(\mu)\big| \\ &- \big\langle 217a^2 + 84a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big) + 16\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big) \\ &\qquad + 10c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle.\end{aligned}$$

d	2	3	4	5	6	7
$n_{3,d}$	0	0	14,280	9,469,152	6,573,686,112	6,289,178,278,656

TABLE 1. The Numbers $n_{3,d}$ for Small Values of d

The notation used in Theorem 1.1 is the same as in [Z2]; see Subsection 2.2 for more details. For now, it is sufficient to say that the intersection numbers in Theorem 1.1 can be expressed in terms of intersection numbers of tautological classes in the space of stable rational maps into $\mathbb{P}^2$. The latter are shown to be computable in [P2]; see also Subsection 5.7 in [Z2]. Thus, Theorem 1.1 gives a practical formula for computing the numbers $n_{3,d}$. Our number $n_{3,4}$ agrees with that of [AF1].

Along the way, we also enumerate rational curves with certain singularities. In particular, let $\mathcal{S}_{1;2}(\mu)$ denote the set of plane rational degree-d curves that have a $(3,4)$-cusp and pass through the $3d-4$ points $\mu_1,\ldots,\mu_{3d-4}$. Corollary 3.5 expresses the number $|\mathcal{S}_{1;2}(\mu)|$ in terms of intersections of tautological classes in the space of rational maps. See also Lemmas 3.1, 3.2, and 3.3. The degree-four numbers of Lemma 3.1 and Corollary 3.5 agree with the numbers computed by P. Aluffi from a formula in [AF2].

The argument of this paper can be easily modified to enumerate genus-3 plane curves with any fixed non-hyperelliptic smooth complex structure on the normalization. Suppose (Σ, j) is not hyperelliptic and has n "hyperflexes" in the sense of [AF1], i.e. exactly n Weierstrass points with gap values $(1,2,5)$ and $24-2n$ Weierstrass points with gap values $(1,2,4)$; see [GH, p. 273]. Then the number of genus-3 degree-d plane curves passing through $3d-4$ points and with normalization (Σ, j) is $\big(n_{3,d}-2n|S_{1;2}(\mu)|\big)/\mathrm{Aut}(\Sigma, j)$; see the remarks following Corollary 2.13. In the $d=4$ case, the same correction is obtained in Subsection 3.6 of [AF1]. If (Σ, j) is hyperelliptic, the modifications required to compute the corresponding enumerative number would be significant; see Remark 3 after Corollary 2.13.

The previous paragraph raises the following natural question. Suppose (Σ_r, j_r) is a sequence of general genus-three complex surfaces that converges to a non-general non-hyperelliptic smooth surface (Σ, j). In other words, each surface Σ_r is smooth, non-hyperelliptic, and free of hyperflexes, while the surface Σ is smooth and non-hyperelliptic, but has $n>0$ hyperflexes. For each r, let $\{u_{r;k}\colon \Sigma_r \longrightarrow \mathbb{P}^2\}$ be the $n_{3,d}$-element set of holomorphic maps of degree-d that pass through the points $p_1, \ldots, p_{3d-4}$. The previous paragraph implies that $2n|S_{1;2}(\mu)|$ of the elements of the set $\{u_{r;k}\}$ "disappear" as r tends to infinity. The natural question then is what is the limit of these $2n|S_{1;2}(\mu)|$-element sets with respect to the stable-map, or Gromov-convergence, compactification of the space of holomorphic maps $\Sigma \longrightarrow \mathbb{P}^2$. The answer is the set of stable maps described as follows. The domain of each map has two irreducible components, Σ and $\mathbb{P}^1$. The singular point of Σ is one of the n hyperflexes. The map is constant on Σ and thus has degree d on the rational component. The image of each map is a rational curve that passes through the $3d-4$ points and has a $(3,4)$-cusp at the image of the principal component. A natural extension of the arguments in [P1], [KQR], and [Z1] to the genus-three case shows that every two-component stable map that appears in the limit must have the above form. By symmetry, we can then conclude that each stable map described above appears in the limit exactly twice. It should also be possible to reach the latter conclusion by adopting the argument of Section 5 of [Z1] to the present situation.

The author thanks P. Aluffi, T. Mrowka, and R. Vakil for helpful conversations. In particular, the question of enumerating genus-3 curves with a fixed non-general complex structure was posed to the author by P. Aluffi, who also pointed out that the conclusion of the original limiting argument for obtaining counts of such curves directly from the numbers $n_{3,d}$ could not have been be valid. The author also thanks the referee for very detailed comments and corrections concerning the original version of this paper.

1.2. Summary. We now outline the proof of Theorem 1.1. If $\nu \in \Gamma(\Sigma \times \mathbb{P}^2; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^2}^* T\mathbb{P}^2)$, let $\mathcal{M}_{\Sigma,\nu,d}$ denote the set of all smooth maps u from Σ to $\mathbb{P}^2$ of degree d such that $\bar{\partial} u|_z = \nu|_{(z,u(z))}$ for all $z \in \Sigma$. If μ is as above and $N = 3d-4$, put

$$\mathcal{M}_{\Sigma,\nu,d}(\mu) = \big\{(y_1, \ldots, y_N; u)\colon u \in \mathcal{M}_{\Sigma,\nu,d};\ y_l \in \Sigma,\ u(y_l) = \mu_l\ \forall l = 1, \ldots, N\big\}.$$

For a generic ν, $\mathcal{M}_{\Sigma,\nu,d}$ is a smooth finite-dimensional oriented manifold, and $\mathcal{M}_{\Sigma,\nu,d}(\mu)$ is a zero-dimensional finite submanifold of $\mathcal{M}_{\Sigma,\nu,d}\times\Sigma^N$, whose cardinality (with sign) is the symplectic invariant $\mathrm{RT}_{3,d}(;\mu)$. This number depends only on the degree d; see Section 2 of [RT].

If $\|\nu_i\|_{C^0}\longrightarrow 0$ and $(\underline{y}_i;u_i)\in\mathcal{M}_{\Sigma,\nu_i,d}(\mu)$, then a subsequence of $\{(\underline{y}_i;u_i)\}_{i=1}^{\infty}$ must converge in the Gromov topology to one of the following:
(1) an element of $\mathcal{H}_{\Sigma,d}(\mu)$;
(2) $(\Sigma_b,\underline{y},u_b)$, where Σ_b is a bubble tree of S^2's attached to Σ with marked points $y_1,\ldots,y_N$, and $u_b\colon\Sigma_b\longrightarrow\mathbb{P}^2$ is a holomorphic map such that $u_b(y_l)=\mu_l$ for $l=1,\ldots,N$, and
(2a) $u_b|\Sigma$ is simple and the tree contains at least one S^2;
(2b) $u_b|\Sigma$ is multiply-covered;
(2c) $u_b|\Sigma$ is constant and the tree contains at least one S^2.
By a dimension-counting argument similar to the proof of Proposition 6.6 in [Z2], the cases (2a) and (2b) cannot occur. As in [Z2], our approach will be to take $t>0$ very small and to determine the number $CR_3(\mu)$ of the elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near the maps of type (2c). The rest of the elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ must lie near the space $\mathcal{H}_{\Sigma,d}(\mu)$. By Proposition 3.30 in [Z3] and Corollary 6.5 in [Z2], there is a one-to-one correspondence between the elements of $\mathcal{H}_{\Sigma,d}(\mu)$ and the nearby elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$, at least if $d\geq 5$. In fact, a standard argument shows that Corollary 6.5 remains valid if $d=4$ and (Σ,j) is general. If $d=1,2,3$, $\mathcal{H}_{\Sigma,d}(\mu)=\emptyset$ by [ACGH, p. 116]. Thus, we are able to compute the cardinality of $\mathcal{H}_{\Sigma,d}(\mu)$ by computing the total number $CR_3(\mu)$ of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near the maps of type (2c).

The space of stable maps of type (2c) is stratified by smooth, usually non-compact, manifolds $\mathcal{M}_{\mathcal{T}}(\mu)$. The set of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near each space $\mathcal{M}_{\mathcal{T}}(\mu)$ corresponds to the zero set of a map between two bundles over $\mathcal{M}_{\mathcal{T}}(\mu)$. By extracting dominant terms from each such map, the signed cardinality $N(\mathcal{T})$ of the zero set of the map can be identified with the signed cardinality of the zero set of an affine map between vector bundles over a closure of $\mathcal{M}_{\mathcal{T}}(\mu)$ or of a certain submanifold of $\mathcal{M}_{\mathcal{T}}(\mu)$. The argument is nearly the same as in Sections 2 and 4 of [Z2]. It is summarized briefly at the end of Subsection 2.4. The number $CR_3(\mu)$, the sum of the numbers $N(\mathcal{T})$, can then be expressed as the sum of the cardinalities of the zero sets of affine vector bundles over compact manifolds; see Corollary 2.13. Topological formulas for the six numbers $n_m^{(k)}(\mu)$ of Corollary 2.13 are obtained in Section 4; see Lemmas 4.9, 4.8, 4.1, 4.4, 4.2, and 4.3, respectively. Using the results of Lemmas 3.1, 3.3, and 3.4, we obtain the expression for $CR_3(\mu)$ given in

Theorem 1.1.

In Subsection 2.1, we review the topological tools to be used in Sections 3 and 4. We summarize our notation for spaces of bubble maps and vector bundles over them in Subsection 2.2. Subsection 2.3 describes the structure of spaces of rational maps and the behavior of certain bundle sections over them near the boundary strata. These descriptions are needed to implement the topological tools of Subsection 2.1 in Sections 3 and 4. In Subsection 2.4, we describe the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ near each given strata $\mathcal{M}_{\mathcal{T}}(\mu)$ of bubble maps of type (2c) in terms of the zero sets of affine maps between finite-rank vector bundles over relatively simple topological spaces.

In Section 3, we enumerate rational curves with certain singularities and also compute the intersection numbers used in Section 4. Rational curves with singularities can be identified with the zeros of bundle sections over spaces of stable rational maps that lie in the main stratum. We use the topological tools of Subsection 2.1 to determine the contribution from the boundary strata of such spaces to the euler class of the bundle. Finally, in Section 4, we derive topological formulas for the six numbers of Corollary 2.13 using the same approach as in Section 3.

In a certain sense, the number $CR_3(\mu)$ of Theorem 1.1 can be viewed as the infinite-dimensional analogue of the boundary contribution to the euler class of a vector bundle. Similarly to the finite-dimensional case, $CR_3(\mu)$ is the number of zeros of a small perturbation of the section $\bar{\partial}$ that lie near the boundary strata. However, an important difference with the finite-dimensional case is that "the euler class" of the infinite-rank vector bundle involved depends on the Fredholm-homotopy class of the section $\bar{\partial}$. Construction of such an euler class, in a much more general setting, is the subject of [FO] and [LT].

2. The Computational Setting

2.1. Topology. We begin by describing the topological tools used in the next two sections. In particular, we review the notion of contribution to the euler class of a vector bundle from a (not necessarily closed) subset of the zero set of a section. We also recall how one can enumerate the zeros of an affine map between vector bundles. These concepts are closely intertwined. Details can be found in Section 3 of [Z2].

Throughout this paper, all vector bundles are assumed to be complex and normed. If $F \longrightarrow \mathcal{M}$ is a smooth vector bundle, closed subset Y of F is *small* if it contains no fiber of F and is preserved under scalar multiplication. If $\mathcal{Z}$ is a compact oriented zero-dimensional manifold, we denote the signed cardinality of $\mathcal{Z}$ by ${}^{\pm}|\mathcal{Z}|$.

Definition 2.1. *Suppose $F, \mathcal{O} \longrightarrow \mathcal{M}$ are smooth vector bundles.*
(1) If $F = \bigoplus_{i=1}^{i=k} F_i$, bundle map $\alpha\colon F \longrightarrow \mathcal{O}$ is a polynomial of degree $d_{[k]}$ if for each $i \in [k]$ there exists

$$p_i \in \Gamma(\mathcal{M}; F_i^{*\otimes d_i} \otimes \mathcal{O}) \text{ for } i \in [k]$$

such that

$$\alpha(\upsilon) = \sum_{i=1}^{i=k} p_i\big(\upsilon_i^{d_i}\big) \quad \forall \upsilon = (\upsilon_i)_{i\in[k]} \in \bigoplus_{i=1}^{i=k} F_i.$$

(2) If $\alpha\colon F \longrightarrow \mathcal{O}$ is a polynomial, the rank of α is the number

$$rk\ \alpha \equiv \max\{rk_b\alpha : b \in \mathcal{M}\}, \quad \text{where} \quad rk_b\alpha = \dim_{\mathbb{C}}\big(Im\ \alpha_b\big).$$

The polynomial $\alpha\colon F \longrightarrow \mathcal{O}$ is of constant rank if $rk_b\alpha = rk\ \alpha$ for all $b \in \mathcal{M}$; α is nondegenerate if $rk_b\alpha = rk\ F$ for all $b \in \mathcal{M}$.
(3) If Ω is an open subset of F and $\phi : \Omega \longrightarrow \mathcal{O}$ is a smooth bundle map, bundle map $\alpha\colon F \longrightarrow \mathcal{O}$ is a dominant term of ϕ if there exists $\varepsilon \in C^0(F;\mathbb{R})$ such that

$$\big|\phi(\upsilon) - \alpha(\upsilon)\big| \le \varepsilon(\upsilon)|\alpha(\upsilon)| \quad \forall \upsilon \in \Omega \quad \text{and} \quad \lim_{\upsilon \longrightarrow 0} \varepsilon(\upsilon) = 0.$$

Dominant term $\alpha\colon F \longrightarrow \mathcal{O}$ of ϕ is the resolvent of ϕ if α is a polynomial of constant rank.
(4) $\phi\colon \Omega \longrightarrow \mathcal{O}$ is hollow if there exist a dominant term α of ϕ and splittings $F = F^- \oplus F^+$ and $\mathcal{O} = \mathcal{O}^- \oplus \mathcal{O}^+$ such that $\alpha(F^+) \subset \mathcal{O}^+$, $\alpha^- \equiv \pi^- \circ (\alpha|F^-)$ is a constant-rank polynomial, where $\pi^- : \mathcal{O} \longrightarrow \mathcal{O}^-$ is the projection map, and $(rk\ \alpha^- + \frac{1}{2}\dim \mathcal{M}) < rk\ \mathcal{O}^-$.

The base spaces we work with in the next two sections are closely related to spaces of rational maps into $\mathbb{P}^2$ of total degree d that pass through the N points $\mu_1, \ldots, \mu_N$, where $N = 3d-4$ as before. From the algebraic geometry point of view, spaces of rational maps are algebraic stacks, but with a fairly obscure local structure. We view these spaces as *mostly smooth*, or *ms*-, manifolds: compact oriented topological manifolds stratified by smooth manifolds, such that the boundary strata have (real) codimension at least two. Subsection 2.3 gives explicit descriptions of neighborhoods of boundary strata and of the behavior of certain bundle sections near such strata. We call the main stratum $\mathcal{M}$ of ms-manifold $\bar{\mathcal{M}}$ the *smooth base* of $\bar{\mathcal{M}}$. Definition 3.7 in [Z2] also introduces the natural notions of *ms-maps* between ms-manifolds, *ms-bundles* over ms-manifolds, and *ms-sections* of ms-bundles.

Definition 2.2. *Let $\bar{\mathcal{M}} = \mathcal{M}_n \sqcup \bigsqcup_{i=0}^{n-2} \mathcal{M}_i = \mathcal{M} \sqcup \bigsqcup_{i=0}^{n-2} \mathcal{M}_i$ be an ms-manifold of dimension n.*
(1) If $\mathcal{Z} \subset \mathcal{M}_i$ is a smooth oriented submanifold, a normal-bundle model for $\mathcal{Z}$ is an ordered triple (F, Y, ϑ), where
(1a) $F \longrightarrow \mathcal{Z}$ is a smooth vector bundle and Y is a small subset of F;
(1b) for some $\delta \in C^\infty(\mathcal{Z}; \mathbb{R}^+)$, $\vartheta\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow \bar{\mathcal{M}}$ is a continuous map such that
(1b-i) $\vartheta\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow \bar{\mathcal{M}}$ is a homeomorphism onto an open neighborhood of $\mathcal{Z}$ in $\mathcal{M} \cup \mathcal{Z}$;
(1b-ii) $\vartheta|_{\mathcal{Z}}$ is the identity map, and $\vartheta\colon F_\delta - Y - \mathcal{Z} \longrightarrow \mathcal{M}$ is an orientation-preserving diffeomorphism on an open subset of $\mathcal{M}$.
(2) A closure of a normal-bundle model (F, Y, ϑ) for $\mathcal{Z}$ is an ordered triple $(\bar{\mathcal{Z}}, \tilde{F}, \pi)$, where
(2a) $\bar{\mathcal{Z}}$ is an ms-manifold with smooth base $\mathcal{Z}$;
(2b) $\pi\colon \bar{\mathcal{Z}} \longrightarrow \bar{\mathcal{M}}$ is an ms-map such that $\pi|_{\mathcal{Z}}$ is the identity;
(2c) $\tilde{F} \longrightarrow \bar{\mathcal{Z}}$ is an ms-bundle such that $\tilde{F}|_{\mathcal{Z}} = F$.

We use a normal-bundle model for $\mathcal{Z}$ to describe the behavior of bundle sections over $\bar{\mathcal{M}}$ near $\mathcal{Z}$. In particular, if $\alpha\colon E \longrightarrow \mathcal{O}$ is an ms-polynomial, we call $\mathcal{Z}$ an *α-regular* subset of $\bar{\mathcal{M}}$ if for some normal-bundle model (F, Y, ϑ) for $\mathcal{Z}$, $\vartheta^*\alpha$ can be approximated, by a constant-rank polynomial $p\colon F \oplus E \longrightarrow \mathcal{O}$; see Definition 3.9 in [Z2]. The polynomial $\alpha\colon E \longrightarrow \mathcal{O}$ is *regular* if $\bar{\mathcal{M}}$ can be decomposed into finitely many α-regular subsets. If $\text{rk } E + \frac{1}{2}\dim \bar{\mathcal{M}} = \text{rk } \mathcal{O}$, for a generic $\nu \in \Gamma(\bar{\mathcal{M}}; \mathcal{O})$, the zero set of the polynomial map

$$\psi_{\alpha,\nu}\colon E \longrightarrow \mathcal{O}, \qquad \psi_{\alpha,\nu}(\upsilon) = \nu_\upsilon + \alpha(\upsilon),$$

is a zero-dimensional oriented submanifold of $E|\mathcal{M}$. By Lemma 3.10 in [Z2], if α is a regular polynomial, $\psi_{\alpha,\nu}^{-1}(0)$ is a finite set for a generic choice of ν, and $N(\alpha) \equiv^{\pm} \big|\psi_{\alpha,\nu}^{-1}(0)\big|$ is independent of such a choice of ν.

In Section 3, rational curves with pre-specified singularities are identified with the zero set of a section s of a vector bundle V over a smooth manifold $\mathcal{M}$. The section s extends over a compactification $\bar{\mathcal{M}}$ of $\mathcal{M}$. Thus, the subset of $\mathcal{M}$ we are interested in can be identified with the euler class $e(V)$ of V minus the s-contribution $\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{M}}(s)$ to $e(V)$ from $\bar{\mathcal{M}} - \mathcal{M}$. In the cases we encounter in Sections 3 and 4, $s^{-1}(0)\cap(\bar{\mathcal{M}}-\mathcal{M})$ decomposes into disjoint, and usually non-compact, manifolds $\mathcal{Z}_i$ near which the behavior of s can be understood. Then $\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{M}}(s)=\sum\mathcal{C}_{\mathcal{Z}_i}(s)$, where $\mathcal{C}_{\mathcal{Z}_i}(s)$ is the *s-contribution of* $\mathcal{Z}_i$ *to* $e(V)$. This is the signed number of elements of $\{s+\nu\}^{-1}(0)$ that lie very close to $\mathcal{Z}_i$, where $\nu\in\Gamma(\bar{\mathcal{M}};V)$ is a small generic perturbation of s. The manifolds $\mathcal{Z}_i$ we encounter fall in one of the two categories described below.

Definition 2.3. *Suppose $\bar{\mathcal{M}}$ is an ms-manifold of dimension $2n$, $V\longrightarrow\bar{\mathcal{M}}$ is an ms-bundle of rank n, $s\in\Gamma(\bar{\mathcal{M}};V)$, and $\mathcal{Z}\subset s^{-1}(0)$.*
(1) $\mathcal{Z}$ is <u>s-hollow</u> if there exist a normal-bundle model (F,Y,ϑ) for $\mathcal{Z}$ and a bundle isomorphism $\vartheta_V\colon\vartheta^*V\longrightarrow\pi_F^*V$, covering the identity on $F_\delta-(Y-\mathcal{Z})$, such that
(1a) $\vartheta_V|_{F_\delta-Y-\mathcal{Z}}$ is smooth and $\vartheta_V|_{\mathcal{Z}}$ is the identity;
(1b) the map $\phi\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-\mathcal{Z})\longrightarrow V$ is hollow.
(2) $\mathcal{Z}$ is <u>s-regular</u> if there exist a normal-bundle model (F,Y,ϑ) for $\mathcal{Z}$ with closure $(\bar{\mathcal{Z}},\tilde{F},\pi)$, a regular polynomial $\alpha\colon\tilde{F}\longrightarrow\pi^*V$, and a bundle isomorphism $\vartheta_V\colon\vartheta^*V\longrightarrow\pi_F^*V$ covering the identity on $F_\delta-(Y-\mathcal{Z})$, such that
(2a) $\vartheta_V|_{F_\delta-Y-\mathcal{Z}}$ is smooth and $\vartheta_V|_{\mathcal{Z}}$ is the identity;
(2b) $\alpha|_{\mathcal{Z}}$ is nondegenerate and is the resolvent for

$$\phi\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-\mathcal{Z})\longrightarrow V,$$

and Y is preserved under scalar multiplication in each of the components of F for the splitting corresponding to α as in (1) of Definition 2.1.

Proposition 2.4. *Let $V\longrightarrow\bar{\mathcal{M}}$ be an ms-bundle of rank n over an ms-manifold of dimension $2n$. Suppose $\mathcal{U}$ is an open subset of $\mathcal{M}$ and $s\in\Gamma(\bar{\mathcal{M}};V)$ is such that $s|_{\mathcal{U}}$ is transversal to the zero set.*
(1) If $s^{-1}(0)\cap\mathcal{U}$ is a finite set, $^{\pm}|s^{-1}(0)\cap\mathcal{U}|=\langle e(V),[\bar{\mathcal{M}}]\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)$.
(2) If $\bar{\mathcal{M}}-\mathcal{U}=\bigsqcup_{i=1}^{i=k}\mathcal{Z}_i$, where each $\mathcal{Z}_i$ is s-regular or s-hollow, then $s^{-1}(0)\cap\mathcal{U}$ is finite, and

$$^{\pm}\big|s^{-1}(0)\cap\mathcal{U}\big|=\big\langle e(V),[\bar{\mathcal{M}}]\big\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)=\big\langle e(V),[\bar{\mathcal{M}}]\big\rangle-\sum_{i=1}^{i=k}\mathcal{C}_{\mathcal{Z}_i}(s).$$

If $\mathcal{Z}_i$ is s-hollow, $\mathcal{C}_{\mathcal{Z}_i}(s)=0$. If $\mathcal{Z}_i$ is s-regular and $\alpha_i\colon\tilde{F}_i\longrightarrow V$ is the corresponding polynomial,

$$\mathcal{C}_{\mathcal{Z}_i}(s)={}^{\pm}\big|\{\upsilon\in\tilde{F}_i\colon\bar{\nu}_\upsilon+\alpha_i(\upsilon)=0\}\big|\equiv N(\alpha_i),$$

where $\bar{\nu} \in \Gamma(\bar{\mathcal{Z}}_i; V)$ *is a generic section. Finally, if* $\alpha_i \in \Gamma(\bar{\mathcal{Z}}_i; \tilde{F}_i^{*\otimes k} \otimes \pi^* V)$ *has constant rank over* $\bar{\mathcal{Z}}_i$ *and factors through a* $\tilde{k}$*-to-*1 *cover* $\rho_i \colon \tilde{F}_i \longrightarrow \tilde{F}_i^{\otimes k}$,

$$\mathcal{C}_{\mathcal{Z}_i}(s) = \tilde{k}\big\langle e\big(\pi^* V/\alpha_i(\tilde{F}_i)\big), [\bar{\mathcal{Z}}_i]\big\rangle.$$

This is Corollary 3.13 in [Z2]. Proposition 2.4 reduces the problem of computing $\mathcal{C}_{\mathcal{Z}_i}(s)$ for an s-regular manifold $\mathcal{Z}_i$ to counting the zeros of a polynomial map between two vector bundles. The general setting for the latter problem is the following. Suppose $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles such that $\text{rk } E + \frac{1}{2}\dim \bar{\mathcal{M}} = \text{rk } \mathcal{O}$, and $\alpha : E \longrightarrow \mathcal{O}$ is a regular polynomial. Let $\bar{\nu} \in \Gamma(\bar{\mathcal{M}}; \mathcal{O})$ be such that the map

$$\psi_{\alpha,\bar{\nu}} \equiv \bar{\nu} + \alpha \colon E \longrightarrow \mathcal{O}$$

is transversal to the zero set in $\mathcal{O}$ on $E|\mathcal{M}$, and all its zeros are contained in $E|\mathcal{M}$. Then $N(\alpha) \equiv^{\pm} |\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ depends only on α. If the rank of E is zero, then clearly

$$N(\alpha) =^{\pm} \big|\psi_{\alpha,\bar{\nu}}^{-1}(0)\big| = \big\langle e(\mathcal{O}), [\bar{\mathcal{M}}]\big\rangle.$$

If the rank of E is positive and $\bar{\nu}$ is generic, it does not vanish and thus determines a trivial line subbundle $\mathbb{C}\bar{\nu}$ of $\mathcal{O}$. Let $\mathcal{O}^{\perp} = \mathcal{O}/\mathbb{C}\bar{\nu}$ and denote by $\alpha^{\perp}$ the composition of α with the quotient projection map. If E is a line bundle and α is linear,

$$N(\alpha) =^{\pm} \big|\psi_{\alpha,\bar{\nu}}^{-1}(0)\big| = \big\langle e(E^* \otimes \mathcal{O}^{\perp}), [\bar{\mathcal{M}}]\big\rangle - \mathcal{C}_{\alpha^{-1}(0)}(\alpha^{\perp}).$$

By Proposition 2.4, the computation of $\mathcal{C}_{\alpha^{-1}(0)}(\alpha^{\perp})$ again involves counting the zeros of polynomial maps, but with the rank of the new target bundle, i.e. $E^* \otimes \mathcal{O}^{\perp}$, one less than the rank of the original one, i.e. $\mathcal{O}$. Subsection 3.3 in [Z2] reduces the problem of determining $N(\alpha)$ in all other cases to the case where E is a line bundle and α is linear. Thus, at least in reasonably good cases, $N(\alpha)$ can be determined after a finite number of steps.

The next lemma summarizes the results of Subsection 3.3 in [Z2] in the case where the original map $\alpha : E \longrightarrow \mathcal{O}$ is linear. This case suffices for our purposes. We denote by $\alpha_E \in \Gamma(\mathbb{P}E; \gamma_E^* \otimes \pi_E^* \mathcal{O})$ the section induced by α. Let $\lambda_E = c_1(\gamma_E^*)$.

Lemma 2.5. *Suppose* $\bar{\mathcal{M}}$ *is an ms-manifold and* $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ *are ms-bundles such that*

$$\textit{rk } E + \frac{1}{2}\dim \bar{\mathcal{M}} = \textit{rk } \mathcal{O}.$$

If $\alpha\in\Gamma(\bar{\mathcal{M}};E^*\otimes\mathcal{O})$ *and* $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ *are such that* α *is regular,* $\bar{\nu}$ *has no zeros, the map*

$$\psi_{\alpha,\bar{\nu}}\equiv\bar{\nu}+\alpha\colon E\longrightarrow\mathcal{O}$$

is transversal to the zero set on $E|\mathcal{M}$*, and all its zeros are contained in* $E|\mathcal{M}$*, then* $\psi_{\alpha,\bar{\nu}}^{-1}(0)$ *is a finite set,* ${}^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ *depends only on* α*, and*

$$N(\alpha)\equiv^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|=\big\langle c(\mathcal{O})c(E)^{-1},[\bar{\mathcal{M}}]\big\rangle-\mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}).$$

2.2. Review of Notation. In this subsection, we give a brief description of the most important notation used in this paper. See Section 2 in [Z3] for more details.

Let $q_N,q_S\colon\mathbb{C}\longrightarrow S^2\subset\mathbb{R}^3$ be the stereographic projections mapping the origin in $\mathbb{C}$ to the north and south poles, respectively. Denote the south pole of S^2, i.e. the point $(0,0,-1)\in\mathbb{R}^3$, by ∞. We identify $\mathbb{C}$ with $S^2-\{\infty\}$ via the map q_N.

If N is any nonnegative integer, let $[N]=\{1,\ldots,N\}$. If I_1 and I_2 are two sets, we denote the disjoint union of I_1 and I_2 by I_1+I_2.

Definition 2.6. *A finite partially ordered set* I *is a* <u>*linearly ordered set*</u> *if for all* $i_1,i_2,h\in I$ *such that* $i_1,i_2<h$*, either* $i_1\le i_2$ *or* $i_2\le i_1$*.*
A linearly ordered set I *is a* <u>*rooted tree*</u> *if* I *has a unique minimal element, i.e. there exists* $\hat{0}\in I$ *such that* $\hat{0}\le i$ *for all* $i\in I$*.*

If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h\in\hat{I}$, denote by $\iota_h\in I$ the largest element of I which is smaller than h. We call $\iota:\hat{I}\longrightarrow I$ the attaching map of I. Suppose $I=\bigsqcup_{k\in K}I_k$ is the splitting of I into rooted trees such that k is the minimal element of I_k. If $\hat{1}\notin I$, we define the linearly ordered set $I+_k\hat{1}$ to be the set $I+\hat{1}$ with all partial-order relations of I along with the relations

$$k<\hat{1},\qquad \hat{1}<h \text{ if } h\in\hat{I}_k.$$

If I is a rooted tree, we write $I+\hat{1}$ for $I+_k\hat{1}$.

If $S=\Sigma$ or $S=S^2$ and M is a finite set, a $\mathbb{P}^2$*-valued bubble map with* M*-marked points* is a tuple

$$b=\big(S,M,I;x,(j,y),u\big),$$

where I is a linearly ordered set, and

$$x\colon \hat{I}\longrightarrow S\cup S^2,\ j\colon M\longrightarrow I,\ y\colon M\longrightarrow S\cup S^2,$$
$$\text{and}\qquad u\colon I\longrightarrow C^\infty(S;\mathbb{P}^2)\cup C^\infty(S^2;\mathbb{P}^2)$$

are maps such that

$$x_h\in\begin{cases}S^2-\{\infty\}, & \text{if } \iota_h\in\hat{I};\\ S, & \text{if } \iota_h\notin\hat{I},\end{cases}\qquad y_l\in\begin{cases}S^2-\{\infty\}, & \text{if } j_l\in\hat{I};\\ S, & \text{if } j_l\notin\hat{I},\end{cases}$$
$$u_i\in\begin{cases}C^\infty(S^2;\mathbb{P}^2), & \text{if } i\in\hat{I};\\ C^\infty(S;\mathbb{P}^2), & \text{if } i\notin\hat{I},\end{cases}$$

and $u_h(\infty)=u_{\iota_h}(x_h)$ for all $h\in\hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b=\Big(\bigsqcup_{i\in I}\Sigma_{b,i}\Big)\Big/\sim,\qquad\text{where}$$
$$\Sigma_{b,i}=\begin{cases}\{i\}\times S^2, & \text{if } i\in\hat{I};\\ \{i\}\times S, & \text{if } i\notin\hat{I},\end{cases}\quad\text{and}\quad (h,\infty)\sim(\iota_h,x_h)\ \forall h\in\hat{I},$$

with marked points $(j_l,y_l)\in\Sigma_{b,j_l}$, and continuous map $u_b\colon\Sigma_b\longrightarrow\mathbb{P}^2$, given by $u_b|\Sigma_{b,i}=u_i$ for all $i\in I$. We require that all the singular points of Σ_b, i.e. $(\iota_h,x_h)\in\Sigma_{b,\iota_h}$ for $h\in\hat{I}$, and all the marked points be distinct. Furthermore, if $S=S^2$, all these points are to be different from the special marked point $(\hat{0},\infty)\in\Sigma_{b,\hat{0}}$. In addition, if $\Sigma_{b,i}=S^2$ and $u_{i*}[S^2]=0\in H_2(\mathbb{P}^2;\mathbb{Z})$, then $\Sigma_{b,i}$ must contain at least two singular and/or marked points of Σ_b other than (i,∞). Two bubble maps b and b' are *equivalent* if there exists a homeomorphism $\phi\colon\Sigma_b\longrightarrow\Sigma_{b'}$ such that $u_b=u_{b'}\circ\phi$, $\phi(j_l,y_l)=(j_l',y_l')$ for all $l\in M$, $\phi|_{\Sigma_{b,i}}$ is holomorphic for all $i\in I$, and $\phi|_{\Sigma_{b,i}}=Id$ if $S=\Sigma$ and $i\in I-\hat{I}$.

The general structure of bubble maps is described by tuples

$$\mathcal{T}=(S,M,I;j,\underline{d}),$$

with $d_i\in\mathbb{Z}$ describing the degree of the map u_b on $\Sigma_{b,i}$. We call such tuples *bubble types*. Bubble type $\mathcal{T}$ is *simple* if I is a rooted tree; $\mathcal{T}$ is *is basic* if $\hat{I}=\emptyset$; $\mathcal{T}$ is *semiprimitive* if $\iota_h\notin\hat{I}$ for all $h\in\hat{I}$. We call the semiprimitive bubble type $\mathcal{T}$ *primitive* if $j_l\in\hat{I}$ for all $j_l\in M$. The above equivalence relation on the set of bubble maps induces an equivalence relation on the set of bubble types.

For each $h,i\in I$, let

$$H_i\mathcal{T}=\{h\in\hat{I}:\iota_h=i\},\quad M_i\mathcal{T}=\{l\in M:j_l=i\},$$

$$\chi_{\mathcal{T}}h=\begin{cases}0, & \text{if } d_i=0\ \forall i\in I \text{ s.t. } i\le h;\\ 1, & \text{if } d_h\neq 0, \text{ but } d_i=0\ \forall i\in I \text{ s.t. } i<h;\\ 2, & \text{otherwise};\end{cases}$$

$$\chi(\mathcal{T})=\big\{h\in I:\chi_{\mathcal{T}}h=1\big\}.$$

Denote by $\mathcal{H}_{\mathcal{T}}$ the space of all holomorphic bubble maps with structure $\mathcal{T}$.

The automorphism group of every bubble type $\mathcal{T}$ we encounter in the next two sections is trivial. Thus, every bubble type discussed below is presumed to be automorphism-free.

If $S=\Sigma$, we denote by $\mathcal{M}_{\mathcal{T}}$ the set of equivalence classes of bubble maps in $\mathcal{H}_{\mathcal{T}}$. Then there exists $\mathcal{M}_{\mathcal{T}}^{(0)}\subset\mathcal{H}_{\mathcal{T}}$ such that $\mathcal{M}_{\mathcal{T}}$ is the quotient of $\mathcal{M}_{\mathcal{T}}^{(0)}$ by an $(S^1)^{\hat{I}}$-action. Corresponding to this action, we obtain $|\hat{I}|$ line orbi-bundles $\{L_h\mathcal{T}\longrightarrow\mathcal{M}_{\mathcal{T}}:h\in\hat{I}\}$. The bundle of gluing parameters in the case $S=\Sigma$ is

$$F\mathcal{T}=\bigoplus_{h\in\hat{I}}F_h\mathcal{T},\ \text{where}\qquad F_{h,[b]}\mathcal{T}=\begin{cases}L_{h,[b]}\mathcal{T}\otimes L^*_{\iota_h,[b]}\mathcal{T}, & \text{if } \iota_h\in\hat{I};\\ L_{h,[b]}\mathcal{T}\otimes T_{x_h}\Sigma, & \text{if } \iota_h\notin\hat{I}.\end{cases}$$

Let $F^{\emptyset}\mathcal{T}=\{\upsilon=(\upsilon_h)_{h\in\hat{I}}\in F\mathcal{T}:\upsilon_h\neq 0\ \forall h\in\hat{I}\}$.

For each bubble type $\mathcal{T}=(S^2,M,I;j,\underline{d})$, let

$$\mathcal{U}_{\mathcal{T}}=\big\{[b]:b=\big(S^2,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}},\ u_{i_1}(\infty)=u_{i_2}(\infty)\ \forall i_1,i_2\in I-\hat{I}\big\}.$$

Similarly to the $S=\Sigma$ case above, $\mathcal{U}_{\mathcal{T}}$ is the quotient of a subset $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{H}_{\mathcal{T}}$ by a $\tilde{G}_{\mathcal{T}}\equiv(S^1)^I$-action. Denote by $\mathcal{U}_{\mathcal{T}}^{(0)}$ the quotient of $\mathcal{B}_{\mathcal{T}}$ by $G_{\mathcal{T}}\equiv(S^1)^{\hat{I}}\subset\tilde{G}_{\mathcal{T}}$. Then $\mathcal{U}_{\mathcal{T}}$ is the quotient of $\mathcal{U}_{\mathcal{T}}^{(0)}$ by the residual $G^*_{\mathcal{T}}\equiv(S^1)^{I-\hat{I}}\subset\tilde{G}_{\mathcal{T}}$ action. Corresponding to these quotients, we obtain line orbi-bundles $\{L_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}:h\in\hat{I}\}$ and $\{L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}:i\in I\}$. Let

$$F\mathcal{T}=\bigoplus_{h\in\hat{I}}F_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)},\ \text{where}\quad F_{h,[b]}\mathcal{T}=\begin{cases}L_{h,[b]}\mathcal{T}\otimes L^*_{\iota_h,[b]}\mathcal{T}, & \text{if } \iota_h\in\hat{I};\\ L_{h,[b]}\mathcal{T}, & \text{if } \iota_h\notin\hat{I};\end{cases}$$

$$\mathcal{F}\mathcal{T}=\bigoplus_{h\in\hat{I}}\mathcal{F}_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}},\ \text{where}\quad \mathcal{F}_{h,[b]}\mathcal{T}=L_{h,[b]}\mathcal{T}\otimes L^*_{\iota_h,[b]}\mathcal{T}.$$

The bundle of gluing parameters in the case $S=S^2$ is $\mathcal{F}\mathcal{T}$.

The Gromov-convergence topology on the space of equivalence classes of bubble maps induces a partial ordering on the set of bubble types and their equivalence classes such that the spaces

$$\bar{\mathcal{M}}_{\mathcal{T}} = \bigcup_{\mathcal{T}'\le\mathcal{T}} \mathcal{M}_{\mathcal{T}'}, \quad \bar{\mathcal{U}}^{(0)}_{\mathcal{T}} = \bigcup_{\mathcal{T}'\le\mathcal{T}} \mathcal{U}^{(0)}_{\mathcal{T}'}, \quad \text{and} \quad \bar{\mathcal{U}}_{\mathcal{T}} = \bigcup_{\mathcal{T}'\le\mathcal{T}} \mathcal{U}_{\mathcal{T}'}$$

are compact and Hausdorff. The $G^*_{\mathcal{T}}$-action on $\mathcal{U}^{(0)}_{\mathcal{T}}$ extends to an action on $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}}$, and thus line orbi-bundles $L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}$ with $i\in I-\hat{I}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}$. The evaluation maps

$$\mathrm{ev}_l\colon \mathcal{H}_{\mathcal{T}} \longrightarrow \mathbb{P}^2, \quad \mathrm{ev}_l\big((S,M,I;x,(j,y),u)\big) = u_{j_l}(y_l),$$

descend to all the quotients and induce continuous maps on $\bar{\mathcal{M}}_{\mathcal{T}}, \bar{\mathcal{U}}_{\mathcal{T}}$, and $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}}$. If $\mu=\mu_M$ is an M-tuple of submanifolds of $\mathbb{P}^2$, let

$$\mathcal{M}_{\mathcal{T}}(\mu) = \big\{b\in\mathcal{M}_{\mathcal{T}}\colon \mathrm{ev}_l(b)\in\mu_l \ \forall l\in M\big\}$$

and define spaces $\mathcal{U}_{\mathcal{T}}(\mu)$, $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, etc. in a similar way. If $S=S^2$, we define another evaluation map,

$$\mathrm{ev}\colon \mathcal{B}_{\mathcal{T}} \longrightarrow \mathbb{P}^2 \quad \text{by} \quad \mathrm{ev}\big((S^2,M,I;x,(j,y),u)\big) = u_{\hat{0}}(\infty),$$

where $\hat{0}$ is any minimal element of I. This map descends to $\mathcal{U}^{(0)}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{T}}$. If $\mu=\mu_{\tilde{M}}$ is an $\tilde{M}$-tuple of constraints, let

$$\mathcal{U}_{\mathcal{T}}(\mu) = \big\{b\in\mathcal{U}_{\mathcal{T}}\colon \mathrm{ev}_l(b)\in\mu_l \ \forall l\in M\cap\tilde{M}, \ \mathrm{ev}(b)\in\mu_l \ \forall l\in M-\tilde{M}\big\}$$

and define $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$, etc. similarly. If $S=\Sigma$, $\mathcal{T}$ is a simple bubble type, and $d_{\hat{0}}=0$, define

$$\mathrm{ev}\colon \mathcal{H}_{\mathcal{T}} \longrightarrow \mathbb{P}^2 \quad \text{by} \quad \mathrm{ev}\big((\Sigma,M,I;x,(j,y),u)\big) = u_{\hat{0}}(\Sigma).$$

This map is well-defined, since $u_{\hat{0}}$ is a degree-zero holomorphic map and thus is constant.

Suppose $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a bubble type, $\big\{\mathcal{T}_k=(S^2,M_k,I_k;j_k,\underline{d}_k)\big\}$ are the corresponding simple types (see [Z3]), $k\in I-\hat{I}$, and M_0 is a nonempty subset of $M_k\mathcal{T}$. Let

$$\mathcal{T}/M_0 = \big(S^2, I, M-M_0; j|(M-M_0), \underline{d}\big).$$

Define $\mathcal{T}(M_0)\equiv(S^2,M,I+_k\hat{1};j',\underline{d}')$ by

$$j'_l = \begin{cases} k, & \text{if } l\in M_0;\\ \hat{1}, & \text{if } l\in M_k\mathcal{T}-M_0;\\ j_l, & \text{otherwise;}\end{cases} \qquad d'_i = \begin{cases} 0, & \text{if } i=k;\\ d_k, & \text{if } i=\hat{1};\\ d_i, & \text{otherwise.}\end{cases}$$

The tuples $\mathcal{T}/M_0$ and $\mathcal{T}(M_0)$ are bubble types as long as $d_k\neq 0$ or $M_0\neq M_{\hat{0}}\mathcal{T}$. Then,

$$\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)=\bar{\mathcal{M}}_{0,\{\hat{1}\}+M_0}\times\bar{\mathcal{U}}_{\mathcal{T}/M_0}(\mu), \tag{2.1}$$

where $\bar{\mathcal{M}}_{0,\{\hat{1}\}+M_0}$ denotes the Deligne-Mumford moduli space of rational curves with $(\{\hat{0},\hat{1}\}+M_0)$-marked points. If $l\in M_k\mathcal{T}$ for some $k\in I-\hat{I}$, we denote $\mathcal{T}(\{l\})$ by $\mathcal{T}(l)$. Let

$$c_1(\mathcal{L}_k^*\mathcal{T})\equiv c_1(L_k^*\mathcal{T})-\sum_{M_0\subset M_k,M_0\neq\emptyset}PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]\in H^2\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big). \tag{2.2}$$

If the constraints μ are disjoint, $\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)=\emptyset$ if $|M_0|\geq 2$ and

$$\begin{aligned}\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1(\mathcal{L}_k^*\mathcal{T})&=\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1(L_{\hat{1}}^*\mathcal{T}(l))\\&=\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1(\mathcal{L}_{\hat{1}}^*\mathcal{T}(l)).\end{aligned} \tag{2.3}$$

See Subsection 5.2 in [Z2].

We are now ready to explain the statement of Theorem 1.1. Let d and μ be as in Subsection 1.1. If k is a positive integer, let $\bar{\mathcal{V}}_k(\mu)$ denote the disjoint union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over equivalence classes of basic bubble types $\mathcal{T}=(S^2,[N],I;j,\underline{d})$ with $|I|=k$ and $\sum d_k=d$. Similarly, we denote by $\mathcal{V}_k(\mu)$ the subspace of $\bar{\mathcal{V}}_k(\mu)$ consisting of the spaces $\mathcal{U}_{\mathcal{T}}(\mu)$ with $\mathcal{T}$ as above. Note that the dimension of $\bar{\mathcal{V}}_k(\mu)$ over $\mathbb{R}$ is $12-4k$. Let

$$a=\mathrm{ev}^*c_1(\gamma_{\mathbb{P}^2}^*)\in H^2(\bar{\mathcal{V}}_k(\mu);\mathbb{Z}),\qquad c_1(\mathcal{L}^*)=c_1\big(\mathcal{L}_{\hat{0}}^*\mathcal{T}\big)\in H^2(\bar{\mathcal{V}}_1(\mu);\mathbb{Z}).$$

where $\mathcal{T}=(S^2,[N],\{\hat{0}\};\hat{0},d)$. While the components of $\bar{\mathcal{V}}_2(\mu)$ are unordered, we can still define the chern classes

$$c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*),\ c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*),\ c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)\in H^*\big(\bar{\mathcal{V}}_2(\mu)\big).$$

In the notation of the previous paragraph, $c_1(\mathcal{L}_i^*)$ denotes the cohomology class $c_1\big(\mathcal{L}_{k_i}^*\mathcal{T}_{k_i}\big)$, where we write $I=\{k_1,k_2\}$.

There are generalizations of the splitting (2.1) that are useful in computations. Let $\mathcal{T}$ and $\{\mathcal{T}_k\}$ be as above. Suppose $k\in I-\hat{I}$ and $d_k=0$. Denote by $\bar{\mathcal{T}}$ the bubble type obtained from $\mathcal{T}$ by removing k from I and $M_k\mathcal{T}$ from M. Then,

$$\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{M}_{0,H_k\mathcal{T}+M_k\mathcal{T}}\times\mathcal{U}_{\bar{\mathcal{T}}}(\mu), \tag{2.4}$$

where $\mathcal{M}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ denotes the main stratum of $\bar{\mathcal{M}}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$. It is shown in [Z3], that $\mathcal{M}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ is the quotient of a subset $\mathcal{M}^{(0)}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ of $\mathbb{C}^{H_k\mathcal{T}+M_k\mathcal{T}}$

by the diagonal S^1-action. The closure $\tilde{\mathcal{M}}^{(0)}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ of $\mathcal{M}^{(0)}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ is S^1-equivariantly diffeomorphic to $S^{2(|H_k\mathcal{T}+M_k\mathcal{T}|)-3}$; see Subsection 5.7 for the case $|H_k\mathcal{T}+M_k\mathcal{T}|=3$. Thus, $\mathcal{M}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ admits a compactification

$$\tilde{\mathcal{M}}_{0,H_k\mathcal{T}+M_k\mathcal{T}} = \tilde{\mathcal{M}}^{(0)}_{0,H_k\mathcal{T}+M_k\mathcal{T}}/S^1 \approx \mathbb{P}^{|H_k\mathcal{T}+M_k\mathcal{T}|-2}.$$

Via the splitting (2.4), we obtain a compactification of $\mathcal{U}_{\mathcal{T}}(\mu)$:

$$\tilde{\mathcal{U}}_{\mathcal{T}}(\mu) \equiv \tilde{\mathcal{M}}_{0,H_k\mathcal{T}+M_k\mathcal{T}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu). \tag{2.5}$$

Note that $\tilde{\mathcal{U}}_{\mathcal{T}}(\mu)=\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ if the cardinality of $H_k\mathcal{T}+M_k\mathcal{T}$ is two or three. Furthermore, in all cases, the restrictions of $L_k\mathcal{T}$ and the tautological line bundle of $\tilde{\mathcal{M}}_{0,H_k\mathcal{T}+M_k\mathcal{T}}$ to $\mathcal{U}_{\mathcal{T}}(\mu)$ agree.

Finally, if X is any space, $F\longrightarrow X$ is a normed vector bundle, and $\delta\colon X\longrightarrow\mathbb{R}$ is any function, let

$$F_\delta = \big\{(b,v)\in F\colon |v|_b < \delta(b)\big\}.$$

Similarly, if Ω is a subset of F, let $\Omega_\delta = F_\delta\cap\Omega$. If $\upsilon=(b,v)\in F$, denote by b_υ the image of υ under the bundle projection map, i.e. b in this case.

2.3. Spaces of Rational Maps. In this subsection, we describe the structure of various spaces of bubble maps passing through the points $\mu_1,\ldots,\mu_N$. The main goal is to describe the behavior of certain bundle sections over such spaces near the boundary strata.

Lemma 2.7. *There exist $r_{\mathbb{P}^2}>0$ and a smooth family of Kahler metrics $\{g_{\mathbb{P}^2,q}: q\in\mathbb{P}^n\}$ on $\mathbb{P}^2$ with the following property. If $B_q(q',r)\subset\mathbb{P}^2$ denotes the $g_{\mathbb{P}^2,q}$-geodesic ball about q' of radius r, the triple $(B_q(q,r_{\mathbb{P}^2}),J,g_{\mathbb{P}^2,q})$ is isomorphic to a ball in $\mathbb{C}^2$ for all $q\in\mathbb{P}^2$.*

This is the $n=2$ case of Lemma 2.1 in [Z2]. If $b=\big(S^2,M,I;x,(j,y),u\big)\in\mathcal{B}_{\mathcal{T}}$, $m\geq1$, and $k\in I$, let

$$\mathcal{D}^{(m)}_{\mathcal{T},k}b = \frac{2}{(m-1)!}\frac{D^{m-1}}{ds^{m-1}}\frac{d}{ds}(u_k\circ q_S)\Big|_{(s,t)=0},$$

where the covariant derivatives are taken with respect to the metric $g_{\mathbb{P}^2,b}\equiv g_{\mathbb{P}^2,\mathrm{ev}(b)}$ and $s+it\in\mathbb{C}$. If $\tilde{\mathcal{T}}$ is a basic bubble type, the maps $\mathcal{D}^{(m)}_{\mathcal{T},k}$, with $\mathcal{T}<\tilde{\mathcal{T}}$ and $k\in I-\hat{I}$, induce a continuous section of $\mathrm{ev}^*T\mathbb{P}^2$ over $\bar{\mathcal{U}}^{(0)}_{\tilde{\mathcal{T}}}$ and a continuous section of the bundle $L_k^*\tilde{\mathcal{T}}^{\otimes m}\otimes\mathrm{ev}^*T\mathbb{P}^2$ over $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}$, described by

$$\mathcal{D}^{(m)}_{\tilde{\mathcal{T}},k}[b,c_k] = c_k^m\mathcal{D}^{(m)}_{\mathcal{T},k}b, \quad \text{if } b\in\mathcal{U}^{(0)}_{\mathcal{T}},\ c_k\in\mathbb{C}.$$

We will often write $\mathcal{D}_{\mathcal{T},k}$ instead of $\mathcal{D}^{(1)}_{\mathcal{T},k}$. If $\mathcal{T}$ is simple, we will abbreviate $\mathcal{D}^{(m)}_{\mathcal{T},k}$ as $\mathcal{D}^{(m)}$. If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type and $k\in\hat{I}$, let $\mathcal{D}^{(m)}_{\mathcal{T},k}$ denote the section $\mathcal{D}^{(m)}_{\bar{\mathcal{T}},k}$. Finally, fix a real number $p>2$.

Theorem 2.8. *Suppose d is a positive integer, $N=3d-4$, M_0 is a subset of $[N]$, and $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of points in general position in $\mathbb{P}^2$. If $\tilde{\mathcal{T}}=(S^2,[N]-M_0,\tilde{I};\tilde{j},\tilde{d})$ is a basic bubble type such that $\sum\tilde{d}_i=d$, the space $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ is an ms-manifold of dimension $2(6-2|\tilde{I}|-|M_0|)$ and $L_k\tilde{\mathcal{T}}$ for $k\in\tilde{I}$ and $ev^*T\mathbb{P}^2$ are ms-bundles over $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$. If $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})<\tilde{\mathcal{T}}$, there exist $\delta,C\in C^\infty\big(\mathcal{U}_{\mathcal{T}}(\mu);\mathbb{R}^+\big)$ and a homeomorphism*

$$\gamma^\mu_{\mathcal{T}}\colon \mathcal{F}\mathcal{T}_\delta\longrightarrow\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu),$$

onto an open neighborhood of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ such that $\gamma^\mu_{\mathcal{T}}|\mathcal{U}_{\mathcal{T}}(\mu)$ is the identity and $\gamma^\mu_{\mathcal{T}}|\mathcal{F}^\emptyset\mathcal{T}_\delta$ is an orientation-preserving diffeomorphism onto an open subset of $\mathcal{U}_{\mathcal{T}}(\mu)$. Furthermore, with appropriate identifications,

$$\Big|\mathcal{D}_{\tilde{\mathcal{T}},k}\gamma^\mu_{\mathcal{T}}(\upsilon)-\alpha_{\mathcal{T},k}\big(\rho_{\mathcal{T}}(\upsilon)\big)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big|\quad\forall\upsilon\in\mathcal{F}\mathcal{T}_\delta,\qquad\textit{where}$$

$$\rho_{\mathcal{T}}(\upsilon)=\big(b,(\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)\in\tilde{\mathcal{F}}\mathcal{T}\equiv\bigoplus_{h\in\chi(\mathcal{T})}L_h\mathcal{T}\otimes L^*_{\tilde{\iota}_h}\mathcal{T};$$

$$\tilde{\upsilon}_h=\prod_{i\in\hat{I},i\le h}\upsilon_i;\ \ \tilde{\iota}_h\in I-\hat{I},\ \ \tilde{\iota}_h\le h;\quad \alpha_{\mathcal{T},k}\big(b,(\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)=\sum_{h\in I_k\cap\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},h}\tilde{\upsilon}_h,$$

and $I_k\subset I$ is the rooted tree containing k.

This is a special case of Theorem 2.8 in [Z2]; see also the remark following the theorem. The analytic estimate on $\mathcal{D}_{\tilde{\mathcal{T}},k}$ is used frequently in the next two sections. If $\mathcal{T}$ is semiprimitive, the bundle $\mathcal{F}\mathcal{T}=\tilde{\mathcal{F}}\mathcal{T}$ and the section $\alpha_{\mathcal{T}}=\alpha_{\mathcal{T}}\circ\rho_{\mathcal{T}}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ via the decomposition (2.5). In terms of the notions of Subsection 2.1, $(\mathcal{F}\mathcal{T},\mathcal{F}\mathcal{T}-\mathcal{F}^\emptyset\mathcal{T},\gamma^\mu_{\mathcal{T}})$ is a normal-bundle model for $\mathcal{U}_{\mathcal{T}}(\mu)\subset\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$. This normal-bundle model admits a closure if $\mathcal{T}$ is semiprimitive.

We will need a similar description for spaces of stable maps corresponding to the rational degree-d curves with certain singularities that pass through the $3d-4$ points $\mu_1,\ldots,\mu_N$. If $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a bubble type and $\chi_{\mathcal{T}}h=1$, let

$$E_h\mathcal{T}=\begin{cases}L_h\mathcal{T}, & \text{if } h\in I-\hat{I};\\ \tilde{\mathcal{F}}_h\mathcal{T}, & \text{if } h\in\hat{I};\end{cases}\qquad E\mathcal{T}=\bigoplus_{h\in\chi(\mathcal{T})}E_h\mathcal{T}.$$

If $M=[N]-M_0$ and $\sum d_i=d$, put

$$\mathcal{S}_{\mathcal{T};1}(\mu)=\big\{b\in\mathcal{U}_{\mathcal{T}}(\mu)\colon \mathcal{D}_{\mathcal{T},h}b=0 \text{ for some } h\in\chi(\mathcal{T})\big\}.$$

If $|\chi(\mathcal{T})|\geq 2$, let

$$\mathcal{S}_{\mathcal{T};2}(\mu)=\big\{\big[b,(\tilde{v})_{h\in\chi(\mathcal{T})}\big]\in\mathbb{P}E\mathcal{T}\colon \sum_{h\in\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},h}\tilde{v}_h=0\big\}-\mathcal{S}_{\mathcal{T};1}(\mu).$$

If $\mathcal{T}$ is basic and $|I|=1$, the set $\mathcal{S}_1(\mu)\equiv\mathcal{S}_{\mathcal{T};1}(\mu)$ of maps can be identified with a dense open subset of the set of irreducible rational degree-d curves that pass through the $3d-4$ points and have a cusp. We denote the closure of $\mathcal{S}_1(\mu)$ in $\bar{\mathcal{V}}_1(\mu)$ by $\bar{\mathcal{S}}_1(\mu)$. Let $\mathcal{S}_{2;2}(\mu)$ be the disjoint union of the spaces $\mathcal{S}_{\mathcal{T};2}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}$ with $|I|=2$. The set $\mathcal{S}_{2;2}(\mu)$ can be identified with a dense open subset of the set of two-component rational degree-d curves that pass through the $3d$–4 points and have a tacnode as a node common to both components. We denote by $\bar{\mathcal{S}}_{2;2}(\mu)$ the closure of $\mathcal{S}_{2;2}(\mu)$ in $\mathbb{P}E_2$, where $E_2\longrightarrow\bar{\mathcal{V}}_2(\mu)$ is the bundle such that $E_2|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)=E\mathcal{T}$. Similarly, we denote by $\mathcal{S}_{2;1}(\mu)$ the disjoint union of the spaces $\mathcal{S}_{\mathcal{T};1}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}$ with $|I|=2$. This finite set can be identified with a subset of $\bar{\mathcal{S}}_{2;2}(\mu)$ as well as with the set of two-component rational degree-d curves passing through the $3d-4$ points such that the two components meet at a node at which one of them has a cusp.

Suppose $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})$ is a simple bubble type with $\sum d_i=d$. By Corollary 6.3 in [Z2], the space $\mathcal{S}_{\mathcal{T};1}(\mu)$ is a smooth complex submanifold of $\mathcal{U}_{\mathcal{T}}(\mu)$. Let

$$\mathcal{N}S_{\mathcal{T};1}=L_{h_1}^*\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^2\longrightarrow\mathcal{S}_{\mathcal{T};1}(\mu)$$

denote its normal bundle, where $h_1\in\chi(\mathcal{T})$ is such that $\mathcal{D}_{\mathcal{T},h_1}b=0$. Put

$$\mathcal{F}S_{\mathcal{T};1}=\bigoplus_{h\in\hat{I}-\chi(\mathcal{T})}\mathcal{F}_h\mathcal{T}\ \oplus\begin{cases}\{0\}, & \text{if } h_1\not\in\hat{I};\\ \mathcal{F}_{h_1}\mathcal{T}, & \text{if } h_1\in\hat{I}\ \&\ \chi(\mathcal{T})=\{h_1\};\\ \mathcal{F}_{h_1}\mathcal{T}\oplus L_{h_1}^*\mathcal{T}\otimes L_{h_2}\mathcal{T}, & \text{if } \chi(\mathcal{T})=\{h_1,h_2\};\end{cases}$$

$$\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1}=\begin{cases}\mathcal{F}_{h_1}\mathcal{T}^{\otimes 2}, & \text{if } h_1\in\hat{I}\ \&\ \chi(\mathcal{T})=\{h_1\};\\ \mathcal{F}_{h_1}\mathcal{T}^{\otimes 2}\oplus\mathcal{F}_{h_2}\mathcal{T}, & \text{if } \chi(\mathcal{T})=\{h_1,h_2\}.\end{cases}$$

If $d_{\hat{0}}=0$, we define $\rho_{\mathcal{T};1}\colon\mathcal{F}S_{\mathcal{T};1}\longrightarrow\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1}$ and

$$\alpha_{\mathcal{T};1}\in\Gamma\big(\bar{\mathcal{S}}_{\mathcal{T};1}(\mu);\mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1};L_{\hat{0}}^*\mathcal{T}^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)$$

by

$$\rho_{\mathcal{T};1}(\upsilon) = \begin{cases} \upsilon_{h_1}\otimes\upsilon_{h_1}, & \text{if } \chi(\mathcal{T})=\{h_1\};\\ \big(\upsilon_{h_1}\otimes\upsilon_{h_1}, \upsilon_{h_1}\otimes u\big), & \text{if } \chi(\mathcal{T})=\{h_1,h_2\} \ \&\ u\in L^*_{h_1}\mathcal{T}\otimes L_{h_2}\mathcal{T}; \end{cases}$$

$$\alpha_{\mathcal{T};1}(\varpi) = \begin{cases} \mathcal{D}^{(2)}_{\mathcal{T},h_1}\varpi, & \text{if } \chi(\mathcal{T})=\{h_1\};\\ \mathcal{D}^{(2)}_{\mathcal{T},h_1}\varpi_1 + x_{h_2}\mathcal{D}^{(1)}_{\mathcal{T},h_2}\varpi_2, & \text{if } \chi(\mathcal{T})=\{h_1,h_2\},\ \varpi=(\varpi_1,\varpi_2),\\ & \quad b_\varpi=\big(S^2,M,I;x,(j,y),u\big). \end{cases}$$

If $|\chi(\mathcal{T})|\ge 2$, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is a smooth submanifold of $\mathbb{P}E\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu)$. We identify it with a subset of $\mathcal{U}_{\mathcal{T}}(\mu)$ via the bundle projection map $\pi_{E\mathcal{T}}:\mathbb{P}E\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu)$. If $|\chi(\mathcal{T})|=3$, $\mathcal{S}_{\mathcal{T};2}(\mu)=\mathcal{U}_{\mathcal{T}}(\mu)$, and we put $\mathcal{N}S_{\mathcal{T};2}=\{0\}$. If $\chi(\mathcal{T})=\{h_1,h_2\}$, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is a smooth submanifold of $\mathcal{U}_{\mathcal{T}}(\mu)$ with normal bundle

$$\mathcal{N}S_{\mathcal{T};2} = L^*_{h_2}\mathcal{T}\otimes\big(\text{Im}\ \mathcal{D}_{\mathcal{T},h_1}\big)^{\perp}.$$

If $\iota_{h_1}=\iota_{h_2}$ for all $h_1,h_2\in\chi(\mathcal{T})$, put

$$\mathcal{F}S_{\mathcal{T};2} = \gamma_{\underset{h\in\chi(\mathcal{T})}{\bigoplus}\mathcal{F}_h\mathcal{T}} \oplus \bigoplus_{h\notin\chi(\mathcal{T})}\mathcal{F}_h\mathcal{T},$$

$$\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2} = \gamma_{\underset{h\in\chi(\mathcal{T})}{\bigoplus}\mathcal{F}_h\mathcal{T}} \otimes \begin{cases} \mathbb{C}, & \text{if } \iota_h=\hat{0}\ \forall h\in\chi(\mathcal{T});\\ \mathcal{F}_{\hat{1}}\mathcal{T}^{\otimes 2}, & \text{if } \iota_h=\hat{1}\neq\hat{0}\ \forall h\in\chi(\mathcal{T}). \end{cases}$$

Let $\rho_{\mathcal{T};2}:\mathcal{F}S_{\mathcal{T};2}\longrightarrow\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}$ be the projection map followed by multiplication. Define

$$\alpha_{\mathcal{T};2}\in\Gamma\big(\bar{\mathcal{S}}_{\mathcal{T};2}(\mu);\text{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2};L^*_{\hat{0}}\mathcal{T}^{\otimes 2}\otimes\text{ev}^*T\mathbb{P}^2)\big) \qquad \text{by}$$

$$\alpha_{\mathcal{T};2}\big((\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big) = \sum_{h\in\chi(\mathcal{T})} x_h\mathcal{D}^{(1)}_{\mathcal{T},h}\tilde{\upsilon}_h \qquad \text{if } b_\upsilon=\big(S^2,M,I;x,(j,y),u\big).$$

If $\chi_{\mathcal{T}}\hat{1}=0$ for some $\hat{1}\in\hat{I}-\chi(\mathcal{T})$, $H_{\hat{0}}\mathcal{T}=\{\hat{1},h_1\}$, and $h_2\in\chi(\mathcal{T})-\{h_1\}$, let

$$\mathcal{F}S_{\mathcal{T};2} = \mathcal{F}_{\hat{1}}\mathcal{T}\oplus\mathcal{F}_{h_2}\mathcal{T}, \qquad \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2} = \mathcal{F}_{\hat{1}}\mathcal{T}\otimes\mathcal{F}_{h_2}\mathcal{T}, \quad \text{and} \quad \rho_{\mathcal{T};2}:\mathcal{F}S_{\mathcal{T};2}\longrightarrow\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}$$

be the multiplication map. Define $\alpha_{\mathcal{T};2}\in\Gamma\big(\mathcal{S}_{\mathcal{T};2}(\mu);\text{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2};L^*_{\hat{0}}\mathcal{T}^{\otimes 2}\otimes\text{ev}^*T\mathbb{P}^2)\big)$ by

$$\alpha_{\mathcal{T};2}\big(\upsilon_{\hat{1}}\otimes\upsilon_{h_2}\big) = x_{\hat{1}}\upsilon_{\hat{1}}\sum_{h\in\chi(\mathcal{T})-\{h_1\}}\mathcal{D}_{\mathcal{T},h}\upsilon'_h + x_{h_1}\mathcal{D}_{\mathcal{T},h_1}\upsilon_{h_1},$$

$$\text{if} \qquad b_{\upsilon_{\hat{1}}\otimes\upsilon_{h_2}}=\big(S^2,M,I;x,(j,y),u\big),\ \big[\upsilon_{\hat{1}}\otimes\upsilon',\upsilon_{h_1}\big]\in\mathcal{S}_{\mathcal{T};2}(\mu),$$
$$\text{where} \qquad \upsilon'=(\upsilon_h)_{h\in\chi(\mathcal{T})-\{h_1\}}.$$

In all cases, denote by $\mathcal{F}^{\emptyset}\mathcal{S}_{\mathcal{T};k}$ the subset of $\mathcal{F}S_{\mathcal{T};k}$ consisting of vectors with all components nonzero.

Proposition 2.9. *If* $\tilde{\mathcal{T}}=(S^2,[N]-M_0,\{\hat{0}\};\hat{0},d)$ *and* $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})<\tilde{\mathcal{T}}$ *are simple bubble types,*

$$\bar{\mathcal{S}}_{\tilde{\mathcal{T}};1}(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\begin{cases}\mathcal{S}_{\mathcal{T};1}(\mu), & \textit{if } |\chi(\mathcal{T})|=1;\\ \mathcal{S}_{\mathcal{T};1}(\mu)\cup\mathcal{S}_{\mathcal{T};2}(\mu), & \textit{if } |H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2\ \&\ M_{\hat{0}}\mathcal{T}=\emptyset;\\ \mathcal{S}_{\mathcal{T};2}(\mu), & \textit{otherwise.}\end{cases}$$

In addition, there exist $\delta,C\in C^\infty(\mathcal{S}_{\mathcal{T};k}(\mu);\mathbb{R}^+)$ *and a continuous map*

$$\gamma_{\mathcal{T};k}\colon \mathcal{F}S_{\mathcal{T};k;\delta}\longrightarrow\bar{\mathcal{S}}_{\tilde{\mathcal{T}};1}(\mu)$$

onto an open neighborhood of $\mathcal{S}_{\mathcal{T};k}(\mu)$ *in* $\bar{\mathcal{S}}_{\tilde{\mathcal{T}};1}(\mu)$ *such that* $\gamma_{\mathcal{T};k}|\mathcal{S}_{\mathcal{T};k}(\mu)$ *is the identity and* $\gamma_{\mathcal{T};k}|\mathcal{F}^\emptyset\mathcal{S}_{\mathcal{T};k;\delta}$ *is an orientation-preserving diffeomorphism onto an open subset of* $\mathcal{S}_{\tilde{\mathcal{T}};1}(\mu)$. *Furthermore, if* $d_{\hat{0}}\neq 0$, $\mathcal{D}^{(2)}_{\tilde{\mathcal{T}},\hat{0}}$ *does not vanish on* $\mathcal{S}_{\mathcal{T};1}(\mu)$. *If* $d_{\hat{0}}=0$, *with appropriate identifications,*

$$\Big|\mathcal{D}^{(2)}_{\tilde{\mathcal{T}},\hat{0}}\big(\gamma_{\mathcal{T};k}(\upsilon)\big)-\alpha_{\mathcal{T};k}\big(\rho_{\mathcal{T};k}(\upsilon)\big)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T};k}(\upsilon)\big|\qquad\forall\upsilon\in\mathcal{F}S_{\mathcal{T};k;\delta}.$$

Lemma 2.10. *If* $\tilde{\mathcal{T}}$ *and* $\mathcal{T}$ *are as in Proposition 2.9, there exist* $\delta\in C^\infty(\mathcal{S}_{\mathcal{T};k}(\mu);\mathbb{R}^+)$ *and a continuous map*

$$\tilde{\gamma}_{\mathcal{T};k}\colon\big(\mathcal{N}S_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}\big)_\delta\longrightarrow\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$$

onto an open neighborhood of $\mathcal{S}_{\mathcal{T};k}(\mu)$ *in* $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ *such that* $\gamma_{\mathcal{T};k}|\mathcal{S}_{\mathcal{T};k}(\mu)$ *is the identity and*

$$\gamma_{\mathcal{T};k}\big|\mathcal{N}S_{\mathcal{T};k,\delta}\times_{\mathcal{S}_{\mathcal{T};k}(\mu)}\mathcal{F}^\emptyset\mathcal{T}_\delta$$

is smooth and orientation-preserving an open subset of $\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$. *Furthermore, there exists a subbundle* $\mathcal{O}_{\mathcal{T}}$ *of* $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ *of rank less than the rank of* $\mathcal{N}S_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}$ *such that, with appropriate identifications,*

$$\mathcal{D}_{\tilde{\mathcal{T}},\hat{0}}\big|\tilde{\gamma}_{\mathcal{T};k}(X,\upsilon)\subset\mathcal{O}_{\mathcal{T}}\qquad\forall(X,\upsilon)\in\big(\mathcal{N}S_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}\big)_\delta.$$

The proof of Proposition 2.9, with the exception of the estimate on $\mathcal{D}^{(2)}_{\tilde{\mathcal{T}},\hat{0}}$, is either the same or very similar to the proof of Lemma 5.8 in [Z2], depending on the bubble type $\mathcal{T}$. If $d_{\hat{0}}\neq 0$, we apply the analytic estimate of Theorem 2.8 and the Implicit Function Theorem to the section $\gamma_{\mathcal{T}}^{\mu*}\mathcal{D}_{\tilde{\mathcal{T}},\hat{0}}$. If $d_{\hat{0}}=0$ and $k=1$, the two theorems are applied to a section of $(L_{\hat{0}}\mathcal{T}\otimes\mathcal{F}_{h_1}\mathcal{T})^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ induced by $\gamma_{\mathcal{T}}^{\mu*}\mathcal{D}_{\tilde{\mathcal{T}},\hat{0}}$. If $|\chi(\mathcal{T})|\ge 2$ and $H_{\hat{0}}\mathcal{T}=\chi(\mathcal{T})$, we work with a section of $(L_{\hat{0}}\mathcal{T}\otimes\gamma_{E\mathcal{T}})^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ over the blowup of $E\mathcal{T}$ along $\mathcal{U}_{\mathcal{T}}(\mu)$. The case $\chi_{\mathcal{T}}h=\hat{1}$ for all $h\in\chi(\mathcal{T})$ is similar. If $\chi(\mathcal{T})=\{h_1,h_2\}$ is a two-element set, and $H_{\hat{0}}\mathcal{T}=\{\hat{1},h_1\}$, we use the same section, but given a small element

$(\upsilon_{\hat{1}},\upsilon_{h_2})\in\mathcal{FS}_{\mathcal{T};2}$, we start with the approximate solution $(\upsilon_{\hat{1}},\kappa\upsilon_{\hat{1}}\upsilon_{h_2},\upsilon_{h_2})$, with

$$\kappa\in(L_{\hat{1}}\mathcal{T}\otimes L_{h_2}\mathcal{T})^*\otimes L_{h_1}\mathcal{T}\quad\text{s.t.}\quad\big[\upsilon_{\hat{1}}\upsilon_{h_2},\kappa\upsilon_{\hat{1}}\upsilon_{h_2}\big]\in\mathcal{S}_{\mathcal{T};2}(\mu).$$

The approach to the remaining case is analogous. The estimate on $\mathcal{D}^{(2)}_{\tilde{\mathcal{T}},\hat{0}}$ is obtained by the same argument as in the proof of Lemma 5.10 in [Z2]. The proof makes use of the construction of $\gamma^{\mu}_{\mathcal{T}}$ in [Z3], which involves a modification of the pregluing step of standard gluing procedures for pseudoholomorphic curves. The first statement of Lemma 2.10 is an immediate consequence of Theorem 2.8. The second statement makes no claim unless $|\hat{I}|=1$. In such a case, the proof is exactly the same as the proof of Corollary 5.9 in [Z2].

We next describe the behavior of the section

$$\mathcal{D}_{2;2}\equiv c_1\mathcal{D}_1+c_2\mathcal{D}_2\in\Gamma\big(\bar{\mathcal{S}}_{2;2}(\mu);\gamma^*_{E_2}\otimes\text{ev}^*T\mathbb{P}^2\big),$$

for $c_1,c_2\in\mathbb{C}$ distinct, near $\partial\bar{\mathcal{S}}_{2;2}(\mu)\equiv\bar{\mathcal{S}}_{2;2}(\mu)-\mathcal{S}_{2;2}(\mu)$. As before, we identify $\mathcal{S}_{2;1}(\mu)$ with a subset of $\bar{\mathcal{S}}_{2;2}(\mu)$. Similarly, if $\mathcal{T}=\big(S^2,[N],I;j,\underline{d}\big)$ is a bubble type such that $I-\hat{I}=\{k_1,k_2\}$ is a two-element set and $\sum d_i=d$, let

$$\begin{aligned}\mathcal{S}_{\mathcal{T};2}(\mu)=&\big\{\big[b,L_{k_1}\mathcal{T}\big]\colon b\in\mathcal{U}_{\mathcal{T}}(\mu),\ \mathcal{D}_{\mathcal{T},k_1}b=0\big\}\\&\cup\big\{\big[b,L_{k_2}\mathcal{T}\big]\colon b\in\mathcal{U}_{\mathcal{T}}(\mu),\ \mathcal{D}_{\mathcal{T},k_2}b=0\big\}\subset\mathbb{P}E_2.\end{aligned}$$

Proposition 2.11. *Suppose d is a positive integer and μ is a tuple of $3d$–4 points in general position in $\mathbb{P}^2$. Then*

$$\partial\bar{\mathcal{S}}_{2;2}(\mu)=\mathcal{S}_{2;1}(\mu)\cup\bigcup_{[\mathcal{T}]}\mathcal{S}_{\mathcal{T};2}(\mu),$$

where the union is taken over all equivalence classes of non-basic types

$$\mathcal{T}=\big(S^2,[N],I;j,\underline{d}\big)$$

such that $I-\hat{I}=\{k_1,k_2\}$ is a two-element set and $\sum d_i=d$. Furthermore, there exist $\delta,C>0$ and homeomorphism

$$\gamma_{2;2}\colon\big\{u\in\gamma^*_{E_2}\otimes(E_2/\gamma_{E_2})\big|\partial\bar{\mathcal{S}}_{2;2}(\mu)\colon|u|<\delta\big\}\longrightarrow\bar{\mathcal{S}}_{2;2}(\mu)$$

onto an open neighborhood of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ in $\bar{\mathcal{S}}_{2;2}(\mu)$ such that $\gamma_{2;2}|\partial\bar{\mathcal{S}}_{2;2}(\mu)$ is the identity and $\gamma_{2;2}$ restricts to an orientation-preserving diffeomorphism from the complement of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ onto an open subset of $\mathcal{S}_{2;2}(\mu)$. Finally, with appropriate identifications,

$$\begin{aligned}&\big|\mathcal{D}_{2;2}\gamma_{2;2}(u)-\alpha_{2;2}(u)\big|\le C|u|^{1+\frac{1}{p}}\\&\qquad\forall u\in\big\{u\in\gamma^*_{E_2}\otimes(E_2/\gamma_{E_2})\big|\partial\bar{\mathcal{S}}_{2;2}(\mu)\colon|u|<\delta\big\}\longrightarrow\bar{\mathcal{S}}_{2;2}(\mu),\end{aligned}$$

where $\alpha_{2;2}\in\Gamma\big(\partial\bar{\mathcal{S}}_{2;2}(\mu); Hom(\gamma_{E_2}^*\otimes(E_2/\gamma_{E_2}),\gamma_{E_2}^*\otimes ev^*T\mathbb{P}^2)\big)$ *is an injection on every fiber.*

This proposition follows from Theorem 2.8 and the Implicit Function Theorem by an argument similar to the proof of Proposition 2.9. Note that with our choice of constraints, $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ is a finite set. Thus, we are able to take δ and C to be positive real numbers rather than continuous functions $\partial\bar{\mathcal{S}}_{2;2}(\mu)\longrightarrow\mathbb{R}^+$.

2.4. Description of $CR_3(\mu)$. In this subsection, we describe the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near each strata $\mathcal{M}_{\mathcal{T}}(\mu)$ of bubble maps of type (2c) in terms of the zeros of affine maps between vector bundles over closures of certain subspaces of $\mathcal{M}_{\mathcal{T}}(\mu)$. These results are proved by an argument similar to Sections 2 and 4 in [Z2], which is outlined briefly at the end of this subsection.

We start by recalling more notation used in [Z2]. If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, let $\big\{g_{b,\hat{0}}: b\in\mathcal{M}_{\mathcal{T}}(\mu)\big\}$ be a smooth family of Kahler metrics on (Σ,j) such that
(1) for all $b=(\Sigma,[N],I;x,(j,y),u)\in\mathcal{M}_{\mathcal{T}}(\mu)$ and $h\in H_{\hat{0}}\mathcal{T}$, $g_{b,\hat{0}}$ is flat on a neighborhood of x_h in Σ;
(2) the metric $g_{b,\hat{0}}$ is determined by the tuple $(x_h)_{h\in H_{\hat{0}}\mathcal{T}}$.
We denote by $\mathcal{H}_\Sigma^{0,1}$ the 3-dimensional space of harmonic $(0,1)$-forms on Σ.

If $\psi\in\mathcal{H}_\Sigma^{0,1}$, $b\in\mathcal{M}_{\mathcal{T}}$, $m\geq 1$, and the metric $g_{b,\hat{0}}$ is flat near x, we define $D_{b,x}^{(m)}\psi\in T_x^{0,1}\Sigma^{\otimes m}$ as follows. If (s,t) are conformal coordinates centered at x such that s^2+t^2 is the square of the $g_{b,\hat{0}}$-distance to x, let

$$\{D_{b,x}^{(m)}\psi\}\Big(\frac{\partial}{\partial s}\Big)\equiv\{D_{b,x}^{(m)}\psi\}\bigg(\underbrace{\frac{\partial}{\partial s},\ldots,\frac{\partial}{\partial s}}_{m}\bigg)=\frac{\pi}{m!}\Big\{\frac{D^{m-1}}{ds^{m-1}}\psi_j\Big|_{(s,t)=0}\Big\}\Big(\frac{\partial}{\partial s}\Big),$$

where the covariant derivatives are taken with respect to the metric $g_{b,\hat{0}}$. If $\{\psi_j\}$ is an orthonormal basis for $\mathcal{H}_\Sigma^{0,1}$, let $s_{b,x}^{(m)}\in T_x^*\Sigma^{\otimes m}\otimes\mathcal{H}_\Sigma^{0,1}$ be given by

$$s_{b,x}^{(m)}(v)\equiv s_{b,x}^{(m)}(\underbrace{v,\ldots,v}_{m})=\sum\overline{\Big\{D_{b,x}^{(m)}\psi_j\Big\}(v)}\psi_j.$$

The section $s_{b,x}^{(m)}$ is always independent of the choice of a basis for $\mathcal{H}_\Sigma^{0,1}$, but is dependent on the choice of the metric $g_{b,\hat{0}}$ if $m>1$. By [GH, p. 246], $s_x\equiv s_{b,x}$ does not vanish and thus determines a line subbundle $\mathcal{H}_\Sigma^+$ of $\Sigma\times\mathcal{H}_\Sigma^{0,1}\longrightarrow\Sigma$.

We denote its orthogonal complement by $\mathcal{H}_\Sigma^-$. Let

$$\pi^- \in \Gamma\big(\Sigma; (\Sigma\times\mathcal{H}_\Sigma^{0,1})^* \otimes \mathcal{H}_\Sigma^-\big)$$

be the orthogonal projection map onto $\mathcal{H}_\Sigma^-$. While the section $s_{b,x}^{(2)}$ depends on the choice of the metric $g_{b,\hat{0}}$, $s_x^{(2)} \equiv \pi_x^- \circ s_{b,x}^{(2)}$ does not and thus is globally defined on Σ. If Σ is not hyperelliptic, as we assume to be the case, $s_x^{(2)}$ does not vanish and thus determines a line subbundle $\mathcal{H}_\Sigma^{-+}$ of $\mathcal{H}_\Sigma^-$. We denote its orthogonal complement by $\mathcal{H}_\Sigma^{--}$ and the corresponding orthogonal projection map by π^{--}. The composition $s_x^{(3)} \equiv \pi_x^{--} \circ s_{b,x}^{(3)}$ is again independent of the choice of the metric $g_{b,\hat{0}}$. If Σ is general, the section

$$s^{(3)} \in \Gamma\big(\Sigma; T^*\Sigma^{\otimes 3} \otimes \mathcal{H}_\Sigma^{--}\big)$$

vanishes transversally at 24 distinct points $z_1, \ldots, z_{24}$ of Σ. These points correspond to the flexes of Σ under the canonical embedding into $\mathbb{P}^2$.

Theorem 2.12. *Suppose d is a positive integer, $N = 3d-4$, μ is an N-tuple of points in general position in $\mathbb{P}^2$, $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type such that $d_{\hat{0}} = 0$ and $\sum d_i = d$. If*

$$\nu \in \Gamma(\Sigma\times\mathbb{P}^2; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^2}^* T\mathbb{P}^2)$$

is a generic section, there exist a compact subset $K_{\mathcal{T},\nu}$ of $\mathcal{M}_\mathcal{T}(\mu)$ and integer $N(\mathcal{T})$ with the following property. If K is a compact subset of $\mathcal{M}_\mathcal{T}(\mu)$ containing $K_{\mathcal{T},\nu}$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$,

$$^{\pm}\big|U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)\big| = N(\mathcal{T}).$$

If $\mathcal{T}$ is not primitive, $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$. If $\mathcal{T}$ is primitive, $N(\mathcal{T})$ is the number of zeros of the affine maps between vector bundles described below.

Above $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ denotes the space of all stable maps from Σ to $\mathbb{P}^2$ that map the marked points to $\mu_1, \ldots, \mu_N$. For each primitive bubble type $\mathcal{T}$, we now describe the number $N(\mathcal{T})$ as the sum of numbers $N(\alpha)$, where each α is a regular ms-polynomial between two ms-bundles over an ms-manifold; see Subsection 2.1.

If $|\hat{I}|>3$, $\mathcal{M}_{\mathcal{T}}(\mu)=\emptyset$. If $|\hat{I}|=1,2,3$, define

$$\alpha_{|\hat{I}|}\in\Gamma\big(\Sigma^{\hat{I}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu);\mathrm{Hom}(\bigoplus_{h\in\hat{I}}T\Sigma_h\otimes L_h\mathcal{T},\mathcal{H}_{\Sigma}^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big),$$

$$\text{by}\qquad \alpha_{|\hat{I}|}\big(b,(x_h)_{h\in\hat{I}};(v_h\otimes v_h)_{h\in\hat{I}}\big)=\sum_{h\in\hat{I}}\big(\mathcal{D}_{\mathcal{T},h}v_h\big)\big(s_{x_h}v_h\big). \tag{2.6}$$

If $|\hat{I}|=3$, $N(\mathcal{T})=n^{(1)}(\mathcal{T})=N(\alpha_3)$. If $|\hat{I}|=2$,

$$N(\mathcal{T})=n^{(1)}(\mathcal{T})+2n^{(2)}(\mathcal{T})=N(\alpha_2)+2N(\alpha_{2;1}),$$

with $\alpha_{2;1}$ defined as follows. If $\hat{I}=\{\hat{1},\hat{2}\}$, $b\in\mathcal{S}_{\bar{\mathcal{T}};1}(\mu)$, and $\mathcal{D}_{\bar{\mathcal{T}};\hat{1}}(b)=0$,

$$\begin{aligned}&\alpha_{2;1}\big(b,(x_h)_{h\in\hat{I}};(v_h\otimes v_h)_{h\in\hat{I}}\big)\\ &\qquad=\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}v_{\hat{1}}\big)\big(s^{(2)}_{x_{\hat{1}}}v_{\hat{1}}\big)+\big(\mathcal{D}_{\mathcal{T},\hat{2}}v_{\hat{2}}\big)\big(\pi^-_{x_{\hat{1}}}s_{x_{\hat{2}}}v_{\hat{2}}\big)\in\mathcal{H}^-_{\Sigma}(x_{\hat{1}})\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2,\\ &\text{if}\qquad v_{\hat{1}}\otimes v_{\hat{1}}\in T\Sigma_{\hat{1}}^{\otimes 2}\otimes L_{\hat{1}}\mathcal{T}^{\otimes 2}\big|_{(b,x_{\hat{1}})},\quad v_{\hat{2}}\otimes v_{\hat{2}}\in T\Sigma_{\hat{2}}\otimes L_{\hat{2}}\mathcal{T}\big|_{(b,x_{\hat{2}})}.\end{aligned} \tag{2.7}$$

If $|\hat{I}|=1$,

$$\begin{aligned}N(\mathcal{T})&=n^{(1)}(\mathcal{T})+2n^{(2)}(\mathcal{T})+3n_3(\mathcal{T})+4n_4(\mathcal{T})\\ &=N(\alpha_1)+2N(\alpha_{1;1})+3N(\alpha_{1;2})+96|\mathcal{S}_{1;2}(\mu)|,\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&\alpha_{1;1}\in\Gamma\big(\Sigma\times\bar{\mathcal{S}}_1(\mu);\mathrm{Hom}(T\Sigma^{\otimes 2}\otimes L_{\hat{1}}\mathcal{T}^{\otimes 2},\mathcal{H}^-_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)\quad\text{and}\\ &\alpha_{1;2}\in\Gamma\big(\Sigma\times\mathcal{S}_{1;2}(\mu);\mathrm{Hom}(T\Sigma^{\otimes 3}\otimes L_{\hat{1}}\mathcal{T}^{\otimes 3},\mathcal{H}^{--}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)\end{aligned}$$

are defined by

$$\alpha_{1;k}\big(x,b,v\otimes v\big)=\big(\mathcal{D}^{(k+1)}_{\mathcal{T},\hat{1}}v\big)\big(s^{(k+1)}_x v\big). \tag{2.9}$$

Finally, for each $m=1,2,3$, we denote by $n^{(k)}_m(\mu)$ the sum of the numbers $n^{(k)}(\mathcal{T})$ taken over all equivalence classes of primitive bubble types $\mathcal{T}$ with $|\hat{I}|=m$.

Corollary 2.13. *With notation as above,*

$$\begin{aligned}CR_3(\mu)=&\big(n^{(1)}_1(\mu)+2n^{(2)}_1(\mu)+3n^{(3)}_1(\mu)+96|\mathcal{S}_{1;2}(\mu)|\big)\\ &+\big(n^{(1)}_2(\mu)+2n^{(2)}_2(\mu)\big)+n^{(1)}_3(\mu).\end{aligned}$$

Corollary 2.13 follows immediately from the preceding paragraph, Theorem 2.12, and the definition of $CR_3(\mu)$ in Subsection 1.2.

Remark 1: The multiplicity k for $n_m^{(k)}(\mu)$ is the degree of a polynomial map between two vector spaces of small dimensions. In the cases under consideration, these degrees are clear. In the genus-two case, they are described in Section 4 of [Z2].

Remark 2: The number 96 in (2.8) arises because each element of $\mathcal{M}_{\mathcal{T}}(\mu)$ corresponding to a simple flex of Σ and a rational map in $\mathcal{S}_{2;2}(\mu)$ enters with a multiplicity of 4. A hyperflex of Σ would result in a multiplicity of 10, at least if $d\geq 4$; see the last paragraph of this subsection for more details. Thus, if Σ has n hyperflexes and $24-2n$ simple flexes, the number 96 in equation (2.8) and in Corollary 2.13 should be replaced by $96+2n$. No other changes are needed. The analogue of the term $n_1^{(4)}(\mu)=24|\mathcal{S}_{1;2}(\mu)|$ in the genus-two case is $n_1^{(3)}(\mu)=6|\mathcal{S}_1(\mu)|$, as each of the six hyperelliptic points of Σ and a cuspidal map through the fixed $3d-2$ points enters with a multiplicity of 3; see Subsection 4.5 in [Z2].

Remark 3: It should be possible to adapt this approach to the case where Σ is a hyperelliptic genus-three surface, but significant changes would be required. In particular, there will likely be a contribution to $CR_3(\mu)$ from an affine map over the space $\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu)$, as is the case in the genus-two case in $\mathbb{P}^3$; see Subsection 4.6 in [Z2]. Furthermore, higher-order contributions $n_m^{(k)}(\mu)$, $k\geq 3$, will have a very different description, which will involve the hyperelliptic and Weierstrass points of Σ.

We now outline the proof of Theorem 2.12 following [Z2]. For each $b\in\mathcal{M}_{\mathcal{T}}$ and $\upsilon\in F^{\emptyset}\mathcal{T}$ sufficiently small, we first construct a nearly holomorphic map $u_\upsilon:\Sigma\longrightarrow\mathbb{P}^n$. We then attempt to solve the equation

$$\bar{\partial}\exp_{u_\upsilon}\xi = t\nu \Longleftrightarrow \bar{\partial}u_\upsilon + D_\upsilon\xi + N_\upsilon\xi = t\nu \in \Gamma^{0,1}(u_\upsilon) \tag{2.10}$$

for a small vector field $\xi\in\Gamma(u_\upsilon)$ along u_υ. Since we need to count the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near $\mathcal{M}_{\mathcal{T}}(\mu)$, we require that ξ lie in a subspace $\tilde{\Gamma}_+(\upsilon)$ complementary to the "tangent bundle" $\Gamma_-(\upsilon)$ of the space $\{u_\upsilon:\upsilon\in F^{\emptyset}\mathcal{T}_\delta\}$. The cokernel of the operator D_b is $\mathcal{H}^{0,1}_\Sigma\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2$. It induces an orthogonal splitting

$$\Gamma^{0,1}(\upsilon)=\Gamma^{0,1}_-(\upsilon)\oplus\Gamma^{0,1}_+(\upsilon)$$

such that $\pi^{0,1}_+\circ D_\upsilon:\tilde{\Gamma}_+(\upsilon)\longrightarrow\Gamma^{0,1}_+(\upsilon)$ is an isomorphism and $\Gamma^{0,1}_-(\upsilon)$ is isomorphic to $\mathcal{H}^{0,1}_\Sigma\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2$. If υ and t are sufficiently small, depending on b, the $\pi^{0,1}_+$-part of equation (2.10) has a unique solution ξ_υ, which also solves the entire equation (2.10) if and only if

$$\pi_-^{0,1}\big(t\nu-\bar\partial u_\upsilon-D_\upsilon\xi_\upsilon-N_\upsilon\xi_\upsilon\big)=0\in\Gamma_-^{0,1}(u_\upsilon)\approx\mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2. \tag{2.11}$$

It then remains to adjust for the constraints and extract the leading-order term from (2.11). The latter part depends on the choices of the above splittings of $\Gamma(\upsilon)$ and $\Gamma^{0,1}(\upsilon)$. The spaces $\Gamma_-(\upsilon)$ and $\Gamma_-^{0,1}(\upsilon)$ are constructed from the kernel and cokernel of D_b fairly explicitly in Subsections 2.2 and 2.3 of [Z2]. In order to extract the leading-order term from (2.11), we need the composite $\pi_-^{0,1}\circ D_\upsilon$ to be sufficiently small on $\tilde\Gamma_+(\upsilon)$. In [Z2], this is insured by choosing $\tilde\Gamma_+(\upsilon)$ so that its image under D_υ is orthogonal to $\mathcal{H}_\Sigma^+(x_{\hat 1})\otimes\mathrm{ev}^*T\mathbb{P}^2$ if $b=(S^2,[N],I;x,(j,y),u)$ and $\hat 1$ is an element of $H_{\hat 0}\mathcal{T}$. By an argument similar to Subsection 2.3 in [Z2], in the given case we can choose $\tilde\Gamma_+(\upsilon)$ so that its image under D_υ is orthogonal to $\big(\mathcal{H}_\Sigma^+(x_{\hat 1})\oplus\mathcal{H}_\Sigma^{-+}(x_{\hat 1})\big)\otimes\mathrm{ev}^*T\mathbb{P}^2$, provided $d_{\hat 1}\ge 2$. Then in all cases, $\pi_-^{0,1}\circ D_\upsilon|\tilde\Gamma_+(\upsilon)$ will be sufficiently small for the purposes of extracting dominant terms from (2.11) as in Section 4 of [Z2]. Polynomial maps between vector bundles arise from the power series expansion for $\pi_-^{0,1}\bar\partial u_\upsilon$ given in Proposition 4.4 of [Z2].

As an example, we now describe the expansion of $\pi_-^{0,1}\bar\partial u_\upsilon$ near an element

$$b=(\Sigma,[N],I;x,(j,y),u)\in\mathcal{M}_{\mathcal{T}}(\mu)$$

such that $\hat I=\{\hat 1\}$ is a single-element set, $d_{\hat 1}=d$, $u_{\hat 1}(\infty)$ is a $(3,4)$-cusp, or equivalently $\mathcal{D}_{\mathcal{T},\hat 1}^{(1)}b=\mathcal{D}_{\mathcal{T},\hat 1}^{(2)}b=0$, and $x_{\hat 1}\in\Sigma$ is a flex or a hyperflex of Σ. A neighborhood of such an element in $\mathcal{M}_{\mathcal{T}}(\mu)$ can be modeled on small elements

$$(X,Y,w)\in L_{\hat 1}^*\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^2\oplus L_{\hat 1}^*\mathcal{T}^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2\oplus T_{x_{\hat 1}}\Sigma\big|_b.$$

For a small nonzero $\upsilon\in\mathcal{F}\mathcal{T}\big|_b\approx L_{\hat 1}\mathcal{T}\otimes T\Sigma\big|_b$,

$$\pi_{x_{\hat 1}}^+\pi_-^{0,1}\bar\partial u_\upsilon\approx X\otimes s_{x_{\hat 1}}^{(1)}\upsilon,\qquad \pi_{x_{\hat 1}}^{-+}\pi_-^{0,1}\bar\partial u_\upsilon\approx Y\otimes s_{x_{\hat 1}}^{(2)}\upsilon^{\otimes 2},$$

$$\pi_{x_{\hat 1}}^{--}\pi_-^{0,1}\bar\partial u_\upsilon\approx\begin{cases}\big\{\pi_{x_{\hat 1}}^{--}s_b^{(4)}(w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat 1}^{(3)}\big\}\upsilon^{\otimes 3}+\big\{\pi_{x_{\hat 1}}^{--}s_b^{(4)}\otimes\mathcal{D}_{\mathcal{T},\hat 1}^{(4)}\big\}\upsilon^{\otimes 4},\\ \big\{\pi_{x_{\hat 1}}^{--}s_b^{(5)}(w,w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat 1}^{(3)}\big\}\upsilon^{\otimes 3}+\big\{\pi_{x_{\hat 1}}^{--}s_b^{(5)}(w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat 1}^{(4)}\big\}\upsilon^{\otimes 4}\\ \qquad\qquad+\big\{\pi_{x_{\hat 1}}^{--}s_b^{(5)}\otimes\mathcal{D}_{\mathcal{T},\hat 1}^{(5)}\big\}\upsilon^{\otimes 5}.\end{cases}$$

The two cases above correspond to $x_{\hat 1}$ being a flex and a hyperflex, respectively. One difference between the cases is that $\pi_{x_{\hat 1}}^{--}s_b^{(4)}$ is zero at a hyperflex, but not at a flex. This is the reason we cannot have $\pi_{x_{\hat 1}}^{--}s_b^{(4)}$ in a full-rank polynomial that approximates $\pi_-^{0,1}\bar\partial u_\upsilon$ near b if $x_{\hat 1}$ is a hyperflex. Note that for every nonzero $\upsilon\in\mathcal{FT}\big|_b$, the maps

$$L_{\hat{1}}^*\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^2\big|_b\longrightarrow\mathcal{H}_\Sigma^+\otimes T\mathbb{P}^2\big|_b, \qquad X\longrightarrow X\otimes s_{x_{\hat{1}}}^{(1)}\upsilon,$$
$$L_{\hat{1}}^*\mathcal{T}^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2\big|_b\longrightarrow\mathcal{H}_\Sigma^{-+}\otimes T\mathbb{P}^2\big|_b, \qquad Y\longrightarrow Y\otimes s_{x_{\hat{1}}}^{(2)}\upsilon^{\otimes 2},$$

are one-to-one. On the other hand, if $x_{\hat{1}}$ is a flex, the map

$$T\Sigma\oplus\mathcal{FT}\big|_b\longrightarrow\mathcal{H}_\Sigma^{--}\otimes T\mathbb{P}^2\big|_b,$$
$$(w,\upsilon)\longrightarrow\big\{\pi_{x_{\hat{1}}}^{--}s_b^{(4)}(w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat{1}}^{(3)}\big\}\upsilon^{\otimes 3}+\big\{\pi_{x_{\hat{1}}}^{--}s_b^{(4)}\otimes\mathcal{D}_{\mathcal{T},\hat{1}}^{(4)}\big\}\upsilon^{\otimes 4},$$

has degree *four*, since the images of $\mathcal{D}_{\mathcal{T},\hat{1}}^{(3)}$ and $\mathcal{D}_{\mathcal{T},\hat{1}}^{(4)}$ at b are linearly independent if $d\geq 4$. If $x_{\hat{1}}$ is a hyperflex, the map

$$T\Sigma\oplus\mathcal{FT}\big|_b\longrightarrow\mathcal{H}_\Sigma^{--}\otimes T\mathbb{P}^2\big|_b,$$
$$(w,\upsilon)\longrightarrow\big\{\pi_{x_{\hat{1}}}^{--}s_b^{(5)}(w,w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat{1}}^{(3)}\big\}\upsilon^{\otimes 3}+\big\{\pi_{x_{\hat{1}}}^{--}s_b^{(5)}(w,\cdot)\otimes\mathcal{D}_{\mathcal{T},\hat{1}}^{(4)}\big\}\upsilon^{\otimes 4}$$
$$+\big\{\pi_{x_{\hat{1}}}^{--}s_b^{(5)}\otimes\mathcal{D}_{\mathcal{T},\hat{1}}^{(5)}\big\}\upsilon^{\otimes 5},$$

has degree *ten*, since the images of $\mathcal{D}_{\mathcal{T},\hat{1}}^{(3)}$ and $\mathcal{D}_{\mathcal{T},\hat{1}}^{(5)}$ at b are linearly independent if $d\geq 4$. These two degrees are the multiplicities mentioned in Remark 2 above. Note that if $d<4$, the set $\mathcal{S}_{1;2}(\mu)$ is empty.

3. Rational Curves with Singularities

3.1. Intersections in $\mathcal{S}_{2;1}(\mu)$, $\bar{\mathcal{S}}_{2;2}(\mu)$, and $\bar{\mathcal{S}}_1(\mu)$. This section is dedicated to computing the intersection numbers of spaces of stable rational maps that are needed in Section 4. We start with the "codimension-one" and "-two" cases.

Lemma 3.1. *If d is a positive integer, the number of two-component rational degree-d curves passing through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$ such that the two components meet at a node at which one of them has a cusp is given by*

$$\big|\mathcal{S}_{2;1}(\mu)\big|=\big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle-3\big|\mathcal{V}_3(\mu)\big|.$$

d	2	3	4	5	6	7
$\lvert\mathcal{S}_{2;1}(\mu)\rvert$	0	0	528	91,872	26,055,360	12,596,219,904

TABLE 2. The Numbers $|\mathcal{S}_{2;1}(\mu)|$ for Small Values of d

Proof: The proof is essentially the same as that of Lemma 5.4 in [Z2], which enumerates irreducible cuspidal curves through $3d-2$ points. The argument uses the analytic estimate of Theorem 2.8 and the topological tools of Subsection 2.1. In fact, the above formula can be deduced from the formula of Lemma 5.4 in [Z2], since its proof applies with no change to enumerate irreducible curves through $3d-3$ points with a cusp on a fixed line in $\mathbb{P}^2$.

Lemma 3.2. *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$\begin{aligned}\big\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\big\rangle &= \big\langle 3a^2 + a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big), [\bar{\mathcal{V}}_2(\mu)]\big\rangle,\\ \big\langle \lambda_{E_2}, [\bar{\mathcal{S}}_{2;2}(\mu)]\big\rangle &= \big\langle 3a^2 + 3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)\\ &\qquad + \big(c_1^2(L_1^*)+c_1^2(L_2^*)\big) + c_1(L_1^*)c_1(L_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle,\end{aligned}$$

with chern classes $c_1^2(L_1^*)+c_1^2(L_2^*)$ and $c_1(L_1^*)c_1(L_2^*)$ defined similarly to $c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)$ and $c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)$.

Proof: We only sketch the argument, since the proof is analogous to that of Lemma 5.13 in [Z2]. Let $\mathcal{D}\in\Gamma\big(\mathbb{P}E_2;\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^n\big)$ be the section induced by the section

$$\mathcal{D}_1+\mathcal{D}_2 \in \Gamma\big(\bar{\mathcal{V}}_2(\mu); E_2^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big),$$

defined similarly to $c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)$; see Subsection 2.2. Then

$$\mathcal{S}_{2;2}(\mu) = \mathcal{D}^{-1}(0)\cap\big(\mathbb{P}E_2|_{\mathcal{V}_2(\mu)-\mathcal{S}_{2;1}(\mu)}\big).$$

Let s be a section of $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)\longrightarrow\mathbb{P}E_2$ such that s is smooth and transversal to the zero set of the bundle on all smooth strata of $\mathbb{P}E_2$ and of $\bar{\mathcal{S}}_{2;2}(\mu)$. In particular, $s^{-1}(0)\cap\mathcal{S}_{2;1}(\mu)=\emptyset$. Then, by Proposition 2.4,

$$\begin{aligned}\big\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\big\rangle &=^{\pm} \big|s^{-1}(0)\cap\mathcal{S}_{2;2}(\mu)\big|\\ &=^{\pm} \big|\mathcal{D}^{-1}(0)\cap\big(\mathbb{P}E_2|_{\mathcal{V}_2(\mu)-\mathcal{S}_{2;1}(\mu)}\big)\cap s^{-1}(0)\big|\\ &= \big\langle e\big(\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big), [s^{-1}(0)]\big\rangle - \mathcal{C}_{\mathbb{P}E_2|_{\partial\bar{\mathcal{V}}_2(\mu)}\cap s^{-1}(0)}(\mathcal{D}).\end{aligned}\tag{3.1}$$

If $\mathcal{U}_{\mathcal{T}}(\mu)\subset\mathcal{V}_2(\mu)$ and $\mathcal{T}$ is not a basic bubble type, it is easy to see that $\mathrm{ev}\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big)$ is a finite set of points. It follows that $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$ is a trivial line bundle and we can assume that s does not vanish on $\mathbb{P}E_2|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$. Thus, the last term in (3.1) is zero, and

$$\big\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\big\rangle = \big\langle ac_2\big(\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big), [\mathbb{P}E_2]\big\rangle.$$

The first claim then follows from the definitions of E_2 and of the relevant chern classes. Note that

$$a\big(c_1(L_1^*)+c_1(L_2^*)\big) = a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big).$$

For the second identity, observe that

$$\gamma_{E_2}\big|_{\mathbb{P}E_2|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\cap\mathcal{D}^{-1}(0)}\approx\mathbb{C}.$$

Thus, the proof is similar.

Lemma 3.3. *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$\begin{aligned}\big\langle a^2,[\bar{\mathcal{S}}_1(\mu)]\big\rangle&=\big\langle a^2c_1^2(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle-\big\langle a^2,[\bar{\mathcal{V}}_2(\mu)]\big\rangle,\\ \big\langle ac_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\big\rangle&=\big\langle 3a^2c_1^2(\mathcal{L}^*)+ac_1^3(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle,\\ \big\langle c_1^2(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\big\rangle&=\big\langle 3a^2c_1^2(\mathcal{L}^*)+3ac_1^3(\mathcal{L}^*)+c_1^4(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle.\end{aligned}$$

Proof: (1) This lemma is proved similarly to Lemma 3.2. Let

$$E=\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1),\qquad E=\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{L}^*,\quad\text{or}\quad E=\mathcal{L}^*\oplus\mathcal{L}^*,$$

depending on which of the three identities we are proving. Let s be a section of $E\longrightarrow\bar{\mathcal{V}}_1(\mu)$ such that s is smooth and transversal to the zero set in E on all smooth strata of $\bar{\mathcal{V}}_1(\mu)$ and of $\bar{\mathcal{S}}_1(\mu)$. Then

$$\begin{aligned}&\big\langle c_2(E),[\bar{\mathcal{S}}_1(\mu)]\big\rangle\\ &\qquad=\big\langle c_2(E)c_2\big(L^*\otimes\mathrm{ev}^*T\mathbb{P}^2),[\bar{\mathcal{V}}_1(\mu)]\big\rangle-\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}).\end{aligned}\tag{3.2}$$

The bundle $L\longrightarrow\bar{\mathcal{V}}_1(\mu)$ and section $\mathcal{D}\in\Gamma\big(\bar{\mathcal{V}}_1(\mu);L^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ in (3.2) are defined as follows. Let $N=3d-4$ and $\tilde{\mathcal{T}}=(S^2,[N],\{\hat{0}\};\hat{0},d)$. Then $L=L_{\hat{0}}\tilde{\mathcal{T}}$ and $\mathcal{D}=\mathcal{D}_{\tilde{\mathcal{T}},\hat{0}}$. Suppose

$$\mathcal{T}=(S^2,[N],I;j,\underline{d})<\tilde{\mathcal{T}}.$$

If $d_{\hat{0}}\neq0$, $\mathcal{D}$ is transversal to the zero set on $\mathcal{U}_{\mathcal{T}}(\mu)$, and

$$s^{-1}(0)\cap\mathcal{D}^{-1}(0)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\emptyset$$

by our assumptions on s. Thus, from now on, we assume that $d_{\hat{0}}=0$. If $\mathcal{T}$ is a semiprimitive bubble type, either $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$ or $\mathcal{L}|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$ is trivial. It follows that if $E=\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{L}^*$, we can choose s so that it does not vanish on $\partial\bar{\mathcal{V}}_1(\mu)$. The second equality is then immediate from (3.2) and

$$ac_1(L^*)=ac_1(\mathcal{L}^*)\in H^4\big(\bar{\mathcal{V}}_1(\mu)\big).\tag{3.3}$$

(2) Suppose $E=\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)$. If the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is at least two and $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$ is not trivial, $|\hat{I}|=\big|H_{\hat{0}}\mathcal{T}\big|=2$. For a good choice of s, the map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 2.8 identifies neighborhoods of $s^{-1}(0)\cap\mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{F}\mathcal{T}$ and in $s^{-1}(0)$. Since $s^{-1}(0)\cap\mathcal{S}_{\mathcal{T};k}(\mu)=\emptyset$, the section $\alpha_{\mathcal{T}}$ of Theorem 2.8 defines an isomorphism between $\mathcal{F}\mathcal{T}$ and $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ over

every point of $s^{-1}(0)\cap\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, by Proposition 2.4 and the analytic estimate of Theorem 2.8,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D}) =^{\pm} \big|\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)\big| = \big\langle a^2, \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle.$$

Summing these contributions over all equivalence classes of such bubble types $\mathcal{T}$, we obtain

$$\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}) = \big\langle a^2, \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle. \tag{3.4}$$

The first identity follows from equations (3.2), (3.3), and (3.4).
(3) Suppose $E=\mathcal{L}^*\oplus\mathcal{L}^*$. If the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is at least two and $\mathcal{L}|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}$ is not trivial, $H_{\hat{0}}\mathcal{T}=\{\hat{1}\}$ is a one-element set and $|\hat{I}|\in\{1,2\}$. If $|\hat{I}|=2$, by an argument similar to (2) above, $\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)$ is $\mathcal{D}$-hollow in the sense of Definition 2.3, where $\mathcal{D}$ is viewed as a section over $s^{-1}(0)\subset\bar{\mathcal{V}}_1(\mu)$. Thus, by Proposition 2.4, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D})=0$. If $|\hat{I}|=1$, $\mathcal{T}=\tilde{\mathcal{T}}(\{l\})$ for some $l\in[N]$. For the purposes of computing $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D})$, it can be assumed that the map of Theorem 2.8 still identifies neighborhoods of $s^{-1}(0)\cap\mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{FT}$ and in $s^{-1}(0)$; see the proof of Lemma 5.13 in [Z2]. Since $\alpha_{\mathcal{T}}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)$, by Proposition 2.4 and the analytic estimate of Theorem 2.8,

$$\begin{aligned}\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D}) &= \big\langle c_1\big(L^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big) - c_1(\mathcal{FT}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s^{-1}(0)\big]\big\rangle\\ &= \big\langle c_1^2(\mathcal{L}^*)c_1(L_{\hat{1}}^*\mathcal{T}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle,\end{aligned}$$

since the restrictions of the bundles L^* and $\mathrm{ev}^*T\mathbb{P}^2$ to $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ are trivial. Summing up these contributions and using equation (2.3), we obtain

$$\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}) = \sum_{l\in[N]}\big\langle c_1^3(\mathcal{L}^*), \big[\bar{\mathcal{U}}_{\tilde{\mathcal{T}}(l)}(\mu)\big]\big\rangle. \tag{3.5}$$

The final identity of the lemma follows from equations (3.2), (3.3), and (3.5); see also equation (2.2).

3.2. Computation of $\big|\mathcal{S}_{1;2}(\mu)\big|$. In this subsection, we enumerate rational plane curves that have a $(3,4)$-cusp. We call point p a $(3,4)$-*cusp* of plane curve $\mathcal{C}$ if for a choice of local coordinates near p, $\mathcal{C}$ is parameterized by a map of the form

$$t\longrightarrow\big(t^3, t^4+o(t^4)\big), \qquad 0\longrightarrow p.$$

Lemma 3.4. *If d is a positive integer, the number of rational degree-d curves that pass through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$*

and have a $(3,4)$*-cusp is given by*

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| = \big\langle 3a^2+6ac_1(\mathcal{L}^*)+4c_1^2(\mathcal{L}^*),\big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle - 2\big|\mathcal{S}_{2;1}(\mu)\big| - 3\big|\mathcal{V}_3(\mu)\big| \\ -\big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.\end{aligned}$$

Proof: (1) We continue with the notation used in the proof of Lemma 3.3. By definition, Proposition 2.4, and equation (3.3),

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| &=^{\pm} \big|\mathcal{D}^{(2)\,-1}(0)\cap\mathcal{S}_1(\mu)\big| \\ &= \big\langle e\big(L^{*\otimes 2}\otimes \mathrm{ev}^*T\mathbb{P}^2\big),\big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_1(\mu)}\big(\mathcal{D}^{(2)}\big) \\ &= \big\langle 3a^2+6ac_1(\mathcal{L}^*)+4c_1^2(L^*),\big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_1(\mu)}\big(\mathcal{D}^{(2)}\big).\end{aligned} \tag{3.6}$$

Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\tilde{\mathcal{T}}$. If $d_{\hat{0}}\neq 0$, $\mathcal{D}^{(2)}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$ by Proposition 2.9. Thus, from now on we consider only bubble types $\mathcal{T}$ such that $d_{\hat{0}}=0$.

(2) Suppose $\chi(\mathcal{T})=\{\hat{1}\}$ is a one-element set. Then $\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};1}(\mu)$ and with appropriate identifications

$$\Big|\mathcal{D}^{(2)}_{\tilde{\mathcal{T}},\hat{0}}\big(\gamma_{\mathcal{T};1}(\upsilon)\big)-\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}\big(\tilde{\upsilon}_{\hat{1}}\otimes\tilde{\upsilon}_{\hat{1}}\big)\Big| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\tilde{\upsilon}_{\hat{1}}\big|^2 \qquad \forall\upsilon\in\mathcal{F}S_{\mathcal{T};1;\delta}=\mathcal{F}\mathcal{T}_\delta,$$

where $\gamma_{\mathcal{T};1}:\mathcal{F}\mathcal{T}_\delta\longrightarrow\bar{\mathcal{S}}_1(\mu)$ is the map of Proposition 2.9. Since $d_{\hat{1}}\neq 0$, $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ does not vanish on $\mathcal{S}_{\mathcal{T};1}(\mu)$. Thus, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{T}$ is $\mathcal{D}^{(2)}$-hollow and $\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=0$. If $\hat{I}=H_{\hat{0}}\mathcal{T}$, i.e. $\mathcal{T}=\tilde{\mathcal{T}}(l)$ for some $l\in[N]$, by Proposition 2.4 and the splitting (2.5),

$$\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=2N\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\big), \tag{3.7}$$

where $\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\in\Gamma\big(\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu);\mathrm{Hom}(L^{\otimes 2}_{\bar{\mathcal{T}},\hat{1}};\mathrm{ev}^*T\mathbb{P}^2)\big)$. By Lemma 2.5,

$$\begin{aligned}N\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\big) &= \big\langle c_1\big(\mathrm{ev}^*T\mathbb{P}^2\big)-c_1\big(L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2}\big),\big[\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)\big]\big\rangle - \mathcal{C}_{\mathcal{D}^{(2)\,-1}_{\bar{\mathcal{T}},\hat{1}}(0)}\big(\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}\big) \\ &= 2\big\langle c_1\big(L^*_{\hat{1}}\mathcal{T}\big),\big[\bar{\mathcal{S}}_{\mathcal{T};1}(\mu)\big]\big\rangle - \mathcal{C}_{\mathcal{D}^{(2)\,-1}_{\bar{\mathcal{T}},\hat{1}}(0)}\big(\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}\big),\end{aligned} \tag{3.8}$$

since $\mathrm{ev}\big|\mathcal{U}_{\tilde{\mathcal{T}}(l)}(\mu)=\mu_l$. If $\mathcal{T}'=(S^2,[N]-\{l\},I';j',\underline{d}')\le\bar{\mathcal{T}}$ and $d'_{\hat{1}}\neq 0$, by Corollary 6.3 in [Z2], $\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}$ does not vanish on $\mathcal{U}_{\mathcal{T}'}(\mu)$ if the constraints μ are in general position. If $d'_{\hat{1}}=0$ and $\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)\cap\mathcal{U}_{\mathcal{T}'}(\mu)\neq\emptyset$, Proposition 2.9 implies that $H_{\hat{1}}\mathcal{T}'=\hat{I}'$ is a two-element set. Furthermore, in such a case,

$$\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)\cap\mathcal{U}_{\mathcal{T}'}(\mu)=\mathcal{S}_{\mathcal{T}';2}(\mu),$$

is a finite set and there exists an identification $\gamma_{\mathcal{T}';2}$ of neighborhoods of $\mathcal{S}_{\mathcal{T}';2}(\mu)$ in $\gamma_{E\mathcal{T}'}$ and in $\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)$ such that

$$\left|\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\gamma_{\mathcal{T}';2}(\upsilon)-\alpha_{\mathcal{T}';k}(\upsilon)\right|\le C|\upsilon|^{1+\frac{1}{p}}\qquad\forall\upsilon\in\gamma_{E\mathcal{T}',\delta},\tag{3.9}$$

where $\alpha_{\mathcal{T}';k}\in\Gamma\big(\mathcal{S}_{\mathcal{T}';2}(\mu);\mathrm{Hom}(\gamma_{E\mathcal{T}'};L_{\hat{1}}^*\mathcal{T}'^{\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)$ is an injection on every fiber. On the other hand, $\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}=\pi^\perp_{\bar{\nu}}\circ\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}$, where

$$\pi^\perp_{\bar{\nu}}:L_{\hat{1}}^*\mathcal{T}'^{\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^2\longrightarrow L_{\hat{1}}^*\mathcal{T}'^{\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$$

is the projection onto the quotient by a trivial subbundle $\mathbb{C}\bar{\nu}$; see Subsection 2.1. Since $\alpha_{\mathcal{T}';2}$ is an injection, $\pi^\perp_{\bar{\nu}}\circ\alpha_{\mathcal{T}';2}$ is an isomorphism between $\gamma_{E\mathcal{T}'}$ and $L_{\hat{1}}^*\mathcal{T}'^{\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ over every point of $\mathcal{S}_{\mathcal{T}';2}(\mu)$ if $\bar{\nu}$ is generic. Thus, by Proposition 2.4 and the estimate (3.9),

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T}';2}(\mu)}\big(\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}\big)=\big|\mathcal{S}_{\mathcal{T}';2}(\mu)\big|.\tag{3.10}$$

Combining equations (3.7)-(3.10), using (2.3), and summing over the bubble types $\mathcal{T}$ with $\chi(\mathcal{T})=\{\hat{1}\}$, we obtain

$$\sum_{|\chi(\mathcal{T})|=1}\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=\sum_{l\in[N]}\big\langle 4c_1(\mathcal{L}^*),\big[\bar{\mathcal{S}}_{\tilde{\mathcal{T}}(l);1}(\mu)\big]\big\rangle-2\big|\mathcal{S}_{2,1;2}(\mu)\big|,\tag{3.11}$$

where $\mathcal{S}_{2,1;2}(\mu)$ denotes the set of two-component rational degree-d curves that pass through the $3d-4$ points and have a tacnode at one of the points, which is a node common to both irreducible components.

(3) Suppose $\chi(\mathcal{T})=\{\hat{1},\hat{2}\}$ is a two-element set. If $\chi(\mathcal{T})=\hat{I}$ and $M_{\hat{0}}\mathcal{T}=\emptyset$,

$$\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};1}(\mu)\cup\mathcal{S}_{\mathcal{T};2}(\mu);$$

see Proposition 2.9. By Corollary 6.3 in [Z2], the images of $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}^{(1)}_{\mathcal{T},\hat{2}}$ are transversal in $\mathrm{ev}^*T\mathbb{P}^2$ over $\mathcal{U}_{\mathcal{T}}(\mu)$. Since $\mathcal{S}_{\mathcal{T};1}(\mu)$ is a finite set, it follows that the section $\alpha_{\mathcal{T};1}$ of Proposition 2.9 defines an isomorphism between $\mathcal{F}S_{\mathcal{T};1}$ and $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ over every point of $\mathcal{S}_{\mathcal{T};1}(\mu)$. Thus, by Proposition 2.4 and the analytic estimate of Proposition 2.9,

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T};1}(\mu)}\big(\mathcal{D}^{(2)}\big)=2\big|\mathcal{S}_{\mathcal{T};1}(\mu)\big|\Longrightarrow\mathcal{C}_{\mathcal{S}_{2;1}(\mu)}\big(\mathcal{D}^{(2)}\big)=2\big|\mathcal{S}_{2;1}(\mu)\big|.$$

Similarly, since $\alpha_{\mathcal{T};2}$ does not vanish on $\mathcal{S}_{\mathcal{T};2}(\mu)$ and extends naturally over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$,

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T};2}(\mu)}\big(\mathcal{D}^{(2)}\big)=N\big(\alpha_{\mathcal{T};2}\big)=N\big(\mathcal{D}_{2;2}\big).$$

If $\chi(\mathcal{T})\neq\hat{I}$ or $M_{\hat{0}}\mathcal{T}\neq\emptyset$, by Proposition 2.9, $\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};2}(\mu)$. The section $\alpha_{\mathcal{T};2}$ has full rank on every fiber in these cases. Thus, by Proposition 2.4 and the analytic estimate of Proposition 2.9, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{T}$ is $\mathcal{D}^{(2)}$-hollow and

$\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=0$. If $\chi(\mathcal{T})=\hat{I}$ and $M_{\hat{0}}\mathcal{T}\neq\emptyset$, $\alpha_{\mathcal{T};2}$ extends over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$ via the splitting (2.5) by

$$\alpha_{\mathcal{T};2}\big[x_{\hat{1}},x_{\hat{2}},y_l,b;\upsilon_{\hat{1}},\upsilon_{\hat{2}}\big]=x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}}\upsilon_{\hat{1}}+x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}\upsilon_{\hat{2}}.$$

This extension vanishes only on the set $x_{\hat{1}}=x_{\hat{2}}$. Thus, by Proposition 2.4 and Lemma 2.5,

$$\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=\big\langle c_1\big(L^{*\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^n\big)-c_1\big(\gamma_{L^*\otimes(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T})}\big),\big[\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)\big]\big\rangle \\ -\mathcal{C}_{\alpha^{-1}_{\mathcal{T};2}(0)}\big(\alpha^{\perp}_{\mathcal{T};2}\big)=2\big|\mathcal{S}_{\bar{\mathcal{T}};2}(\mu)\big|.$$

Here we used $\langle c_1(L^*),[\bar{\mathcal{M}}_{0,4}]\rangle=1$; see Corollary 5.22 in [Z2]. Summing over all bubble types $\mathcal{T}$ as above and using Lemma 3.6, we obtain

$$\sum_{|\chi(\mathcal{T})|=2}\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=2\big|S_{2;1}(\mu)\big|+2\big|\mathcal{S}_{2,1;2}(\mu)\big| \\ \big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle. \tag{3.12}$$

(4) Finally, suppose $\chi(\mathcal{T})=\{\hat{1},\hat{2},\hat{3}\}$ is a three-element set. By Proposition 2.9,

$$\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};2}(\mu)=\mathcal{U}_{\mathcal{T}}(\mu).$$

The section $\alpha_{\mathcal{T};2}$ again has full rank. Thus, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is $\mathcal{D}^{(2)}$-hollow if $\chi(\mathcal{T})\neq\hat{I}$. If $\chi(\mathcal{T})=\hat{I}$, $\alpha_{\mathcal{T};2}$ extends over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$ via the splitting (2.5) by

$$\alpha_{\mathcal{T};2}\big[x_{\hat{1}},x_{\hat{2}},x_{\hat{3}},b;\upsilon_{\hat{1}},\upsilon_{\hat{2}},\upsilon_{\hat{3}}\big]=x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}}\upsilon_{\hat{1}}+x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}\upsilon_{\hat{2}}+x_{\hat{3}}\mathcal{D}_{\mathcal{T},\hat{3}}\upsilon_{\hat{3}}.$$

This extension does not vanish on $\mathcal{F}S_{\mathcal{T};2}(\mu)$, since $x_{\hat{1}},x_{\hat{2}},x_{\hat{3}}$ are never all the same. Thus, by Proposition 2.4,

$$\begin{aligned}&\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)\\ &\quad=\big\langle c_1\big(L^{*\otimes2}\otimes\mathrm{ev}^*T\mathbb{P}^n\big)-c_1\big(\gamma_{L^*\otimes L_{\hat{1}}\oplus L^*\otimes L_{\hat{2}}\oplus L^*\otimes L_{\hat{3}}}\big),\big[\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)\big]\big\rangle\\ &\quad=3\big|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)\big|.\end{aligned}$$

Summing over all such bubble types $\mathcal{T}$, we obtain

$$\sum_{|\chi(\mathcal{T})|=3}\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}^{(2)}\big)=3\big|\mathcal{V}_3(\mu)\big|. \tag{3.13}$$

The claim follows by plugging the sum of equations (3.11), (3.12), and (3.13) into (3.6) and using (2.2).

Corollary 3.5. *If d is a positive integer, the number of rational degree-d curves that pass through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$*

and have a $(3,4)$*-cusp is given by*

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| =&\big\langle 33a^2c_1^2(\mathcal{L}^*)+18ac_1^3(\mathcal{L}^*)+4c_1^4(\mathcal{L}^*),\big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle+3\big|\mathcal{V}_3(\mu)\big|\\ &-\big\langle 21a^2+9a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+2\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big)\\ &\qquad+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle\end{aligned}$$

d	2	3	4	5	6	7
$\lvert\mathcal{S}_{1;2}(\mu)\rvert$	0	0	147	54,612	23,177,124	14,617,373,280

TABLE 3. The Numbers $|\mathcal{S}_{1;2}(\mu)|$ for Small Values of d

This corollary is immediate from Lemmas 3.4, 3.1, and 3.3.

Lemma 3.6. *If* d *is a positive integer and* μ *is a tuple of* $3d-4$ *points in general position in* $\mathbb{P}^2$,

$$N\big(\mathcal{D}_{2;2}\big)=\big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.$$

Proof: (1) Since $\mathcal{D}_{2;2}$ does not vanish on $\mathcal{S}_{2;2}(\mu)$, by Lemmas 2.5 and 3.2,

$$\begin{aligned}N\big(\mathcal{D}_{2;2}\big)&=\big\langle c_1(\mathrm{ev}^*T\mathbb{P}^2)-c_1\big(\gamma_{E_2}\big),\big[\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle-\mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}\big(\mathcal{D}_{2;2}^{\perp}\big)\\ &=\big\langle 3a+\lambda_{E_2},\big[\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle-\mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}\big(\mathcal{D}_{2;2}^{\perp}\big)\\ &=\big\langle 12a^2+6a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)\\ &\qquad+c_1(L_1^*)c_1(L_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle-\mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}\big(\mathcal{D}_{2;2}^{\perp}\big).\end{aligned}\tag{3.14}$$

In (3.14), $\mathcal{D}_{2;2}^{\perp}=\pi_{\bar{\nu}}^{\perp}\circ\mathcal{D}_{2;2}$, where $\pi_{\bar{\nu}}^{\perp}\colon\mathrm{ev}^*T\mathbb{P}^2\longrightarrow\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ is the projection onto the quotient by a trivial subbundle $\mathbb{C}\bar{\nu}$; see Subsection 2.1. If $\bar{\nu}$ is generic, by Proposition 2.11, $\pi_{\bar{\nu}}^{\perp}\circ\alpha_{2;2}$ is an isomorphism between $\gamma_{E_2}^*\otimes\big(E_2/\gamma_{E_2}\big)$ and $\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ over every point of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$. Thus, by Proposition 2.4 and the analytic estimate of Proposition 2.11,

$$\mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}\big(\mathcal{D}_{2;2}^{\perp}\big)=\big|\partial\bar{\mathcal{S}}_{2;2}(\mu)\big|=\big|\mathcal{S}_{2;1}(\mu)\big|+\sum_{[\mathcal{T}]}\big|\mathcal{S}_{\mathcal{T};2}(\mu)\big|,\tag{3.15}$$

where the sum is taken over all equivalence classes of non-basic types

$$\mathcal{T}=\big(S^2,[N],I;j,\underline{d}\big)$$

such that $I-\hat{I}=\{k_1,k_2\}$ is a two-element set and $\sum d_i=d$.
(2) Let $\mathcal{T}_i=\big(S^2,M_{k_i},I_{k_i};j,\underline{d}\big)$, where $i=1,2$, be the simple bubble types

corresponding to a bubble type $\mathcal{T}$ as above. If $\mathcal{S}_{\mathcal{T};2}(\mu)\neq\emptyset$, up to a reordering of indices, $\mathcal{T}$ must have one of two forms. The first possibility is that $\mathcal{T}_2$ is basic, while $d_{k_1}=0$ and $H_{k_1}\mathcal{T}=I_{k_1}$ is a two-element set. Then $\mathcal{S}_{\mathcal{T};2}(\mu)=\mathcal{U}_{\mathcal{T}}(\mu)$. The sum of the cardinalities of the sets $\mathcal{S}_{\mathcal{T};2}(\mu)$ taken over all equivalence classes of such bubble types is then $3|\mathcal{V}_3(\mu)|$, since one of the three irreducible components of the image of each map is distinguished. The other possibility is that $\mathcal{T}_2$ is basic, while $d_{k_1}=0$, $H_{k_1}\mathcal{T}=I_{k_1}$ is a one-element set, and $j_l=k_1$ for some $l\in[N]$. Since $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^2}(1)$ is trivial on $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$,

$$\begin{aligned}\big|\mathcal{S}_{\mathcal{T};2}(\mu)\big| = \big\langle c_2\big(\gamma^*_{E\bar{\mathcal{T}}}\otimes\mathrm{ev}^*T\mathbb{P}^2\big),\big[E\bar{\mathcal{T}}\big]\big\rangle &= \big\langle c_1(L^*_{\hat{1}}\bar{\mathcal{T}})+c_1(L^*_{k_2}\bar{\mathcal{T}}),\big[\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)\big]\big\rangle\\ &= \big\langle c_1(L^*_{\hat{1}}\mathcal{T})+c_1(L^*_{k_2}\mathcal{T}),\big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle,\end{aligned}$$

if $I=\{k_1,k_2,\hat{1}\}$. Summing over all equivalence classes of such bubble types and using equations (2.2) and (2.3), we obtain

$$\sum_{[\mathcal{T}]}\big|\mathcal{S}_{\mathcal{T};2}(\mu)\big| = \sum_{[\tilde{\mathcal{T}}]}\sum_{l\in[N]}\big\langle c_1(\mathcal{L}^*_1)+c_1(\mathcal{L}^*_2),\big[\bar{\mathcal{U}}_{\tilde{\mathcal{T}}(l)}(\mu)\big]\big\rangle+3\big|\mathcal{V}_3(\mu)\big|,$$

where the second sum is taken over equivalence classes of basic bubble types $\tilde{\mathcal{T}}=(S^2,[N],I;j,\tilde{d})$ such that $|\tilde{I}|=2$ and $\sum\tilde{d}_i=d$. Combing equations (3.14) and (3.15), we get

$$\begin{aligned}N\big(\mathcal{D}_{2;2}\big) = \big\langle 12a^2+6a\big(c_1(\mathcal{L}^*_1)+c_1(\mathcal{L}^*_2)\big)+\big(c_1^2(\mathcal{L}^*_1)+c_1^2(\mathcal{L}^*_2)\big)&\\ +c_1(\mathcal{L}^*_1)c_1(\mathcal{L}^*_2),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle-\big|S_{2;1}(\mu)\big|-3\big|\mathcal{V}_3(\mu)\big|.&\end{aligned}$$

The claim now follows by using Lemma 3.1.

4. Computation of $CR_3(\mu)$

4.1. The Numbers $n_1^{(3)}(\mu)$, $n_2^{(2)}(\mu)$, and $n_3^{(1)}(\mu)$. The goal of this section is to give topological formulas for the six numbers $n_m^{(k)}(\mu)$ of Corollary 2.13. We start with the three numbers that involve zero-dimensional spaces of rational maps, i.e. $\mathcal{S}_{1;2}(\mu)$, $\mathcal{S}_{2;1}(\mu)$, and $\mathcal{V}_3(\mu)$.

Lemma 4.1. $n_1^{(3)}(\mu)=12\big|\mathcal{S}_{1;2}(\mu)\big|$.

Proof: By Subsection 2.4, $n_1^{(3)}(\mu)=N(\alpha_{1;2})$, where

$$\begin{aligned}&\alpha_{1;2}\in\Gamma\big(\Sigma\times\mathcal{S}_{1;2}(\mu);\mathrm{Hom}(T\Sigma^{\otimes3}\otimes L^{\otimes3},\mathcal{H}_\Sigma^{-\,-}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big),\\ &\alpha_{1;2}\big(x,b,v\otimes\upsilon\big)=\big(\mathcal{D}^{(3)}\upsilon\big)\big(s_x^{(3)}v\big).\end{aligned}\tag{4.1}$$

The section $s^{(3)}\in\Gamma\big(\Sigma;\mathrm{Hom}(T\Sigma^{\otimes3},\mathcal{H}_\Sigma^{--})\big)$ has simple zeros at $z_1,\ldots,z_{24}$. Thus, $s^{(3)}$ induces a non-vanishing section

$$\tilde{s}^{(3)}\in\Gamma\big(\Sigma;\mathrm{Hom}(\tilde{T}\Sigma,\mathcal{H}_\Sigma^{--})\big),\quad \text{where}\quad \tilde{T}\Sigma=T\Sigma^{\otimes3}\otimes\mathcal{O}(z_1)\otimes\ldots\otimes\mathcal{O}(z_{24}).$$

Furthermore, $N(\tilde{\alpha}_{1;2})=N(\alpha_{1;2})$, where

$$\tilde{\alpha}_{1;2}\in\Gamma\big(\Sigma\times\mathcal{S}_{1;2}(\mu);\mathrm{Hom}(\tilde{T}\Sigma\otimes L^{\otimes3},\mathcal{H}_\Sigma^{--}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)$$

is the section obtained by replacing $s^{(3)}$ by $\tilde{s}^{(3)}$ in (4.1). See Subsection 5.2 in [Z2] for a similar argument in the genus-two case. Since $\mathcal{D}^{(3)}$ does not vanish on $\mathcal{S}_{1;2}(\mu)$ by Corollary 6.3 in [Z2], $\tilde{\alpha}_{1;2}$ does not vanish on $\Sigma\times\mathcal{S}_{1;2}(\mu)$. Thus, by Lemma 2.5,

$$n_1^{(3)}(\mu)=N\big(\tilde{\alpha}_{1;2}\big)=\big\langle c_1\big(\mathcal{H}_\Sigma^{--}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)-c_1(\tilde{T}\Sigma),\big[\Sigma\times\mathcal{S}_{1;2}(\mu)\big]\big\rangle=12\big|\mathcal{S}_{1;2}(\mu)\big|.$$

since the euler characteristic of Σ is -4.

Remark: Note that this argument remains valid even if Σ has hyperflexes, i.e. the points $z_1,\ldots,z_{24}$ are not all distinct.

Lemma 4.2. $n_2^{(2)}(\mu)=36\big|\mathcal{S}_{2;1}(\mu)\big|$.

Proof: (1) By Subsection 2.4, $n_2^{(2)}(\mu)=N(\alpha_{2;1})$, where

$$\begin{gathered}\alpha_{2;1}\in\Gamma\big(\Sigma_1\times\Sigma_2\times\mathcal{S}_{2;1}(\mu);\mathrm{Hom}(E,\mathcal{O})\big),\\ E=T\Sigma_1^{\otimes2}\otimes L_1^{\otimes2}\oplus T\Sigma_2\otimes L_2,\quad \mathcal{O}=\mathcal{H}_{\Sigma_1}^-\otimes\mathrm{ev}^*T\mathbb{P}^2,\end{gathered}$$

$$\begin{aligned}&\alpha_{2;1}\big(x_1,x_2,b;v_1\otimes v_1,v_2\otimes v_2\big)\\ &\qquad=\big(\mathcal{D}_1^{(2)}v_1\big)\big(s_{x_1}^{(2)}v_1\big)+\big(\mathcal{D}_2^{(1)}v_2\big)\big(\pi_{x_1}^-s_{x_2}v_2\big)\in\mathcal{H}_\Sigma^-(x_1)\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.\end{aligned}$$

Here we define the line bundles $L_1,L_2\longrightarrow\mathcal{S}_{2;1}(\mu)$ and the sections $\mathcal{D}_1^{(2)}$ and $\mathcal{D}_2^{(1)}$ as follows. If $b\in\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)\cap\mathcal{S}_{2;1}(\mu)$, $\tilde{\mathcal{T}}=\big(S^2,[N],\tilde{I};\tilde{j},\tilde{d}\big)$, $\tilde{I}=\{k_1,k_2\}$, and $\mathcal{D}_{\tilde{\mathcal{T}},k_1}b=0$, we take

$$L_1\big|_b=L_{k_1}\tilde{\mathcal{T}}\big|_b,\quad L_2\big|_b=L_{k_2}\tilde{\mathcal{T}}\big|_b,\quad \mathcal{D}_1^{(2)}\big|_b=\mathcal{D}_{\tilde{\mathcal{T}},k_1}^{(2)}\big|_b,\quad \mathcal{D}_2^{(1)}\big|_b=\mathcal{D}_{\tilde{\mathcal{T}},k_2}^{(1)}\big|_b.$$

(2) By Lemma 2.5,

$$\begin{aligned}N(\alpha_{2;1})&=\sum_{k=0}^{k=2}\big\langle\lambda_E^{3-k}c_k\big(\mathcal{H}_{\Sigma_1}^-\otimes\mathbb{C}^2\big),\big[\mathbb{P}E\big]\big\rangle-\mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^\perp\big)\\ &=64\big|\mathcal{S}_{2;1}(\mu)\big|-\mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^\perp\big).\end{aligned}\tag{4.2}$$

Since Σ is not hyperelliptic by assumption, $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in C^*$ if and only if $x_1=x_2$. Thus, $\pi_{x_1}^-s_{x_2}=0$ if and only if $x_1=x_2$. Since the images of $\mathcal{D}_1^{(2)}\big|_b$ and $\mathcal{D}_2^{(1)}\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{S}_{2;1}(\mu)$ by Corollary 6.3 in [Z2], it follows that

$$\alpha_E^{-1}(0)=\mathcal{Z}\equiv\big\{\big(x,x,b;T\Sigma_2\otimes L_2\big)\colon x\in\Sigma,\ b\in\mathcal{S}_{2;1}(\mu)\big\}.$$

The normal bundle of $\mathcal{Z}$ in $\mathbb{P}E_2$ is

$$\mathcal{N}Z=T\Sigma_2\oplus\big(T\Sigma_2\otimes L_2\big)^*\otimes T\Sigma_1^{\otimes 2}\otimes L_1^{\otimes 2}\approx T\Sigma\oplus T\Sigma\longrightarrow\mathcal{Z}.$$

With appropriate identifications,

$$\big|\alpha_E(x,x,b;w,u)-\alpha_{\mathcal{Z}}(x,b;w,u)\big|\le C|w|^2\qquad\forall(x,x,b;w,u)\in\mathcal{N}Z,\tag{4.3}$$

where $\alpha_{\mathcal{Z}}\in\Gamma\big(\mathcal{Z};\mathrm{Hom}(\mathcal{N}Z;\gamma_E^*\otimes\mathcal{O})\big)$ is defined by

$$\alpha_{\mathcal{Z}}(x,b;w,u)=\big\{\mathcal{D}_1^{(2)}\otimes s_x^{(2)}\big\}\circ u+\mathcal{D}_2^{(1)}\otimes s_x^{(2)}(w,\cdot).$$

Since the images of $\mathcal{D}_1^{(2)}\big|_b$ and $\mathcal{D}_2^{(1)}\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{S}_{2;1}(\mu)$, $\alpha_{\mathcal{Z}}$ has full rank over all of $\mathcal{Z}$. If $\bar{\nu}$ is generic, $\pi_{\bar{\nu}}^{\perp}\circ\alpha_{\mathcal{Z}}$ also has full rank on every fiber, where $\pi_{\bar{\nu}}^{\perp}\colon\mathcal{O}\longrightarrow\mathcal{O}/\mathbb{C}\bar{\nu}$ is the quotient projection as before. Then by the analytic estimate (4.3) and Proposition 2.4,

$$\begin{aligned}\mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^{\perp}\big)&=\big\langle c_1\big(T^*\Sigma\otimes\mathcal{O}^{\perp}\big)-c_1\big(\mathcal{N}Z\big),\big[\Sigma\times\mathcal{S}_{2;1}(\mu)\big]\big\rangle\\&=\big\langle\big(3c_1(T^*\Sigma)+2c_1(T^*\Sigma)\big)+2c_1(T^*\Sigma),[\Sigma]\big\rangle\big|\mathcal{S}_{2;1}(\mu)\big|\\&=28\big|\mathcal{S}_{2;1}(\mu)\big|.\end{aligned}\tag{4.4}$$

The claim is obtained by plugging (4.4) into (4.2).

Lemma 4.3. $n_3^{(1)}(\mu)=36\big|\mathcal{V}_3(\mu)\big|$.

Proof: (1) By Subsection 2.4, $n_3^{(1)}(\mu)=N(\alpha_3)$, where

$$\alpha_3\in\Gamma\big(\Sigma_1\times\Sigma_2\times\Sigma_3\times\mathcal{V}_3(\mu);\mathrm{Hom}(E,\mathcal{O})\big),$$
$$E=\bigoplus_i T\Sigma_i\otimes L_i,\quad \mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2,$$

$$\alpha_3\big(x_1,x_2,x_3,b;v_1\otimes\upsilon_1,v_2\otimes\upsilon_2,v_3\otimes\upsilon_3\big)=\sum_i\big(\mathcal{D}_i\upsilon_i\big)\big(s_{x_i}v_i\big)\in\mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

Here the bundles $L_i\longrightarrow\mathcal{V}_3(\mu)$ and the sections

$$\mathcal{D}_i\in\Gamma\big(\mathcal{V}_3(\mu);L_i^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$$

are defined as follows. If $b\in\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)\subset\mathcal{V}_3(\mu)$, $\tilde{\mathcal{T}}=(S^2,[N],\tilde{I};\tilde{j},\tilde{d})$, and $\tilde{I}=\{k_1,k_2,k_3\}$, we let $L_i\big|_b=L_{k_i}\tilde{\mathcal{T}}$ and $\mathcal{D}_i=\mathcal{D}_{\tilde{\mathcal{T}},k_i}$. These bundles and sections are well-defined once we fix a representative for each equivalence class of such

bubble types $\tilde{\mathcal{T}}$ and order the elements of the corresponding set $\tilde{I}$.
(2) By Lemma 2.5,

$$\begin{aligned} N(\alpha_3) &= \sum_{k=0}^{k=3}\big\langle \lambda_E^{5-k}c_k(\mathbb{C}^5),[\mathbb{P}E]\big\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^{\perp}\big)\\ &= 64\big|\mathcal{V}_3(\mu)\big| - \mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^{\perp}\big). \end{aligned} \tag{4.5}$$

Since Σ is not hyperelliptic, $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in C^*$ if and only if $x_1=x_2$. Since the images of $\mathcal{D}_i\big|_b$ and $\mathcal{D}_j\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{V}_3(\mu)$ and $i\neq j$ by Corollary 6.3 in [Z2], it follows that

$$\begin{aligned} \alpha_E^{-1}(0) = \mathcal{Z} \equiv \Big\{\big(x,x,x,b;[v\otimes\upsilon_1,v\otimes\upsilon_2,v\otimes\upsilon_3]\big)\in\mathbb{P}E:&\\ x\in\Sigma,\ b\in\mathcal{V}_3(\mu),\ \sum_i \mathcal{D}_i\upsilon_i=0\Big\}.& \end{aligned}$$

The normal bundle of $\mathcal{Z}$ in $\mathbb{P}E_2$ is

$$\begin{aligned} \mathcal{N}Z &= T\Sigma_2\oplus T\Sigma_3\oplus\big(T\Sigma_1\otimes L_1\big)^*\otimes\big(T\Sigma_2\otimes L_2\oplus T\Sigma_3\otimes L_3\big)\\ &\approx T\Sigma\oplus T\Sigma\oplus\mathbb{C}^2\longrightarrow\mathcal{Z}. \end{aligned}$$

With appropriate identifications,

$$\big|\alpha_E(x,x,x,b;w_2,w_3,u_2,u_3)-\alpha_{\mathcal{Z}}(x,b;w_2,w_3,u_2,u_3)\big|\le C\big|(w_2,w_3)\big|^2 \tag{4.6}$$

for all $(w_2,w_3,u_2,u_3)\in\mathcal{N}Z$, where $\alpha_{\mathcal{Z}}\in\Gamma\big(\mathcal{Z};\mathrm{Hom}(\mathcal{N}Z;(T\Sigma_1\otimes L_1)^*\otimes\mathcal{O})\big)$ is defined by

$$\begin{aligned} \alpha_{\mathcal{Z}}\big(x,b;w_2,w_3,u_2,u_3\big) = &\big\{\mathcal{D}_2\otimes s_x\big\}\circ u_2+\big\{\mathcal{D}_3\otimes s_x\big\}\circ u_3\\ &+\mathcal{D}_2\otimes s^{(2)}_{g_x,x}(w_2,\cdot)+\mathcal{D}_3\otimes s^{(2)}_{g_x,x}(w_3,\cdot) \end{aligned}$$

and $\big\{g_x\colon x\in\Sigma\big\}$ is a smooth family of metrics on Σ such that g_x is flat on a neighborhood of x. Since the images of $\mathcal{D}_2\big|_b$ and $\mathcal{D}_3\big|_b$ are linearly independent in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ for all $b\in\mathcal{V}_3(\mu)$ and the section $s_x^{(2)}=\pi_x^-\circ s_{g_x,x}$ does not vanish on Σ, the linear map $\alpha_{\mathcal{Z}}$ is injective over $\mathcal{Z}$. Thus, by the analytic estimate (4.6) and Proposition 2.4,

$$\begin{aligned} \mathcal{C}_{\alpha_E^{-1}(0)}\big(\alpha_E^{\perp}\big) &= \big\langle c_1\big(T^*\Sigma\otimes\mathcal{O}^{\perp}\big)-c_1\big(\mathcal{N}Z\big),\big[\Sigma\times\mathcal{V}_3(\mu)\big]\big\rangle\\ &= \big\langle 5c_1(T^*\Sigma)+2c_1(T^*\Sigma),[\Sigma]\big\rangle\big|\mathcal{V}_3(\mu)\big| = 28\big|\mathcal{V}_3(\mu)\big|. \end{aligned} \tag{4.7}$$

The claim is obtained by plugging (4.7) into (4.5).

4.2. The Number $n_2^{(1)}(\mu)$. We now use the topological tools of Subsection 2.1 along with the analytic estimates of Subsection 2.3 to give a topological formula for the number $n_2^{(1)}(\mu)$ of Corollary 2.13. The computation involved is long, but fairly straightforward.

Lemma 4.4. *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_2^{(1)}(\mu) = 12\big\langle 10a^2 + 3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_1^*)\big) + c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_1^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.$$

Proof: (1) By Subsection 2.4, $n_2^{(1)}(\mu)=N(\alpha_2)$, where

$$\begin{aligned}
&\alpha_2\in\Gamma\big(\Sigma_1\times\Sigma_2\times\bar{\mathcal{V}}_2(\mu);\mathrm{Hom}(\tilde{E},\mathcal{O})\big),\\
&\tilde{E} = T\Sigma_1\otimes L_1 \oplus T\Sigma_2\otimes L_2,\ \ \mathcal{O} = \mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2,\\
&\alpha_2\big(x_1,x_2,b;v_1\otimes\upsilon_1,v_2\otimes\upsilon_2\big)\\
&\qquad = \big(\mathcal{D}_1\upsilon_1\big)\big(s_{x_1}v_1\big) + \big(\mathcal{D}_2\upsilon_2\big)\big(s_{x_2}v_2\big) \in \mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2,
\end{aligned}$$

with the bundles $L_i \longrightarrow \bar{\mathcal{V}}_2(\mu)$ and the sections $\mathcal{D}_i \in \Gamma\big(\bar{\mathcal{V}}_2(\mu); L_i^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ defined as in the proof of Lemma 4.3. By Lemma 2.5,

$$\begin{aligned}
N(\alpha_2) &= \sum_{k=0}^{k=3}\big\langle\lambda_{\tilde{E}}^{5-k}c_k\big(\mathcal{O}\big), \big[\mathbb{P}\tilde{E}\big]\big\rangle - \mathcal{C}_{\alpha_{\tilde{E}}^{-1}(0)}\big(\alpha_{\tilde{E}}^{\perp}\big)\\
&= 16\big\langle 36a^2+18a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+3\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)\\
&\qquad + 4c_1(L_1^*)c_1(L_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle - \mathcal{C}_{\alpha_{\tilde{E}}^{-1}(0)}\big(\alpha_{\tilde{E}}^{\perp}\big).
\end{aligned} \tag{4.8}$$

Since $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in\mathbb{C}$ if and only if $x_1=x_2$,

$$\begin{aligned}
\alpha_{\tilde{E}}^{-1}(0) =&\big\{\big(x,x,b;[\upsilon_1,\upsilon_2]\big) : \mathcal{D}_1\upsilon_1+\mathcal{D}_2\upsilon_2=0\big\}\\
&\cup\bigcup_{i=1,2}\big\{\big(x_1,x_2,b;T_{x_i}\Sigma_i\otimes L_i|_b\big) : \mathcal{D}_ib=0\big\}.
\end{aligned}$$

We now partition these sets further and apply the topological tools of Subsection 2.1. As usually, we denote by $\bar{\nu}\in\Gamma\big(\mathbb{P}\tilde{E};\mathcal{O}\big)$ a generic non-vanishing section.

(2) We start with the spaces $(\Sigma^2-\Delta)\times\mathcal{S}_{2;1}(\mu)$ and $\Delta\times\mathcal{S}_{2;1}(\mu)$. For notational simplicity, we assume that $\mathcal{D}_1b=0$ for $b\in\mathcal{S}_{2;1}(\mu)$. The normal bundle of the subspace

$$\mathcal{Z}_{2;1}(\mu) = \big\{\big(x_1,x_2,b;T_{x_1}\Sigma_1\otimes L_1|_b\big) : x_1\neq x_2,\ b\in\mathcal{S}_{2;1}(\mu)\big\}$$

in $\mathbb{P}\tilde{E}$ is given by

$$\mathcal{N}Z_{2;1} = \pi_{\tilde{E}}^*\big(L_1^*\otimes \mathrm{ev}^*T\mathbb{P}^2 \oplus T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big) \approx \mathbb{C}^2\otimes T\Sigma_1^*\otimes T\Sigma_2.$$

With appropriate identifications, $\mathcal{D}_1(b,X)=X$ for all $X\in(L_1^*\otimes\mathrm{ev}^*T\mathbb{P}^2)_b$ sufficiently small. Thus,

$$\alpha_{\tilde{E}}\big(x_1,x_2,b;X,u\big) = \alpha_{2;1}\big(x_1,x_2,b;X,u\big) \equiv X\otimes s_{\Sigma,x_1} + \big\{\mathcal{D}_2\otimes s_{\Sigma,x_2}\big\}\circ u.$$

Since $\alpha_{2;1}$ has full rank on $\mathcal{Z}_{2;1}(\mu)\approx(\Sigma^2-\Delta)\times\mathcal{S}_{2;1}(\mu)$ and extends over $\Sigma^2\times\mathcal{S}_{2;1}(\mu)$,

$$\mathcal{C}_{(\Sigma^2-\Delta)\times\mathcal{S}_{2;1}(\mu)}\big(\alpha_{\tilde{E}}^\perp\big) = N\big(\pi_{\bar{\nu}}^\perp\circ\alpha_{2;1}\big),$$

as long as $\bar{\nu}$ is generic. In fact, it can be assumed the image of $\bar{\nu}$ is disjoint from $\mathcal{H}_\Sigma^+(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2$. Then $\pi_{\bar{\nu}}^\perp\big(\mathcal{H}_\Sigma^+(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ is a rank-two subbundle of $\mathcal{O}^\perp$, and

$$\pi_{\bar{\nu}}^\perp\circ\alpha_{2;1}\colon \mathbb{C}^2 \longrightarrow \gamma_{\tilde{E}}^*\otimes\pi_{\bar{\nu}}^\perp\big(\mathcal{H}_\Sigma^+(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$$

is an isomorphism. It follows that $N\big(\pi_{\bar{\nu}}^\perp\circ\alpha_{2;1}\big)=N\big(\tilde{\alpha}_{2;1}\big)$, where

$$\begin{gathered}
\tilde{\alpha}_{2;1}\in\Gamma\big(\Sigma^2\times\mathcal{S}_{2;1}(\mu);\mathrm{Hom}(F_2;\mathcal{O}_2)\big),\\
\tilde{\alpha}_{2;1}(x_1,x_2,b;u) = \pi^\perp_{\pi_{x_1}^-\bar{\nu}}\circ\big\{\mathcal{D}_2\otimes\pi_{x_1}^- s_{x_2}\big\}\circ u,\\
F_2 = T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\approx T\Sigma_1^*\otimes T\Sigma_2,\\
\mathcal{O}_2 = T\Sigma_1^*\otimes L_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^-\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^\perp\approx T\Sigma_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^-\otimes\mathbb{C}^2\big)^\perp,
\end{gathered}$$

By Lemma 2.5,

$$\begin{aligned}
N\big(\tilde{\alpha}_{2;1}\big) &= \big\langle c_1^2(F_2^*)+c_1(F_2^*)c_1(\mathcal{O}_2),\big[\Sigma^2\times\mathcal{S}_{2;1}(\mu)\big]\big\rangle - \mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}\big(\tilde{\alpha}_{2;1}^\perp\big)\\
&= 48\big|\mathcal{S}_{2;1}(\mu)\big| - \mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}\big(\tilde{\alpha}_{2;1}^\perp\big).
\end{aligned}$$

The zero set of $\tilde{\alpha}_{2;1}$ is $\Delta\times\mathcal{S}_{2;1}(\mu)$; see the proof of Lemma 4.2. Its normal bundle is $T\Sigma_2\approx T\Sigma$. If $\bar{\nu}$ and $\bar{\nu}_2$ are generic, as in the proof of Lemma 4.2, we obtain

$$\big|\tilde{\alpha}_{2;1}^\perp(x,x,b;w) - \tilde{\alpha}_{2;1;\Delta}(x,b;w)\big| \le C|w|^2 \quad \forall w\in T\Sigma_\delta,$$

where $\tilde{\alpha}_{2;1;\Delta}: T\Sigma\longrightarrow F_2^*\otimes\mathcal{O}_2^\perp$ is an injection on every fiber. Thus, by Proposition 2.4,

$$\mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}\big(\tilde{\alpha}_{2;1}^\perp\big) = \big\langle c_1\big(F_2^*\otimes\mathcal{O}_2^\perp\big) - c_1\big(T\Sigma\big),\big[\Sigma\times\mathcal{S}_{2;1}(\mu)\big]\big\rangle = 24\big|\mathcal{S}_{2;1}(\mu)\big|.$$

We conclude that

$$\mathcal{C}_{(\Sigma^2-\Delta)\times\mathcal{S}_{2;1}(\mu)}\big(\alpha_{\tilde{E}}^\perp\big) = 24\big|\mathcal{S}_{2;1}(\mu)\big|. \tag{4.9}$$

On the other hand, the space $\bar{\mathcal{Z}}_{2;1}-\mathcal{Z}_{2;1}\approx\Delta\times\mathcal{S}_{2;1}(\mu)$ is $\alpha_{\tilde{E}}^\perp$-hollow, and thus

$$\mathcal{C}_{\Delta\times\mathcal{S}_{2;1}(\mu)}\big(\alpha_{\tilde{E}}^\perp\big) = 0.$$

Indeed, its normal bundle in $\mathbb{P}\tilde{E}_2$ is given by

$$\mathcal{N}Z = \pi_{\tilde{E}}^*\big(T\Sigma_2 \oplus L_1^*\otimes \mathrm{ev}^*T\mathbb{P}^2 \oplus T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big).$$

With appropriate identifications,

$$\big|\alpha_{\tilde{E}}(x,x,b;w,X,u) - \tilde{\alpha}(x,b;w,X,u)\big| \le C|w|^2|u| \quad \forall (w,X,u)\in\mathcal{N}Z_\delta,$$

where

$$\tilde{\alpha}(x,b;w,X,u) = X\otimes s_x + \big\{\mathcal{D}_2\otimes s_x\big\}\circ u + \big\{\mathcal{D}_2\otimes s^{(2)}_{g_x,x}(w,\cdot)\big\}\circ u.$$

Since $\pi_x^- s^{(2)}_{g_x,x}$ does not vanish, $\tilde{\alpha}$ is a dominant term for $\alpha_{\tilde{E}}$; the same holds for composites with projection maps. Since

$$\mathrm{rk}\big(\mathcal{H}_\Sigma^-\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^\perp > \mathrm{rk}\, T\Sigma_2\otimes\big(T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big) + \frac{1}{2}\dim\big(\Delta\times\mathcal{S}_{2;1}(\mu)\big),$$

$\Delta\times\mathcal{S}_{2;1}(\mu)$ is $\alpha_{\tilde{E}}^\perp$-hollow.

(3) Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})$ is a non-basic bubble type and $\mathcal{D}_1 b=0$ for some $b\in\mathcal{U}_\mathcal{T}(\mu)\subset\bar{\mathcal{V}}_2(\mu)$. Let I_1 and I_2 be the corresponding rooted trees and $k_1\in I_1$ and $k_2\in I_2$ the minimal elements. Then $d_{k_1}=0$, $d_{k_2}\neq 0$, and $\big|H_{k_1}\mathcal{T}\big|\in\{1,2\}$. Let

$$\mathcal{Z}_\mathcal{T} = \big\{\big(x_1,x_2,b;T_{x_1}\Sigma_1\otimes L_1|_b\big)\in\mathbb{P}\tilde{E} \colon b\in\mathcal{U}_\mathcal{T}(\mu)\big\}.$$

By Theorem 2.8, the normal bundle of $\mathcal{Z}_\mathcal{T}$ in $\mathbb{P}\tilde{E}$ is

$$\mathcal{N}Z_\mathcal{T} = \pi_{\tilde{E}}^*\big(\mathcal{F}\mathcal{T}\oplus T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big) \approx \mathcal{F}\mathcal{T}\oplus T\Sigma_1^*\otimes T\Sigma_2\otimes L_2.$$

First suppose $H_{k_1}\mathcal{T}=\{\hat{1}\}$ is a one-element set. Then, with appropriate identifications,

$$\big|\alpha_{\tilde{E}}\big(x_1,x_2,b;\upsilon,u\big) - \alpha_{\mathcal{Z}_\mathcal{T}}\big(x_1,x_2,b;\upsilon_{\hat{1}},u\big)\big| \le C(b)|\upsilon|^{\frac{1}{p}}\big(|\upsilon_{\hat{1}}|+|u|\big) \tag{4.10}$$

for all $(\upsilon,u)\in\mathcal{N}_b\mathcal{Z}_{\mathcal{T},\delta(b)}$, where

$$\alpha_{\mathcal{Z}_\mathcal{T}}\big(x_1,x_2,b;\upsilon_{\hat{1}},u\big) = \big(\mathcal{D}_{\mathcal{T},\hat{1}}\upsilon_{\hat{1}}\big)\otimes s_{x_1} + \big\{\mathcal{D}_2\otimes s_{x_2}\big\}\circ u.$$

If $H_{k_1}\mathcal{T}\neq\hat{I}$, the images of $\mathcal{D}_{\mathcal{T},\hat{1}}\big|_b$ and of $\mathcal{D}_2\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{U}_\mathcal{T}(\mu)$. Thus, $\alpha_{\mathcal{Z}_\mathcal{T}}$ is injective on every fiber and $\mathcal{Z}_\mathcal{T}$ is $\alpha_{\tilde{E}}^\perp$-hollow by (4.10), provided $\bar{\nu}$ is generic. Then, by Proposition 2.4,

$$\mathcal{C}_{\mathcal{Z}_\mathcal{T}}\big(\alpha_{\tilde{E}}^\perp\big)=0 \qquad \text{if} \quad \big|\mathcal{H}_{k_1}\mathcal{T}\big| = 1 < |\hat{I}|.$$

If $\mathcal{H}_{k_1}\mathcal{T}=\hat{I}$, $\alpha_{\mathcal{Z}_\mathcal{T}}$ has full rank outside of the set

$$\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu) = \big\{\big(x,x,b;T_x\Sigma_1\otimes L_1|_b\big)\in\mathcal{Z}_\mathcal{T} \colon \mathcal{D}_{\mathcal{T},\hat{1}}\upsilon_{\hat{1}}+\mathcal{D}_2\upsilon_2=0 \\ \text{for some } (\upsilon_{\hat{1}},\upsilon_2)\neq 0\big\} \approx \Sigma\times\mathcal{S}_{\mathcal{T};2}(\mu).$$

Since $\alpha_{\mathcal{Z}_\mathcal{T}}$ extends naturally over $\bar{\mathcal{Z}}_\mathcal{T}\subset\mathbb{P}\tilde{E}$, by Proposition 2.4,

$$\mathcal{C}_{\mathcal{Z}_\mathcal{T}-\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}\big(\alpha_{\tilde{E}}^\perp\big) = N\big(\tilde{\alpha}_\mathcal{T}\big),$$

where

$$\tilde{\alpha}_{\mathcal{T}} = \pi_{\tilde{\nu}}^{\perp} \circ \alpha_{\mathcal{Z}_{\mathcal{T}}} \in \Gamma\big(\bar{\mathcal{Z}}_{\mathcal{T}}; \mathrm{Hom}(F_2; \mathcal{O}_2)\big),$$
$$\tilde{\alpha}_{\mathcal{T}} = \pi_{\tilde{\nu}}^{\perp} \circ \big((\mathcal{D}_{\mathcal{T},\hat{1}} \upsilon_{\hat{1}}) \otimes s_{x_1} + \{\mathcal{D}_2 \otimes s_{x_2}\} \circ u\big);$$
$$F_2 = L_1^* \otimes L_{\hat{1}}\mathcal{T} \oplus T\Sigma_1^* \otimes L_1^* \otimes T\Sigma_2 \otimes L_2 \approx L_{\hat{1}}\mathcal{T} \oplus T\Sigma_1^* \otimes T\Sigma_2 \otimes L_2,$$
$$\mathcal{O}_2 = \gamma_{\tilde{E}}^* \otimes \mathcal{O}^{\perp} \approx T\Sigma_1^* \otimes \mathbb{C}^5.$$

By Lemma 2.5,

$$N\big(\tilde{\alpha}_{\mathcal{T}}\big) = \sum_{k=0}^{k=4} \big\langle \lambda_{F_2}^{4-k} c_k(\mathcal{O}_2), [\mathbb{P}F_2] \big\rangle - \mathcal{C}_{\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T};F_2}^{\perp}\big)$$
$$= 16\big\langle 3c_1(\mathcal{L}_1^*) + 4c_1(\mathcal{L}_2^*), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)] \big\rangle - \mathcal{C}_{\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T};F_2}^{\perp}\big),$$

since $a = 0$, $c_1(L_{\hat{1}}\tilde{\mathcal{T}}) = c_1(\mathcal{L}_1^*)$, and $c_1(L_2^*) = c_1(\mathcal{L}_2^*)$ in $H^*\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big)$. Furthermore,

$$\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0) = \big\{\big(x, x, b; [v \otimes \upsilon_{\hat{1}}, v \otimes \upsilon_2]\big) \in \mathbb{P}F_2 : \mathcal{D}_{\mathcal{T},\hat{1}} \upsilon_{\hat{1}} + \mathcal{D}_2 \upsilon_2 = 0\big\} \approx \Sigma \times \mathcal{S}_{\mathcal{T};2}(\mu),$$
$$F_3 \equiv \mathcal{N}\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0) = T\Sigma_2 \oplus \gamma_{E\mathcal{T}} \approx T\Sigma \oplus \mathbb{C}^2,$$
$$\big|\tilde{\alpha}_{\mathcal{T};F_2}^{\perp}(x, x, b; w, X) - \tilde{\alpha}_{\mathcal{T},\Delta}(x, b; w, X)\big| \le C|w|^2 \ \forall (w, X) \in F_{3,\delta},$$

where

$$\tilde{\alpha}_{\mathcal{T},\Delta} \in \Gamma\big(\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0); \mathrm{Hom}(F_3, \mathcal{O}_3)\big), \quad \mathcal{O}_3 = \gamma_{F_2}^* \otimes \mathcal{O}_2^{\perp} \approx \big(T\Sigma^* \otimes \mathbb{C}^5\big)^{\perp},$$
$$\tilde{\alpha}_{\mathcal{T},\Delta}(w, X) = \pi_{\tilde{\nu}_2}^{\perp} \circ \big(\pi_{\tilde{\nu}}^{\perp} \circ \big(X \otimes s_x + \mathcal{D}_2 \otimes s_{g_x,x}^{(2)}(w, \cdot)\big)\big).$$

Since $\tilde{\alpha}_{\mathcal{T},\Delta}$ has full rank on every fiber, by Proposition 2.4,

$$\mathcal{C}_{\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T};F_2}^{\perp}\big) = \big\langle c_1(\mathcal{O}_3) - c_1(F_3), [\tilde{\alpha}_{\mathcal{T};F_2}^{-1}(0)] \big\rangle$$
$$= 24\big|\mathcal{S}_{\mathcal{T};2}(\mu)\big| = 24\big\langle c_1(\mathcal{L}_{\hat{1}}^*\mathcal{T}) + c_1(\mathcal{L}_2^*), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)] \big\rangle.$$

On the other hand, $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha_{\tilde{E}}^{\perp}$-hollow, and thus

$$\mathcal{C}_{\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}\big(\alpha_{\tilde{E}}^{\perp}\big) = 0.$$

Indeed, by Theorem 2.8,

$$\mathcal{N}\tilde{\mathcal{S}}_{\mathcal{T};2} = T\Sigma_2 \oplus L_2^* \otimes \big(\mathrm{Im}\, \mathcal{D}_{\mathcal{T},\hat{1}}\big)^{\perp} \oplus L_1^* \otimes L_{\hat{1}}\mathcal{T} \oplus T\Sigma_1^* \otimes L_1^* \otimes T\Sigma_2 \otimes L_2,$$
$$\big|\alpha_{\tilde{E}}\big(x, x, b; w, X, \upsilon_{\hat{1}}, u\big) - \tilde{\alpha}\big(x, b; w, X, \upsilon_{\hat{1}}, u\big)\big| \le C\big(|w| + |X|\big)|w||u|,$$

for all $(w, X, \upsilon_{\hat{1}}, u) \in \mathcal{N}\tilde{\mathcal{S}}_{\mathcal{T};2;\delta}$, where

$$\pi_x^{-1}\tilde{\alpha}\big(x, b; w, X, \upsilon_{\hat{1}}, u\big) = \big\{\mathcal{D}_2 \otimes s_x^{(2)}(w, \cdot)\big\} \circ u.$$

The claim that $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha_{\tilde{E}}^{\perp}$-hollow then follows as in (2). Summing over bubble types as above, we conclude that

(4.11)
$$\sum_{|\chi(\mathcal{T})|=1} \mathcal{C}_{\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)}\big(\alpha_{\tilde{E}}^{\perp}\big) = 16\sum_{\tilde{\mathcal{T}}}\sum_{\{i,j\}=\{1,2\}}\sum_{l\in M_i^*}\big\langle 3c_1(\mathcal{L}_i^*)+4c_1(\mathcal{L}_j^*), \big[\bar{\mathcal{U}}_{\tilde{\mathcal{T}}(l)}\big]\big\rangle - 24\sum_{\tilde{\mathcal{T}}}\sum_{l\in[N]}\big\langle c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{U}}_{\tilde{\mathcal{T}}(l)}\big]\big\rangle,$$

where the outer sums are taken over all equivalence classes of basic bubble types $\tilde{\mathcal{T}}$ such that $\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$ is contained in $\mathcal{V}_2(\mu)$.

(4) We next consider the case where $H_{k_1}\mathcal{T}=\{\hat{1},\hat{2}\}$ is a two-element set. Then $\hat{I}=H_{k_1}\mathcal{T}$,

$$\big|\alpha_{\tilde{E}}\big(x_1,x_2,b;\upsilon,u\big)-\alpha_{\mathcal{Z}_{\mathcal{T}}}\big(x_1,x_2,b;\upsilon,u\big)\big| \le C(b)|\upsilon|^{\frac{1}{p}}\big(|\upsilon|+|u|\big)$$

for all $(\upsilon,u)\in\mathcal{N}_b\mathcal{Z}_{\mathcal{T},\delta(b)}$, where

$$\alpha_{\mathcal{Z}_{\mathcal{T}}}\big(x_1,x_2,b;\upsilon,u\big)=\big(\mathcal{D}_{\mathcal{T},\hat{1}}\upsilon_{\hat{1}}\big)\otimes s_{x_1}+\big(\mathcal{D}_{\mathcal{T},\hat{2}}\upsilon_{\hat{2}}\big)\otimes s_{x_1}+\big\{\mathcal{D}_2\otimes s_{x_2}\big\}\circ u.$$

The map $\alpha_{\mathcal{Z}_{\mathcal{T}}}$ has full rank outside of the set

$$\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)=\big\{\big(x,x,b;T_x\Sigma_1\otimes L_1|_b\big)\in\mathcal{Z}_{\mathcal{T}}\big\}\approx\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu).$$

Thus, by Proposition 2.4, $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}-\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}\big(\alpha_{\tilde{E}}^{\perp}\big)=N\big(\pi_{\tilde{\nu}}^{\perp}\circ\alpha_{\mathcal{Z}_{\mathcal{T}}}\big)$. By the same argument as in (2) above, $N\big(\pi_{\tilde{\nu}}^{\perp}\circ\alpha_{\mathcal{Z}_{\mathcal{T}}}\big)=N\big(\tilde{\alpha}_{\mathcal{T}}\big)$, where

$$\tilde{\alpha}_{\mathcal{T}}\in\Gamma\big(\mathcal{Z}_{\mathcal{T}};\mathrm{Hom}(F_2;\mathcal{O}_2)\big),\quad \tilde{\alpha}_{\mathcal{T}}(u)=\pi_{\tilde{\nu}}^{\perp}\circ\big(\big\{\mathcal{D}_2\otimes\pi_{x_1}^{-}\circ s_{x_2}\big\}\circ u\big);$$
$$F_2=T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\approx T\Sigma_1^*\otimes T\Sigma_2,$$
$$\mathcal{O}_2=\gamma_{\tilde{E}}^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}\approx T\Sigma_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathbb{C}^2\big)^{\perp}.$$

Thus, applying Lemma 2.5 and again Proposition 2.4, similarly to (2) we obtain

$$\begin{aligned} N\big(\tilde{\alpha}_{\mathcal{T}}\big)&=\big\langle c_1(F_2^*)c_1(\mathcal{O}_2)+c_2(\mathcal{O}_2),\big[\mathcal{Z}_{\mathcal{T}}\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T}}^{\perp}\big)\\ &=48\big|\mathcal{U}_{\mathcal{T}}(\mu)\big|-\mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T}}^{\perp}\big);\\ \mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{-1}(0)}\big(\tilde{\alpha}_{\mathcal{T}}^{\perp}\big)&=\big\langle c_1(T\Sigma^*)+c_1\big(F_2^*\otimes\mathcal{O}_2^{\perp}\big),\big[\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)\big]\big\rangle=24\big|\mathcal{U}_{\mathcal{T}}(\mu)\big|.\end{aligned}$$

On the other hand, by an argument similar to (3) above, $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha_{\tilde{E}}^{\perp}$-hollow. We conclude that

$$\sum_{|\chi(\mathcal{T})|=2}\mathcal{C}_{\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)}\big(\alpha_{\tilde{E}}^{\perp}\big)=24\cdot 3\big|\mathcal{V}_3(\mu)\big|=72\big|\mathcal{V}_3(\mu)\big|. \tag{4.12}$$

(5) We finally compute the $\alpha_{\tilde{E}}^{\perp}$-contribution to $e(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})$ from the space

$$\begin{aligned}\mathcal{Z}_{2;2}(\mu)&=\big\{\big(x,x,b;[v\otimes\upsilon_1,v\otimes\upsilon_2]\big)\in\mathbb{P}\tilde{E}\colon \mathcal{D}_1\upsilon_1+\mathcal{D}_1\upsilon_2=0,\ \upsilon_1,\upsilon_2\neq 0\big\}\\ &\approx\Sigma\times\mathcal{S}_{2;2}(\mu).\end{aligned}$$

Its normal bundle in $\mathbb{P}\tilde{E}$ is

$$\mathcal{N}Z_{2;2}=T\Sigma\oplus\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2,\quad\text{where}\quad E_2=L_1\oplus L_2\longrightarrow\bar{\mathcal{V}}_2(\mu).$$

With appropriate identifications, $\mathcal{D}X=X$ for all $X\in\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2$, where $\mathcal{D}\in\Gamma\big(\mathbb{P}E_2;\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ is the section defined in the proof of Lemma 3.2. Then,

$$\begin{gathered}\big|\alpha_{\tilde{E}}(x,x,b;w,X)-\tilde{\alpha}_{2;2}(x,b;w,X)\big|\le C|w|^2\quad\forall(w,X)\in\mathcal{N}Z_{2;2;\delta},\\ \text{where}\qquad\tilde{\alpha}_{2;2}(x,b;w,X)=X\otimes s_x+\mathcal{D}_{2;2}\otimes s^{(2)}_{g_x,x}(w,\cdot),\end{gathered}$$

and $\mathcal{D}_{2;2}\in\Gamma\big(\bar{\mathcal{S}}_{2;2}(\mu);\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ is the section defined in Subsection 2.3. With the identification of small neighborhoods of Δ in $T\Sigma\longrightarrow\Delta$ and Σ^2 used above, the coefficients defining $\mathcal{D}_{2;2}$ are $c_1=0$ and $c_2=1$. By the same argument as in (2) and (4), $\mathcal{C}_{\mathcal{Z}_{2;2}}\big(\alpha_{\tilde{E}}^{\perp}\big)=N\big(\tilde{\alpha}_{2;2}^-\big)$, where

$$\begin{gathered}\tilde{\alpha}_{2;2}^-\in\Gamma\big(\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu);\mathrm{Hom}(F_2;\mathcal{O}_2)\big),\quad\tilde{\alpha}_{2;2}^-(w)=\pi_{\nu}^{\perp}\circ\big\{\mathcal{D}_{2;2}\otimes s_x^{(2)}(w,\cdot)\big\};\\ F_2=T\Sigma,\quad\mathcal{O}_2=\gamma_{\tilde{E}}^*\otimes\big(\mathcal{H}_{\Sigma}^-\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}\approx T\Sigma^*\otimes\gamma_{E_2}^*\otimes\big(\mathcal{H}_{\Sigma}^-\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}.\end{gathered}$$

By Lemmas 2.5 and 3.2,

$$\begin{aligned}N\big(\tilde{\alpha}_{2;2}^-\big)&=\big\langle c_1(F_2^*)c_1(\mathcal{O}_2)+c_2(\mathcal{O}_2),\big[\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}_{2;2}^{-\,-1}(0)}\big(\tilde{\alpha}_{2;2}^{-\perp}\big)\\ &=4\big\langle120a^2+66a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+13\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big)\\ &\qquad\qquad+13c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}_{2;2}^{-\,-1}(0)}\big(\tilde{\alpha}_{2;2}^{-\perp}\big).\end{aligned}$$

By Proposition 2.4 and an argument similar to the proof of Lemma 3.6,

$$\begin{aligned}\mathcal{C}_{\tilde{\alpha}_{2;2}^{-\,-1}(0)}&=\big\langle c_1\big(F_2^*\otimes\mathcal{O}_2^{\perp}\big),\big[\Sigma\times\partial\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle=28\big|\partial\bar{\mathcal{S}}_{2;2}(\mu)\big|\\ &=28\big|\mathcal{S}_{2;1}(\mu)\big|+84\big|\mathcal{V}_3(\mu)\big|+28\sum_{[\tilde{\mathcal{T}}]}\sum_{l\in[N]}\big\langle c(\mathcal{L}_1^*)+c(\mathcal{L}_2^*),\big[\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)\big]\big\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{C}_{\mathcal{Z}_{2;2}}\big(\alpha_{\tilde{E}}^{\perp}\big)=&4\big\langle120a^2+66a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+7\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big)\\ &+7c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)+6\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)+6c_1(L_1^*)c_1(L_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle\\ &-28\big|\mathcal{S}_{2;1}(\mu)\big|-84\big|\mathcal{V}_3(\mu)\big|.\end{aligned}\tag{4.13}$$

Combining equations (4.8), (4.9), (4.11), (4.12), and (4.13), we obtain

$$\begin{aligned}n_2^{(1)}(\mu)=&4\big\langle24a^2+6a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_1^*)\big)+3c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_1^*)\\ &\qquad-\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_1^*)\big),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle+4\big|\mathcal{S}_{2;1}(\mu)\big|+12\big|\mathcal{V}_3(\mu)\big|.\end{aligned}$$

The claim then follows by using Lemma 3.1.

4.3. The Numbers $n_1^{(2)}(\mu)$ and $n_1^{(1)}(\mu)$. We now give topological formulas for the two remaining numbers of Corollary 2.13. It is possible to obtain the same formulas by going through a lengthy computation like in the proof of Lemma 4.4. Instead we take slightly more geometric approaches.

Suppose $\bar{\mathcal{Z}}$ and Σ are topological spaces, $L_{\mathcal{Z}}, V_{\mathcal{Z}}, E_{\mathcal{Z}} \longrightarrow \bar{\mathcal{Z}}$ and $L_\Sigma, V_\Sigma \longrightarrow \Sigma$ are vector bundles, and $\alpha_{\mathcal{Z}} \in \Gamma\big(\bar{\mathcal{Z}}; \mathrm{Hom}(E_{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}})\big)$, $s \in \Gamma\big(\Sigma; L_\Sigma^* \otimes V_\Sigma\big)$ and $\bar{\nu} \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}}\big)$ are sections such that $\bar{\nu}$ does not vanish. Then we define

$$\alpha^s_{\mathcal{Z},\bar{\nu}} \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; \mathrm{Hom}\big(E_{\mathcal{Z}}; L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp\big)\big),$$
$$\text{where} \qquad \big(V_\Sigma \otimes V_{\mathcal{Z}}\big)^\perp = \big(V_\Sigma \otimes V_{\mathcal{Z}}\big)/\mathbb{C}\bar{\nu}, \qquad \text{by}$$
$$\alpha^s_{\mathcal{Z},\bar{\nu}}\big(e, w \otimes \upsilon\big) = \pi_{\bar{\nu}}^\perp\big(\{\alpha_{\mathcal{Z}}(e)\}(\upsilon) \otimes s(w)\big) \in (V_\Sigma \otimes V_{\mathcal{Z}})^\perp.$$

Lemma 4.5. *Suppose $L_{\mathcal{Z}}, V_{\mathcal{Z}}, E_{\mathcal{Z}} \longrightarrow \bar{\mathcal{Z}}$ are ms-bundles of rank one, two, and $\big(2 - \frac{1}{2}\dim \bar{\mathcal{Z}}\big)$, and*

$$\alpha_{\mathcal{Z}} \in \Gamma\big(\bar{\mathcal{Z}}; Hom(E_{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}})\big)$$

is a regular polynomial. Let Σ be a smooth compact oriented two-manifold, $L_\Sigma, V_\Sigma \longrightarrow \Sigma$ smooth vector bundles of rank one and two, respectively, and $s \in \Gamma\big(\Sigma; L_\Sigma^* \otimes V_\Sigma\big)$ a nonvanishing section. Then for an open collection of nonvanishing sections $\bar{\nu} \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}}\big)$
(1) $\alpha^s_{\mathcal{Z},\bar{\nu}}$ is a regular polynomial;
(2) if $V_{\mathcal{Z}} \approx \bar{\mathcal{Z}} \times \mathbb{C}^2$, $N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big) = N\big(\alpha_{\mathcal{Z}}\big)\langle 3c_1(L_\Sigma^*) + 2c_1(V_\Sigma), [\Sigma]\rangle$.

Proof: (1) The first claim is clear. If $V_{\mathcal{Z}} \approx \bar{\mathcal{Z}} \times \mathbb{C}^2$, we can choose section $\bar{\nu} \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}}\big)$ that does not intersect $V_\Sigma^+ \otimes V_{\mathcal{Z}}$, where $V_\Sigma^+ = \mathrm{Im}\, s$. Then $\pi_{\bar{\nu}}^\perp\big(V_\Sigma^+ \otimes V_{\mathcal{Z}}\big)$ is a rank-two subbundle of $\big(V_\Sigma \otimes V_{\mathcal{Z}}\big)^\perp$. Let

$$\mathcal{O}^+ = L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes \pi_{\bar{\nu}}^\perp\big(V_\Sigma^+ \otimes V_{\mathcal{Z}}\big), \qquad \mathcal{O}^- \equiv \big(L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp\big)/\mathcal{O}^+.$$

We identify $\mathcal{O}^-$ with a complement of $\mathcal{O}^+$ in $L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp$.
(2) By definition, $N\big(\alpha_{\mathcal{Z}}\big)$ is the number of zeros of the affine map

$$\psi_{\alpha_{\mathcal{Z}},\nu} \colon E_{\mathcal{Z}} \longrightarrow L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}}, \qquad \psi_{\alpha_{\mathcal{Z}},\nu}(b; \upsilon) = \nu_b + \alpha_{\mathcal{Z}}(\upsilon),$$

for a generic section $\nu \in \Gamma\big(\bar{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}}^*\big)$. Via the construction preceding the lemma, ν induces a section $\tilde{\nu}^+ \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; \mathcal{O}^+\big)$. If ν and $\tilde{\nu}^- \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; \mathcal{O}^-\big)$ are generic, $N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big)$ is the number of zeros of the affine map

$$\psi_{\alpha^s_{\mathcal{Z},\bar{\nu}},\tilde{\nu}^++\tilde{\nu}^-}: E_{\mathcal{Z}} \longrightarrow \mathcal{O}^+\oplus\mathcal{O}^-,$$
$$\psi_{\alpha^s_{\mathcal{Z},\bar{\nu}},\tilde{\nu}^++\tilde{\nu}^-}\big(x,b;\upsilon\big) = \tilde{\nu}^+_{(x,b)} + \tilde{\nu}^-_{(x,b)} + \alpha^s_{\mathcal{Z},\bar{\nu}}\big(x,b;\upsilon\big).$$

The solution of the $\mathcal{O}^+$-part of this equation is precisely $\Sigma\times\psi^{-1}_{\alpha_{\mathcal{Z}},\nu}(0)$. Thus,

$$N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big) =^{\pm} \big|\psi^{-1}_{\alpha^s_{\mathcal{Z},\bar{\nu}},\tilde{\nu}^++\tilde{\nu}^-}(0)\big| = \langle c_1(\mathcal{O}^-),[\Sigma]\rangle N\big(\alpha_{\mathcal{Z}}\big),$$

as claimed.

Lemma 4.6. *Suppose $\bar{\mathcal{Z}}$ is an ms-manifold of dimension two, and Σ, $L_{\mathcal{Z}}$, $V_{\mathcal{Z}}$, $E_{\mathcal{Z}}$, L_Σ, V_Σ, $\alpha_{\mathcal{Z}}$ and s are in Lemma 4.5. Then for an open collection of sections $\bar{\nu}\in\Gamma\big(\Sigma\times\bar{\mathcal{Z}};V_\Sigma\otimes V_{\mathcal{Z}}\big)$*

$$\mathcal{C}_{\alpha^{s}_{\mathcal{Z},\bar{\nu}}{}^{-1}(0)}\big(\alpha^{s\perp}_{\mathcal{Z},\bar{\nu}}\big) = \mathcal{C}_{\alpha_{\mathcal{Z}}^{-1}(0)}\big(\alpha^\perp_{\mathcal{Z}}\big)\big\langle 3c_1(L^*_\Sigma)+2c_1(V_\Sigma),[\Sigma]\big\rangle;$$
$$N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big) = N\big(\alpha_{\mathcal{Z}}\big)\big\langle 3c_1(L^*_\Sigma)+2c_1(V_\Sigma),[\Sigma]\big\rangle + \big\langle c_1(V_{\mathcal{Z}}),[\bar{\mathcal{Z}}]\big\rangle\big\langle c_1(L^*_\Sigma)+c_1(V_\Sigma),[\Sigma]\big\rangle.$$

Proof: (1) Since $\alpha_{\mathcal{Z}}$ is regular and $\bar{\mathcal{Z}}$ is two-dimensional, $\alpha_{\mathcal{Z}}^{-1}(0)$ is a finite set of points. Thus, we can trivialize the bundles $E_{\mathcal{Z}}$, $L_{\mathcal{Z}}$, and $V_{\mathcal{Z}}$ near $\alpha_{\mathcal{Z}}^{-1}(0)$, so that $\nu_z=(0,1)\in L^*_{\mathcal{Z}}\otimes V_{\mathcal{Z}}$, where ν is as in the proof of Lemma 4.5. Furthermore, for each $z\in\alpha_{\mathcal{Z}}^{-1}(0)$, there exists $d_z\geq1$ such that

$$\big|\alpha_{\mathcal{Z}}(u)-(u^{d_z},0)\big| \leq \varepsilon(u)|u|^{d_z} \qquad \forall u\in\mathbb{C}_\delta, \qquad \text{with } \lim_{u\longrightarrow0}\varepsilon(u)=0.$$

Then $\mathcal{C}_z\big(\alpha^\perp_{\mathcal{Z}}\big)=d_z$.
(2) By transversality and dimension-counting, we can choose

$$\bar{\nu}\in\Gamma\big(\Sigma\times\bar{\mathcal{Z}};V_\Sigma\otimes V_{\mathcal{Z}}\big)$$

such that

$$\text{Im }\bar{\nu}\cap\big(s(L_\Sigma)\otimes\alpha_{\mathcal{Z}}(E_{\mathcal{Z}}\otimes L_{\mathcal{Z}})\big)=\emptyset \qquad \text{and}$$
$$\text{Im }\bar{\nu}\cap\big(s(L_\Sigma)\otimes\big(V_{\mathcal{Z}}|\Sigma\times\alpha_{\mathcal{Z}}^{-1}(0)\big)\big)=\emptyset.$$

Then, $\alpha^{s-1}_{\mathcal{Z},\bar{\nu}}(0)=\Sigma\times\alpha_{\mathcal{Z}}^{-1}(0)$. Furthermore, on a neighborhood of $\Sigma\times\alpha_{\mathcal{Z}}^{-1}(0)$, we can define a splitting

$$L^*_\Sigma\otimes(V_\Sigma\otimes\mathbb{C}^2)^\perp = \mathcal{O}^+\oplus\mathcal{O}^- \approx \mathbb{C}^2\oplus\mathcal{O}^-,$$

as in the proof of Lemma 4.5. Let

$$\tilde{\nu}\in\Gamma\big(\Sigma\times\mathcal{Z};L^*_\Sigma\otimes L^*_{\mathcal{Z}}\otimes(V_\Sigma\otimes V_{\mathcal{Z}})^\perp\big)$$

be a nonvanishing section such that on a neighborhood of $\Sigma\times\alpha_{\mathcal{Z}}^{-1}(0)$, $\tilde{\nu}=\tilde{\nu}^++\tilde{\nu}^-$, with $\tilde{\nu}^+$ as in the proof of Lemma 4.5. Then,

$$\big(\mathcal{O}^+\oplus\mathcal{O}^-\big)/\mathbb{C}\tilde{\nu}\approx\mathbb{C}\oplus\mathcal{O}^-,\quad\text{and}\quad \big|\alpha_{\mathcal{Z},\tilde{\nu}}^{s\,\perp}-(u^{d_z},0)\big|\le\varepsilon(u)|u|^{d_z}\quad\forall u\in\mathbb{C}_\delta.$$

Thus, by Proposition 2.4,

$$\mathcal{C}_{\Sigma\times\{z\}}\big(\alpha_{\mathcal{Z},\tilde{\nu}}^{s\,\perp}\big)=d_z\big\langle c_1(\mathcal{O}^-),[\Sigma]\big\rangle=\mathcal{C}_z\big(\alpha_{\mathcal{Z}}\big)\big\langle 3c_1(L_\Sigma^*)+2c_1(V_\Sigma),[\Sigma]\big\rangle.$$

The second claim follows from the first, since by Lemma 2.5 and Proposition 2.4,

$$\begin{aligned}N\big(\alpha_{\mathcal{Z}}\big)&=\big\langle c_1\big(L_{\mathcal{Z}}^*\otimes V_{\mathcal{Z}}\big)+c_1(E_{\mathcal{Z}}),[\bar{\mathcal{Z}}]\big\rangle-\mathcal{C}_{\alpha_{\mathcal{Z}}^{-1}(0)}\big(\alpha_{\mathcal{Z}}\big);\\ N\big(\alpha_{\mathcal{Z},\tilde{\nu}}^{s}\big)&=\big\langle c_2\big(L_\Sigma^*\otimes L_{\mathcal{Z}}^*\otimes(V_\Sigma\otimes V_{\mathcal{Z}})^\perp\big)\\ &\quad+c_1(E^*)c_1\big(L_\Sigma^*\otimes L_{\mathcal{Z}}^*\otimes(V_\Sigma\otimes V_{\mathcal{Z}})^\perp\big),\big[\Sigma\times\bar{\mathcal{Z}}\big]\big\rangle-\mathcal{C}_{\alpha_{\mathcal{Z},\tilde{\nu}}^{s\,-1}(0)}\big(\alpha_{\mathcal{Z},\tilde{\nu}}^{s\,\perp}\big).\end{aligned}$$

Corollary 4.7. *Suppose $\bar{\mathcal{M}}$ is an ms-manifold of dimension four, $L_{\mathcal{M}},V_{\mathcal{M}}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles of rank one and two, respectively, and $\alpha\in\Gamma\big(\bar{\mathcal{M}};Hom(L_{\mathcal{M}},V_{\mathcal{Z}})\big)$ is a regular polynomial. If Σ is a compact smooth oriented manifold of dimension two, $L_\Sigma,V_\Sigma\longrightarrow\Sigma$ are smooth vector bundles of rank one and two, respectively, and $s\in\Gamma\big(\Sigma;Hom(L_\Sigma,V_\Sigma)\big)$ is a nonvanishing section, then*

$$\begin{aligned}N\big(\alpha\otimes s\big)=&\big\langle\big(c_1(L_\Sigma^*)+c_1(V_\Sigma)\big)\big(c_1(L_{\mathcal{M}}^*)c_1(V_{\mathcal{M}})+c_1^2(V_{\mathcal{M}})\big)\\ &\qquad-c_1(L_\Sigma^*)c_2(V_{\mathcal{M}}),\big[\Sigma\times\bar{\mathcal{M}}\big]\big\rangle\\ &-\big\langle c_1(L_\Sigma^*)+c_1(V_\Sigma),[\Sigma]\big\rangle\sum_{\alpha^{-1}(0)=\bigsqcup\mathcal{Z}_i}\big\langle c_1(V_{\mathcal{M}}),[\bar{\mathcal{Z}}_i]\big\rangle,\end{aligned}$$

where the sum is taken over all α-regular subsets $\mathcal{Z}_i$ in a decomposition of $\alpha^{-1}(0)$ as in Proposition 2.4.

Proof: By Lemma 2.5,

$$N\big(\alpha\otimes s\big)=\big\langle c_3\big(L_\Sigma^*\otimes L_{\mathcal{M}}^*\otimes(V_\Sigma\otimes V_{\mathcal{M}})^\perp\big),\big[\Sigma\times\bar{\mathcal{M}}\big]\big\rangle-\mathcal{C}_{\Sigma\times\alpha^{-1}(0)}\big((\alpha\otimes s)^\perp\big).$$

The last term can be written as the sum of terms as in the second equation of Lemma 4.6. On the other hand, by Proposition 2.4,

$$\sum_{\alpha^{-1}(0)=\bigsqcup\mathcal{Z}_i}N(\alpha_{\mathcal{Z}_i})=\big\langle c_2\big(L_{\mathcal{M}}^*\otimes V_{\mathcal{M}}\big),[\bar{\mathcal{M}}]\big\rangle.$$

Thus, the claim follows from Lemma 4.6.

Lemma 4.8. *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_1^{(2)}(\mu) = 12\big\langle 7a^2+6ac_1(\mathcal{L}^*), \big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle - 12\big\langle 9a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.$$

Proof: (1) By Subsection 2.4, $n_1^{(2)}(\mu)=N(\alpha_{1;1})$, where

$$\alpha_{1;1}\in\Gamma\big(\Sigma\times\bar{\mathcal{S}}_1(\mu);\mathrm{Hom}(T\Sigma^{\otimes 2}\otimes L^{\otimes 2},\mathcal{O})\big), \quad \mathcal{O}=\mathcal{H}_\Sigma^-\otimes \mathrm{ev}^*T\mathbb{P}^2,$$
$$\alpha_{1;1}\big(x,b;v\otimes v\big) = \big(\mathcal{D}^{(2)}v\big)\big(s_x^{(2)}v\big) \in \mathcal{H}_\Sigma^-(x)\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

Thus, we can apply Corollary 4.7 with

$$\bar{\mathcal{M}}=\bar{\mathcal{S}}_1(\mu), \quad L_\Sigma=T\Sigma^{\otimes 2}, \quad V_\Sigma=\mathcal{H}_\Sigma^-, \quad L_{\mathcal{M}}=L^{\otimes 2}, \quad V_{\mathcal{M}}=\mathrm{ev}^*T\mathbb{P}^2,$$
$$s=s^{(2)}, \quad \alpha=\mathcal{D}^{(2)}.$$

The first term of Corollary 4.7 gives the intersection number on $\bar{\mathcal{S}}_1(\mu)$ in the statement of the lemma, since $ac_1(L^*)=ac_1(\mathcal{L}^*)$. A decomposition of the zero set of $\mathcal{D}^{(2)}$ is given in the proof of Lemma 3.4. The only stratum of $\alpha^{-1}(0)$ contributing to the second term of Corollary 4.7 is $\mathcal{S}_{2;2}(\mu)$. Lemma 3.2 reduces this contribution to the intersection number on $\bar{\mathcal{V}}_2(\mu)$ of the lemma.

Lemma 4.9. $n_1^{(1)}(\mu)=0$.

Proof: By Subsection 2.4, $n_1^{(1)}(\mu)=N(\alpha_1)$, where

$$\alpha_1\in\Gamma\big(\Sigma\times\bar{\mathcal{V}}_1(\mu);\mathrm{Hom}(T\Sigma\otimes L,\mathcal{O})\big), \qquad \mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2,$$
$$\alpha_1\big(x,b;v\otimes v\big) = \big(\mathcal{D}v\big)\big(s_x v\big)\in\mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

It will be shown that there exists $\bar{\nu}\in\Gamma\big(\mathbb{P}^2;\mathcal{H}_\Sigma^{0,1}\otimes T\mathbb{P}^2\big)$ such that the affine map

$$\psi_{\alpha_1,\mathrm{ev}^*\bar{\nu}}\colon T\Sigma\otimes L\longrightarrow\mathcal{O}, \qquad \big(x,b;v\otimes v\big)\longrightarrow\bar{\nu}_{\mathrm{ev}(b)}+\alpha_1\big(x,b;v\otimes v\big), \tag{4.14}$$

has no zeros over $\Sigma\times\bar{\mathcal{V}}_1(\mu)$. The map

$$\tilde{\alpha}\colon\Sigma\times\mathbb{P}T\mathbb{P}^2\longrightarrow\mathbb{P}\big(\mathcal{H}_\Sigma^{0,1}\otimes T\mathbb{P}^2\big)\approx\mathbb{P}\big(\mathbb{C}^3\otimes T\mathbb{P}^2\big), \qquad (x,\ell)\longrightarrow(\mathrm{Im}\ s_x)\otimes\ell,$$

is an embedding, since Σ is not hyperelliptic. Let $\mathcal{W}$ denote the image of $\gamma_{T\mathbb{P}^2}\big|\big(\mathrm{Im}\ \tilde{\alpha}\big)$ under the projection map

$$\gamma_{T\mathbb{P}^2}\longrightarrow T\mathbb{P}^2, \qquad (q,\ell,v)\longrightarrow(q,v).$$

Then $\mathcal{W}$ is a closed subspace of $\mathcal{H}_\Sigma^{0,1}\otimes T\mathbb{P}^2$ and $\mathcal{W}\longrightarrow\mathbb{P}^2$ is a bundle of affine varieties of dimension three. Thus, by transversality and dimension-counting, we can choose $\bar{\nu}\in\Gamma\big(\mathbb{P}^2;\mathcal{H}_\Sigma^{0,1}\otimes T\mathbb{P}^2\big)$ such that the image $\bar{\nu}$ does not intersect $\mathcal{W}$. Then the map $\psi_{\alpha_1,\mathrm{ev}^*\bar{\nu}}$ of (4.14) does not vanish.

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