On the Genus-One Gromov-Witten Invariants of Complete Intersections

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Abstract
We state and prove a long-elusive relation between genus-one Gromov-Witten of a complete intersection and twisted Gromov-Witten invariants of the ambient projective space. As shown in a previous paper, certain naturally arising cones of holomorphic vector bundle sections over the main component $\mathcal{M}^0_{1,k}(\mathbb{P}^n, d)$ of the moduli space of stable genus-one holomorphic maps into $\mathbb{P}^n$ have a well-defined euler class. In this paper, we extend this result to moduli spaces of perturbed, in a restricted way, $J$-holomorphic maps. This extension is used to show that these cones are the correct genus-one analogues of the vector bundles relating genus-zero Gromov-Witten invariants of a complete intersection to those of the ambient projective space. A relationship for higher-genus invariants is conjectured as well.

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1 Introduction

1.1 Gromov-Witten Invariants and Complete Intersections

Gromov-Witten invariants of symplectic manifolds have been a subject of much research over the past two decades. A great deal of attention has been devoted in particular to Calabi-Yau manifolds. These manifolds play a prominent role in theoretical physics, and as a result physicists have made a number of important predictions concerning CY-manifolds. Some of these predictions have been verified mathematically; others have not.

If \( Y \) is a compact Kahler submanifold of the complex projective space \( \mathbb{P}^n \), one could try to compute GW-invariants of \( Y \) by relating them to GW-invariants of \( \mathbb{P}^n \). For example, suppose \( Y \) is a hypersurface in \( \mathbb{P}^n \) of degree \( a \). In other words, if \( \gamma : \mathbb{P}^n \rightarrow \mathbb{P}^n \) is the tautological line bundle and \( L = \gamma^* \otimes a \rightarrow \mathbb{P}^n \), then

\[
Y = s^{-1}(0),
\]

for some \( s \in H^0(\mathbb{P}^n; \mathcal{L}) \) such that \( s \) is transverse to the zero set. If \( g \), \( k \), and \( d \) are nonnegative integers, let \( \overline{M}_{g,k}(\mathbb{P}^n, d) \) and \( \overline{M}_{g,k}(Y, d) \) denote the moduli spaces of stable \( J_0 \)-holomorphic degree-\( d \) maps from genus-\( g \) Riemann surfaces with \( k \) marked points to \( \mathbb{P}^n \) and \( Y \), respectively. These moduli spaces determine the genus-\( g \) degree-\( d \) GW-invariants of \( \mathbb{P}^n \) and \( Y \).

By definition, the moduli space \( \overline{M}_{g,k}(Y, d) \) is a subset of the moduli space \( \overline{M}_{g,k}(\mathbb{P}^n, d) \). In fact,

\[
\overline{M}_{g,k}(Y, d) = \{ [C, u] \in \overline{M}_{g,k}(\mathbb{P}^n, d) : s \circ u = 0 \in H^0(C; u^* \mathcal{L}) \}. \tag{1.1}
\]

Here \( [C, u] \) denotes the equivalence class of the holomorphic map \( u : C \rightarrow \mathbb{P}^n \) from a genus-\( g \) curve \( C \) with \( k \) marked points. The relationship (1.1) can be restated more globally as follows. Let

\[
\pi^d_{g,k} : \mathcal{U}_{g,k}(\mathbb{P}^n, d) \rightarrow \overline{M}_{g,k}(\mathbb{P}^n, d)
\]

be the universal family and let

\[
ev^d_{g,k} : \mathcal{U}_{g,k}(\mathbb{P}^n, d) \rightarrow \mathbb{P}^n
\]

be the natural evaluation map. In other words, the fiber of \( \pi^d_{g,k} \) over \( [C, u] \) is the curve \( C \) with \( k \) marked points, while

\[
ev^d_{g,k}([C, u]; z) = u(z) \quad \text{if} \quad z \in C.
\]

We define a section \( s^d_{g,k} \) of the sheaf \( \pi^d_{g,k} \circ \ev^d_{g,k} : \overline{M}_{g,k}(\mathbb{P}^n, d) \rightarrow \overline{M}_{g,k}(\mathbb{P}^n, d) \) by

\[
s^d_{g,k}(C, u) = [s \circ u].
\]

By (1.1), \( \overline{M}_{g,k}(Y, d) \) is the zero set of this section.

The previous paragraph suggests that it should be possible to relate the genus-\( g \) degree-\( d \) GW-invariants of the hypersurface \( Y \) to the moduli space \( \overline{M}_{g,k}(\mathbb{P}^n, d) \) in general and to the sheaf

\[
\pi^d_{g,k} \circ \ev^d_{g,k} : \overline{M}_{g,k}(\mathbb{P}^n, d) \rightarrow \overline{M}_{g,k}(\mathbb{P}^n, d)
\]

in particular. In fact, it can be shown that

\[
GW_{0,k}(Y; d; \psi) = \langle \psi, \left[ \overline{M}_{0,k}(Y, d) \right]_{\text{vir}} \rangle = \langle \psi \cdot e(\pi^d_{0,k} \circ \ev^d_{0,k} \mathcal{L}), \left[ \overline{M}_{0,k}(\mathbb{P}^n, d) \right] \rangle \tag{1.2}
\]
for all $\psi \in H^*(\overline{M}_{0,k}(\mathbb{P}^n,d); \mathbb{Q})$; see [Bea] for example. The moduli space $\overline{M}_{0,k}(\mathbb{P}^n,d)$ is a smooth orbivariety and

$$\pi^d_{0,k,*}\text{ev}^d_{0,k}\mathcal{L} \longrightarrow \overline{M}_{0,k}(\mathbb{P}^n,d) \quad (1.3)$$

is a locally free sheaf, i.e. a vector bundle. The right-hand side of (1.2) can be computed via the classical localization theorem of [ABo], though the complexity of this computation increases rapidly with the degree $d$.

A **hyperplane property**, i.e. a relationship such as (1.2), for positive-genus GW-invariants has been elusive since the early days of the Gromov-Witten theory. If $g > 0$, the sheaf

$$\pi^d_{g,k,*}\text{ev}^d_{g,k}\mathcal{L} \longrightarrow \overline{M}_{g,k}(\mathbb{P}^n,d)$$

is not locally free and need not define an euler class. Thus, the right-hand side of (1.2) may not even make sense if 0 is replaced by $g > 0$. Instead one might try to generalize (1.2) as

$$GW_{Y}^{g,k}(d; \psi) \equiv \langle \psi, [\overline{M}_{g,k}(Y,d)]^{\text{vir}} \rangle \equiv \langle \psi \cdot e(R^0\pi^d_{1,0,*}\text{ev}^d_{1,0}\mathcal{L} - R^1\pi^d_{1,0,*}\text{ev}^d_{1,0}\mathcal{L}), [\overline{M}_{g,k}(\mathbb{P}^n,d)]^{\text{vir}} \rangle, \quad (1.4)$$

where $R^i\pi^d_{g,k,*}\text{ev}^d_{g,k}\mathcal{L} \longrightarrow \overline{M}_{g,k}(\mathbb{P}^n,d)$ is the $i$th direct image sheaf. The right-hand side of (1.4) can be computed via the virtual localization theorem of [GrP1]. However,

$$N_1(d) \equiv GW_{Y}^{1,0}(d; 1) \neq \langle e(R^0\pi^d_{1,0,*}\text{ev}^d_{1,0}\mathcal{L} - R^1\pi^d_{1,0,*}\text{ev}^d_{1,0}\mathcal{L}), [\overline{M}_{1}(\mathbb{P}^4,d)]^{\text{vir}} \rangle,$$

according to a low-degree check of [GrP2] and [K] for a quintic threefold $Y \subset \mathbb{P}^4$.

In this paper we prove a hyperplane property for genus-one GW-invariants. We denote by

$$\overline{M}^0_{1,k}(\mathbb{P}^n,d) \subset \overline{M}_{1,k}(\mathbb{P}^n,d)$$

the closure in $\overline{M}_{1,k}(\mathbb{P}^n,d)$, either in the stable-map or Zariski topology, of the subspace

$$\overline{M}^0_{1,k}(\mathbb{P}^n,d) = \{ [\mathcal{C}, u] \in \overline{M}_{1,k}(\mathbb{P}^n,d) : \mathcal{C} \text{ is smooth} \}.$$

If $Y \subset \mathbb{P}^n$ is a hypersurface as above, let

$$\overline{M}^0_{1,k}(Y,d) = \overline{M}_{1,k}(Y,d) \cap \overline{M}^0_{1,k}(\mathbb{P}^n,d).$$

Since $\overline{M}^0_{1,k}(\mathbb{P}^n,d)$ is a unidimensional orbi-variety, it carries a fundamental class. By Corollary 1.6 in [Z6], $\overline{M}^0_{1,k}(Y,d)$ carries a virtual fundamental class. It can be used to define **reduced** genus-one Gromov-Witten invariants:

$$GW_{1,k}^{0,Y}(d; \psi) \equiv \langle \psi, [\overline{M}^0_{1,k}(Y,d)]^{\text{vir}} \rangle \in \mathbb{Q},$$

where $\psi$ is a tautological (cohomology) class on $\overline{M}^0_{1,k}(Y,d)$; see below. We show in this paper that the reduced genus-one GW-invariants satisfy a natural analogue of (1.2).
Theorem 1.1 Suppose $d$ and $a$ are positive integers, $k$ is a nonnegative integer, $\mathcal{L} = \gamma^{*}\otimes^{a} \to \mathbb{P}^{n}$,
\[
\pi_{1,k}^{d} : U_{1,k}(\mathbb{P}^{n}, d) \to \mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)
\] and $ev_{1,k}^{d} : U_{1,k}(\mathbb{P}^{n}, d) \to \mathbb{P}^{n}$ are the universal family and the natural evaluation map, respectively. If $Y \subset \mathbb{P}^{n}$ is a smooth degree-$a$ hypersurface, then
\[
GW^{0,Y}_{1,k}(d; \psi) = \langle \psi \cdot e(\pi_{1,k}^{d}ev_{1,k}^{d*}\mathcal{L}), [\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)] \rangle
\] (1.5)
for every tautological class $\psi$ on $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$.

The tautological classes on $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$ are certain natural cohomology classes. They include all geometric classes defined in Subsection 1.3. We describe the space of all cohomology classes $\psi$ to which Theorem 1.1 applies in Subsection 2.2.

Implicit in the statement of Theorem 1.1 is that the euler class of the sheaf
\[
\pi_{1,k}^{d*}ev_{1,k}^{d*}\mathcal{L} \to \mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)
\] (1.6)
is well-defined, even though it is not locally free. This is the case by Theorem 1.1 in [Z5].

The right-hand side of (1.5) should in principle be computable via localization directly. However, since the space $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$ is not smooth and the sheaf (1.6) is not locally free, the localization theorem of [ABo] is not immediately applicable. A desingularization of the space $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$, i.e. a smooth orbivariety $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$ and a map
\[
\tilde{\pi} : \mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d) \to \mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d),
\]
which is biholomorphic onto $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$, is constructed in [VaZ]. This desingularization of $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$ comes with a desingularization of the sheaf (1.6), i.e. a vector bundle
\[
\tilde{\nu}_{1,k}^{d} \to \mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)
\]
s.t. $\tilde{\pi}_{*}\tilde{\nu}_{1,k}^{d} = \pi_{1,k}^{d*}ev_{1,k}^{d*}\mathcal{L}$.

In particular,
\[
\langle \psi \cdot e(\pi_{1,k}^{d*}ev_{1,k}^{d*}\mathcal{L}), [\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)] \rangle = \langle \tilde{\pi}_{*}\psi \cdot e(\tilde{\nu}_{1,k}^{d*}), [\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)] \rangle.
\] (1.7)
Since a group action on $\mathbb{P}^{n}$ induces actions on $\mathcal{M}_{1,k}^{0}(\mathbb{P}^{n}, d)$ and on $\tilde{\nu}_{1,k}^{d}$, the localization theorem of [ABo] is directly applicable to the right-hand side of (1.7), for a natural cohomology class $\psi$.

By itself, Theorem 1.1 does not provide a way of computing the standard genus-one GW-invariants of $Y$. However, the reduced genus-one GW-invariants capture the contribution of $\mathcal{M}_{1,k}^{0}(Y, d)$ to the standard genus-one GW-invariants. Thus, the difference between the two invariants is completely determined by the genus-zero invariants of $Y$; see Subsection 1.2 in [Z9]. We give explicit formulas in some special cases in Subsection 1.3 below.

Remark 1: Theorem 1.1 generalizes to arbitrary smooth complete intersections in projective spaces. More precisely, if
\[
\mathcal{L} = \gamma^{*}\otimes^{a_{1}} \oplus \ldots \oplus \gamma^{*}\otimes^{a_{m}} \to \mathbb{P}^{n},
\]

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with \( a_1, \ldots, a_m \in \mathbb{Z}^+ \), \( s \in H^0(\mathbb{P}^n; \mathcal{L}) \) is transverse to the zero set in \( \mathcal{L} \), and \( Y = s^{-1}(0) \), then

\[
GW^0_{1,k}(d; \psi) = \langle \psi \cdot e(\pi_{1,k}^d \ev_{1,k}^d \mathcal{L}), [\mathfrak{M}^0_{1,k}(\mathbb{P}^n, d)] \rangle,
\]

for every geometric cohomology class \( \psi \) on \( \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d) \).

**Remark 2:** In turn, Remark 1 generalizes as follows. Suppose \((X, \omega, J)\) is a compact almost Kahler manifold, \( A \in H_2(X; \mathbb{Z})^* \equiv H_2(X; \mathbb{Z}) - \{0\} \), \((\mathcal{L}, \nabla) \to X\) is a complex vector bundle with connection, and \( s \) is a \( \nabla \)-holomorphic section of \( \mathcal{L} \); see Subsections 1.2 and 2.2 for terminology. If \( J \) is genus-one \( A \)-regular in the sense of Definition 1.4 in [Z4], \( s \) is transverse to the zero set in \( \mathcal{L} \), and \((\mathcal{L}, \nabla)\) splits into line bundles that are \((\omega, A)\)-positive in the sense of Definition 1.2 below, then

\[
GW^0_{1,k}(A; \psi) = \langle \psi \cdot e(V^A_{1,k}), [\mathfrak{M}_{1,k}^0(X, A; J)]^\vir \rangle \equiv \langle \psi, \text{PD}_{\mathfrak{M}_{1,k}^0(X, A; J)} e(V^A_{1,k}) \rangle,
\]

where \( Y = s^{-1}(0) \), \( \psi \) is a tautological class, and the cone

\[
V^A_{1,k} \to \mathfrak{M}_{1,k}^0(X, A; J)
\]

is the geometric analogue of the sheaf \( \pi_{1,k}^d \ev_{1,k}^d \mathcal{L} \). It consists of \( \nabla \)-holomorphic sections of the vector bundle \( \mathcal{L} \), as defined in Subsection 1.2 below. By Corollary 1.4, the Poincare dual of its euler class is well defined as long as \((\mathcal{L}, \nabla)\) is a direct sum of \((\omega, A)\)-positive line bundles.

Theorem 1.1 and Remarks 1 and 2 have a natural, but rather speculative, generalization to higher-genus invariants. Suppose that the main component

\[
\mathfrak{M}_{g,k}^0(X, A; J) \subset \mathfrak{M}_{g,k}(X, A; J)
\]

is well defined and carries a virtual fundamental class. If so, it determines reduced genus-\( g \) GW-invariants \( GW^0_{g,k}(A; \psi) \). Suppose further that (the Poincare dual of) the euler of the cone

\[
V^A_{g,k} \to \mathfrak{M}_{g,k}^0(X, A; J)
\]

is well defined. If constructions of these objects are direct generalizations of the corresponding constructions in Subsection 1.2 and in [Z4]-[Z6], then the proof of Theorem 1.1 can be generalized to show that

\[
GW^0_{g,k}(A; \psi) = \langle \psi \cdot e(V^A_{g,k}), [\mathfrak{M}_{g,k}^0(X, A; J)]^\vir \rangle,
\]

provided appropriate generalizations of the assumptions in Remark 2 hold. Along with an equally speculative generalization of Theorem 1.1 in [Z6] stated in Subsection 1.2 of [Z6], (1.10) would, if true, provide an algorithm for computing arbitrary-genus GW-invariants of complete intersections.

From the point of view of algebraic geometry as described in [BehFa] and in [LT2], the genus-\( g \) degree-\( d \) GW-invariant \( GW^Y_{g,k}(d; \psi) \) is the evaluation of \( \psi \) on the virtual fundamental class
Using the more concrete point of view of symplectic topology as described in [FuO] and [LT1], $GW_{g,k}(d;\psi)$ can be interpreted as the euler class of a vector bundle, albeit of an infinite-rank vector bundle over a space of the “same” dimension. As in the finite-dimensional case, this euler class is the number of zeros, counted with appropriate multiplicities, of a transverse (multi-valued, generic) section. It is shown in [LT3] and in [Si] that the two approaches are equivalent. In this paper, we take the latter point of view. Similarly, we view the euler class of the sheaf (1.6) as the zero set of a generic section of its geometric analogue $V^d_{1,k}$ defined in Subsection 1.2.

Theorem 1.1 and Remark 1 are special cases of Remark 2, which is the same as Theorem 2.3. It is proved in Subsection 2.2 by showing that the zero sets of two bundle sections whose cardinalities are the two expressions in (1.9) are the same set. In fact, Theorem 2.3, just like its genus-zero analogue, follows easily from definitions of the two sides in (1.9), once it is established that these definitions are well-posed.

1.2 Cones of Holomorphic Bundle Sections

Let $(X,\omega,J)$ be a compact almost Kahler manifold. In other words, $(X,\omega)$ is a symplectic manifold and $J$ is an almost complex structure on $X$ tamed by $\omega$, i.e.

$$\omega(v,Jv) > 0 \quad \forall \ v \in TX - X.$$ If $g,k$ are nonnegative integers and $A \in H_2(X;\mathbb{Z})$, we denote by $\mathcal{M}_{g,k}(X,A;\omega,J)$ the moduli space of (equivalence classes of) stable $J$-holomorphic maps from genus-$g$ Riemann surfaces with $k$ marked points in the homology class $A$. Let $\mathcal{M}_{1,k}^0(X,A;\omega,J)$ be the main component of the moduli space $\mathcal{M}_{1,k}(X,A;\omega,J)$ described by Definition 1.1 in [Z4]; see also Definition 2.2 below. This closed subspace of $\mathcal{M}_{1,k}(X,A;\omega,J)$ contains the subspace $\mathcal{M}_{1,k}^0(X,A;\omega,J)$ consisting of the stable maps $[\Sigma,u]$ such that the domain $\Sigma$ is a smooth Riemann surface. If $J$ is sufficiently regular, $\mathcal{M}_{1,k}^0(X,A;\omega,J)$ is the closure of $\mathcal{M}_{1,k}^0(X,A;\omega,J)$ in $\mathcal{M}_{1,k}(X,A;\omega,J)$.

Suppose $\mathcal{L} \rightarrow X$ is a complex line bundle and $\nabla$ is a connection in $\mathcal{L}$. If $(\Sigma,j)$ is a Riemann surface and $u: \Sigma \rightarrow X$ is a smooth map, let

$$\nabla^u: \Gamma(\Sigma;u^*\mathcal{L}) \rightarrow \Gamma(\Sigma;\nabla^u \mathcal{L})$$

be the pull-back of $\nabla$ by $u$. If $b=(\Sigma,j;u)$, we define the corresponding $\bar{\partial}$-operator by

$$\bar{\partial}_{\nabla,b}: \Gamma(\Sigma;\nabla^u \mathcal{L}) \rightarrow \Gamma(\Sigma;\Lambda_{i,j}^{0,1}T^*\Sigma\otimes u^*\mathcal{L}), \quad \bar{\partial}_{\nabla,b}\xi = \frac{1}{2}(\nabla^u \xi + i\nabla^u \xi \circ j), \quad (1.11)$$

where $i$ is the complex multiplication in the bundle $u^*\mathcal{L}$ and

$$\Lambda_{i,j}^{0,1}T^*\Sigma\otimes u^*\mathcal{L} = \{ \eta \in \text{Hom}(T\Sigma,u^*\mathcal{L}) : \eta \circ j = -i\eta \}.$$ The kernel of $\bar{\partial}_{\nabla,b}$ is necessarily a finite-dimensional complex vector space.

We denote by $\mathcal{X}_{1,k}(X,A)$ the space of all degree-$A$ stable smooth maps from genus-one Riemann surfaces with $k$ marked points into $X$ and by

$$V^d_{1,k} \rightarrow \mathcal{X}_{1,k}(X,A)$$
the cone, or the bundle of (orbi-)vector spaces, such that
\[ V^A_{1,k} \mid_{[b]} = \ker \bar{\partial} \nu_{k,b} / \Aut(b) \quad \forall \ [b] \in X_{1,k}(X, A). \]

The spaces \( X_{1,k}(X, A) \) and \( V^A_{1,k} \) have natural topologies; see Subsection 2.1 below. By Theorem 1.1 in [Z5], if \((X, \omega, J)\) is the complex projective space \((\mathbb{P}^n, \omega_0, J_0)\) with its standard Kahler structure and \((\mathcal{L}, \nabla)\) is a positive power of the hyperplane line bundle, i.e. the dual of the tautological line bundle, \( \gamma^* \) with its standard structure, \( \gamma \) is well defined. By Theorem 1.2 in [Z5], this is also the case if \( J \) is an almost complex structure on \( X, \omega, J \) in [Z5], if \((\mathcal{L}, \nabla)\) is well defined without any condition on \( J \), as long as \((\mathcal{L}, \nabla)\) satisfies the requirement of Definition 1.2; see Corollary 1.4 below.

**Definition 1.2** Suppose \((X, \omega)\) is a symplectic manifold and \( A \in H_2(X; \mathbb{Z}) \). A complex line bundle \( \mathcal{L} \rightarrow X \) is \((\omega, A)\)-positive if
\[ \langle c_1(\mathcal{L}), B \rangle > 0 \quad \forall \ B \in H_2(X; \mathbb{Z})^* \text{ s.t. } B = A \text{ or } \langle \omega, B \rangle < \langle \omega, A \rangle. \]

We note that \( V^A_{1,k} \rightarrow \overline{\mathcal{M}}^0_{1,k}(X, A; J) \) is not a vector bundle, as the fibers of \( V^A_{1,k} \) are of two possible dimensions. In Subsections 1.2 and 1.3 of [Z5], the Poincare dual of the euler class of \( V^A_{1,k} \) is defined as the zero set of a generic multisection \( \varphi \) of \( V^A_{1,k} \) over \( \overline{\mathcal{M}}^0_{1,k}(X, A; J) \). This zero set determines a homology class in \( \overline{\mathcal{M}}^0_{1,k}(X, A; J) \) if \( \varphi \) is sufficiently regular. In Section 3 of [Z5], it is shown that \( V^A_{1,k} \) contains a vector subbundle of a sufficiently high rank over a neighborhood of every stratum of \( \overline{\mathcal{M}}^0_{1,k}(X, A; J) \). The existence of such subbundles implies that regular sections of \( V^A_{1,k} \) exist; see Subsection 3.1 in [Z5].

If \( J \) does not satisfy the regularity condition of Definition 1.4 in [Z4], the moduli space \( \overline{\mathcal{M}}^0_{1,k}(X, A; J) \) itself need not carry a fundamental class. In this case, we cannot define the Poincare dual of the euler class of \( V^A_{1,k} \) as the zero set of a section of \( V^A_{1,k} \) over \( \overline{\mathcal{M}}^0_{1,k}(X, A; J) \). On the other hand, in [Z6], the definition of
\[ \overline{\mathcal{M}}^0_{1,k}(X, A; J) \subset \overline{\mathcal{M}}_{1,k}(X, A; J) \]
given in [Z4] is generalized to define the main component \( \overline{\mathcal{M}}^0_{1,k}(X, A; J, \nu) \) of the moduli space \( \overline{\mathcal{M}}_{1,k}(X, A; J, \nu) \) of \((J, \nu)\)-holomorphic maps for an *effectively supported perturbation* \( \nu \) of the \( \bar{\partial}_J \) operator; see Definitions 2.1 and 2.2 below. By Theorem 1.5 in [Z6], if \( \nu \) is sufficiently small
and generic, $\overline{\mathcal{M}}_{1,k}^0(X,A;J,\nu)$ determines a rational homology class in a small neighborhood of $\overline{\mathcal{M}}_{1,k}^0(X,A;J)$ in $\mathcal{X}_{1,k}(X,A)$. This rational homology class is independent of the choice of $\nu$. We will define the Poincare dual of the euler class of $\mathcal{V}_{1,k}^A$ as the zero set of a generic multisection of $\mathcal{V}_{1,k}^A$ over $\overline{\mathcal{M}}_{1,k}^0(X,A;J,\nu)$. Tietze Extension Theorem will be used to show that $\mathcal{V}_{1,k}^A$ admits sections that are sufficiently nice for this purpose.

If $J$ is an almost complex structure on $X$ and $J \equiv (J_t)_{t \in [0,1]}$ is a family of almost complex structures on $X$, we denote by

$$\mathfrak{G}_{1,k}^{cs}(X,A;J) \quad \text{and} \quad \mathfrak{G}_{1,k}^{cs}(X,A;J)$$

the spaces of effectively supported perturbations of the $\bar{\partial}_J$-operator on $\mathcal{X}_{1,k}(X,A)$ and of effectively supported families of perturbations of the $\bar{\partial}_J$-operators on $\mathcal{X}_{1,k}(X,A)$; see Subsection 2.1 for details. If

$$\nu \equiv (\nu_t)_{t \in [0,1]} \in \mathfrak{G}_{1,k}^{cs}(X,A;J),$$

we put

$$\overline{\mathcal{M}}_{1,k}^0(X,A;J,\nu) = \{ (t,b) \in [0,1] \times \mathcal{X}_{1,k}(X,A) : b \in \overline{\mathcal{M}}_{1,k}^0(X,A;J_t,\nu_t) \}.$$

We denote by $\overline{\mathbb{Z}}^+$ the set of nonnegative integers. Let

$$\dim_{1,k}(X,A;\mathcal{L}) = \dim_{1,k}(X,A) - 2\langle c_1(\mathcal{L}), A \rangle = 2\langle (c_1(TX) - c_1(\mathcal{L}), A) + k \rangle.$$

**Theorem 1.3** Suppose $(X,\omega,J)$ is a compact almost Kahler manifold, $A \in H_2(X;\mathbb{Z})^*$, $k \in \overline{\mathbb{Z}}^+$, $(\mathcal{L},\nabla) \rightarrow X$ is an $(\omega,A)$-positive line bundle with connection, $\mathcal{V}_{1,k}^A \rightarrow \mathcal{X}_{1,k}(X,A)$ is the corresponding cone, and $W$ is a neighborhood of $\overline{\mathcal{M}}_{1,k}^0(X,A;J)$ in $\mathcal{X}_{1,k}(X,A)$. If $\nu \in \mathfrak{G}_{1,k}^{cs}(X,A;J)$ is sufficiently small and generic and $\varphi$ is a generic multisection of $\mathcal{V}_{1,k}^A$ over $\overline{\mathcal{M}}_{1,k}^0(X,A;J,\nu)$, then $\varphi^{-1}(0)$ determines a rational homology class in $W$. Furthermore, if $J = (J_t)_{t \in [0,1]}$ is a family of $\omega$-tamed almost complex structures on $X$, such that $J_0 = J$ and $J_t$ is sufficiently close to $J$ for all $t$, $\nu_0$ and $\nu_1$ are sufficiently small generic effectively supported perturbations of $\bar{\partial}_{J_0}$ and $\bar{\partial}_{J_1}$, and $\varphi_0$ and $\varphi_1$ are generic multisections of $\mathcal{V}_{1,k}^A$ over $\overline{\mathcal{M}}_{1,k}^0(X,A;J_0,\nu_0)$ and $\overline{\mathcal{M}}_{1,k}^0(X,A;J_1,\nu_1)$, then there exist homotopies

$$\nu = (\nu_t)_{t \in [0,1]} \in \mathfrak{G}_{1,k}^{cs}(X,A;J) \quad \text{and} \quad \Phi \in \Gamma(\overline{\mathcal{M}}_{1,k}^0(X,A;J,\nu);\mathcal{V}_{1,k}^A)$$

between $\nu_0$ and $\nu_1$ and between $\varphi_0$ and $\varphi_1$ such that $\Phi^{-1}(0)$ determines a chain in $W$ and

$$\partial \Phi^{-1}(0) = \varphi_1^{-1}(0) - \varphi_0^{-1}(0).$$

**Corollary 1.4** If $(X,\omega,J)$, $A$, $k$, and $(\mathcal{L},\nabla)$ are as in Theorem 1.3, the cone $\mathcal{V}_{1,k}^A \rightarrow \mathcal{X}_{1,k}(X,A)$ corresponding to $(\mathcal{L},\nabla)$ determines a well-defined homology class

$$PD_{\overline{\mathcal{M}}_{1,k}^0(X,A;J)}^0(\mathcal{V}_{1,k}^A) \in H_{\dim_{1,k}(X,A;\mathcal{L})}(\overline{\mathcal{M}}_{1,k}^0(X,A;J);\mathbb{Q}).$$

This class is an invariant of $(X,\omega)$ and $(\mathcal{L},\nabla)$.

As in [Z5], we will describe the local structure of the cone $\mathcal{V}_{1,k}^A$. In contrast to [Z5], we will not construct a high-rank vector subbundle of $\mathcal{V}_{1,k}^A$ over a neighborhood of every stratum of
Instead, we will use Tietze Extension Theorem to construct a sufficiently regular multisection of $\mathcal{V}_{1,k}$. Its zero set determines a homology class in a small neighborhood of $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$. For a generic $\nu$, $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$ can be stratified by orbifolds $U_\alpha$ of even dimensions; see Subsection 3.4 and Remark 1 at the end of Subsection 3.3. The main stratum of $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$, $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu) \equiv \overline{\mathcal{M}}_{1,k}(X, A; J, \nu) \cap \mathcal{X}_{1,k}(X, A)$, is of dimension $\dim_{1,k}(X, A)$, where $\mathcal{X}_{1,k}(X, A) \subset \mathcal{X}_{1,k}(X, A)$ is the subspace of stable maps with smooth domains. In Subsection 3.5, we describe a subcone $\mathcal{W}_{1,k}^A$ of $\mathcal{V}_{1,k}^A$ such that $\mathcal{W}_{1,k}^A|_{U_\alpha}$ is a smooth vector bundle for every stratum $U_\alpha$. By analyzing the obstruction to extending holomorphic bundle sections from singular to smooth domains in Section 4, we show that $\mathcal{W}_{1,k}^A$ is a regular obstruction-free cone in the sense of Definition 3.3. By Proposition 3.6, for a generic multisection $\varphi$ of $\mathcal{W}_{1,k}^A \subset \mathcal{V}_{1,k}^A$ over $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$, $\varphi|_{U_\alpha}$ is then transverse to the zero set in $\mathcal{W}_{1,k}^A|_{U_\alpha}$. By the rank statements of Proposition 3.9, $\varphi^{-1}(0)$ is stratified by smooth orbifolds of even dimensions. Furthermore, the main stratum of $\varphi^{-1}(0)$ is of dimension $\dim_{1,k}(X, A; \mathcal{L})$ and is contained in $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$. We can then choose an arbitrarily small neighborhood $U$ of the boundary of $\varphi^{-1}(0)$ such that $H_l(U; \mathbb{Q}) = \{0\}$ $\forall l \geq \dim_{1,k}(X, A; \mathcal{L}) - 1$. Since $\varphi^{-1}(0) - U$ is compact, via the pseudocycle construction of Chapter 7 in [McSa] and Section 1 of [RT], $\varphi^{-1}(0)$ determines a homology class $[\varphi^{-1}(0)] \in H_{\dim_{1,k}(X, A; \mathcal{L})}(W, U; \mathbb{Q}) \approx H_{\dim_{1,k}(X, A; \mathcal{L})}(W; \mathbb{Q})$; see also [Z7]. The second part of Theorem 1.3 is a parametrized version of this construction. Corollary 1.4 is an immediate consequence of Theorem 1.3; see also Remark 2 in Subsection 1.3 of [Z6] and the comments at the end of Subsection 1.4 in [Z6].

The statement of Corollary 1.4 is not needed to show that the expressions on the right-hand sides of (1.5) and (1.8) are well defined, as this is the case by Theorem 1.1 in [Z5]. However, the detailed statement of Theorem 1.3 is useful for proving Theorem 1.1 and its generalizations in Remarks 1 and 2 whenever $Y$ is not a Fano complete intersection. If $Y$ is Fano, Theorem 1.1 can be obtained from [Z5] by working just with $J$-holomorphic, instead of $(J, \nu)$-holomorphic, maps.

Remark: If $\mathcal{L}$ is a direct sum of $(\omega, A)$-positive line bundles, the Poincare dual of the euler class of the corresponding cone is defined to be the intersection product of the Poincare duals of the euler classes of the cones corresponding to the component line bundles. The intersection product can be defined by intersecting pseudocycle representatives for the above homology classes; see Subsection 1.2 in [Z5].
1.3 Some Special Cases

Proposition 3.1 in [Z6] implies that the difference between the standard and reduced genus-one invariants of a symplectic manifold \((Y, \omega)\) is a combination of the genus-zero invariants of \(Y\). The exact form of this combination in general is determined in [Z9].

If \((Y, \omega, J)\) is an almost Kahler manifold, for each \(l = 1, \ldots, k\) let

\[
ev_l : \overline{\mathcal{M}}_{g,k}(Y, A; J) \to Y, \quad \left[\Sigma, y_1, \ldots, y_k; u\right] \to u(y_l),
\]

be the evaluation map at the \(l\)th marked point. We will call a cohomology class \(\psi\) on \(\overline{\mathcal{M}}_{g,k}(Y, A; J)\) geometric if \(\psi\) is a product of the classes \(ev_l^\sharp \mu_l\) for \(\mu_l \in H^*(Y; \mathbb{Z})\). By Theorem 1.1 in [Z6], if \(A \in H_2(Y; \mathbb{Z})^*\), then

\[
GW^{Y}_{1,k}(A; \psi) - GW^0_{1,k}(A; \psi) = \begin{cases} 
0, & \text{if } \dim \mathbb{R}Y = 4; \\
\frac{2-\langle c_1(TY), A\rangle}{24}GW^0_{0,k}(A; \psi), & \text{if } \dim \mathbb{R}Y = 6,
\end{cases}
\]

for every geometric cohomology class \(\psi\) on \(\overline{\mathcal{M}}_{1,k}(Y, A; J)\).

In the rest of this subsection, we discuss some implications of Theorem 1.1 and Remarks 1 and 2, combined with (1.12), focusing on Calabi-Yau complete-intersection threefolds. We note that if \(Y\) is a Calabi-Yau threefold, then the expected dimension of the moduli space \(\overline{\mathcal{M}}_{g,0}(Y, A; J)\) is zero for every \(g\) and \(A\).

With notation as in Theorem (1.3), if \(a = 5\), \(Y\) is a quintic threefold. It can be easily seen that \(c_1(TY) = 0\). Let

\[N_g(d) = GW^{Y}_{g,0}(d; 1)\]

Theorem 1.3 and equation (1.12) then give

**Corollary 1.5** Suppose \(d\) is a positive integer, \(\mathcal{L} = \gamma^\ast \otimes 5 \to \mathbb{P}^4\), and

\[
\pi_1^d : \mathcal{U}_1(\mathbb{P}^4, d) \to \overline{\mathcal{M}}_{1}(\mathbb{P}^4, d) \quad \text{and} \quad ev_1^d : \mathcal{U}_1(\mathbb{P}^4, d) \to \mathbb{P}^4
\]

are the universal family and the natural evaluation map, respectively. If \(Y \subset \mathbb{P}^4\) is a smooth quintic threefold,

\[
N_1(d) = \frac{1}{12}N_0(d) + \langle e(\pi_1^d \ast ev_1^d \ast \mathcal{L}), [\overline{\mathcal{M}}_{1}(\mathbb{P}^4, d)] \rangle.
\]

The middle number in (1.13) can be computed using (1.2). This has been done for every \(d\) in [Ber], [Ga1], [Gi], [Le], and [LL Y]. As mentioned in Subsection 1.1, the last number in (1.13) can be computed, for each given \(d\), via the classical localization theorem of [ABo]. Similarly to the genus-zero case, the complexity of computing the last term in (1.13) increases rapidly with the degree \(d\), but this has been fully carried out in [Z8], finally confirming the genus-one prediction.
The numbers $n_1(d)$ that appear in the last row of this table are defined by

$$N_0(d) = \sum_{k|d} n_0(d/k) k^3, \quad N_1(d) = \frac{1}{12} \sum_{k|d} n_0(d/k) + \sum_{k|d} \frac{\sigma(k)}{k} n_1(d/k), \quad \sigma(k) = \sum_{r|k} r.$$

The numbers $n_0(d)$ and $n_1(d)$ are of importance in theoretical physics. Conjecturally, $n_g(d)$ is a count of $J$-holomorphic degree-$d$ genus-$g$ curves in $Y$ for a generic almost complex structure $J$ on $Y$.

With notation as in Remark 1 in Subsection 1.1, if $a_1 + \ldots + a_n = n+1$ and $Y$ is a corresponding complete intersection, then $Y$ is a Calabi-Yau threefold. Let

$$N^Y_1(d) = GW^Y_{g,0}(d; 1).$$

The identities in Remark 1 and in (1.12) then give

$$N^Y_1(d) = \frac{1}{12} N^Y_0(d) + \langle e(V^A_{1,k}), [\overline{\mathcal{M}}^0_1(Y, A; J)] \rangle.$$

Once again, both terms on the right-hand side are computable via (1.2) and the classical localization theorem.

In the more general case of Remark 2 in Subsection 1.1, $Y$ is a Calabi-Yau threefold if

$$c_1(\mathcal{L}) - c_1(TX) = 0 \quad \text{and} \quad \dim \mathbb{R} X - 2\text{rk} \mathbb{C} \mathcal{L} = 6.$$

In such a case,

$$N^Y_1(A) = \frac{1}{12} N^Y_0(A) + \langle e(V^A_{1,k}), [\overline{\mathcal{M}}^0_1(Y, A; J)] \rangle,$$

where $V^A_{1,k}$ is the cone corresponding to $(\mathcal{L}, \nabla)$ and $N^Y_g(A) = GW^Y_{g,0}(A; 1)$.

Two completely different approaches to computing positive-genus GW-invariants of complete intersections have been proposed in [Ga2] and [MaP]. Both approaches use degenerations and relative Gromov-Witten invariants. The first approach can be used to compute the genus-one and -two GW-invariants of a quintic threefold as well as of some other low-degree low-dimensional complete intersections. In contrast, Theorem 1.1 above and Proposition 3.1 in [Z6] are at the present restricted to genus-one GW-invariants only, but are applicable to arbitrary complete intersections.

We would like to thank D. Maulik, R. Pandharipande, G. Tian, and R. Vakil for a number of helpful discussions.

\footnote{The four numbers $N_1(d)$ shown in Table 1 were obtained by a direct localization computation and predate the desingularization construction of [VaZ] and the complete computation of [Z8].}

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell\ldots\ell$</td>
<td>0</td>
<td>2,875,493,355,350</td>
<td>952,691,384,375</td>
<td></td>
</tr>
<tr>
<td>$N_1(d)$</td>
<td>2,875</td>
<td>81,000,324,952</td>
<td>384,375,000</td>
<td></td>
</tr>
<tr>
<td>$n_1(d)$</td>
<td>0</td>
<td>0</td>
<td>609,250</td>
<td>3,721,431,625</td>
</tr>
</tbody>
</table>

Table 1: Low-degree GW-invariants of a quintic threefold
2 Hyperplane Property for Genus-One GW-Invariants

2.1 Review of Definitions

Suppose $X$ is a compact manifold, $A \in H_2(X; \mathbb{Z})$, and $g, k \in \mathbb{Z}^+$. Let $\mathfrak{X}_{g,k}(X, A)$ denote the space of equivalence classes of stable smooth maps $u : \Sigma \to X$ from genus-$g$ Riemann surfaces with $k$ marked points, which may have simple nodes, to $X$ of degree $A$, i.e.

$$u_*[\Sigma] = A \in H_2(X; \mathbb{Z}).$$

The spaces $\mathfrak{X}_{g,k}(X, A)$ are topologized using $L^p$-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes; see Section 3 in [LT1]. Here and throughout the rest of the paper, $p$ denotes a real number greater than two. The spaces $\mathfrak{X}_{g,k}(X, A)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{T}(X)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components of the domain; see Subsections 3.1 and 3.2. The closure of the main stratum, $\mathfrak{X}_{g,k}^0(X, A)$, is $\mathfrak{X}_{g,k}(X, A)$.

If $J$ is an almost complex structure on $X$, let

$$\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$$

be the bundle of $(TX, J)$-valued $(0, 1)$-forms. In other words, the fiber of $\Gamma_{g,k}^{0,1}(X, A; J)$ over a point $[b] = [\Sigma, j; u]$ in $\mathfrak{X}_{g,k}(X, A)$ is the space

$$\Gamma_{g,k}^{0,1}(X, A; J)|_b = \Gamma_{g,k}^{0,1}(b; J)/\text{Aut}(b), \quad \text{where} \quad \Gamma_{g,k}^{0,1}(b; J) = \Gamma(\Sigma; \Lambda_{J,j}^{0,1}T^*\Sigma \otimes u^*TX).$$

Here $j$ is the complex structure on $\Sigma$, the domain of the smooth map $u$. The bundle $\Lambda_{J,j}^{0,1}T^*\Sigma \otimes u^*TX$ over $\Sigma$ consists of $(J, j)$-antilinear homomorphisms:

$$\Lambda_{J,j}^{0,1}T^*\Sigma \otimes u^*TX = \{ \eta \in \text{Hom}(T\Sigma, u^*TX) : J \circ \eta = -\eta \circ j \}.$$

The total space of the bundle $\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$ is topologized using $L^p$-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma_{g,k}^{0,1}(X, A; J)$ to each stratum $\mathfrak{X}_T(X)$ is a smooth vector orbibundle of infinite rank.

We define a continuous section of the bundle $\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$ by

$$\bar{\partial}_J([\Sigma, j; u]) = \partial_{J,j}u = \frac{1}{2}(du + J \circ du \circ j).$$

By definition, the zero set of this section is the moduli space $\mathfrak{M}_{g,k}(X, A; J)$. The restriction of $\bar{\partial}_J$ to each stratum of $\mathfrak{X}_{g,k}(X, A)$ is smooth. The section $\bar{\partial}_J$ is Fredholm, i.e. the linearization of its restriction to every stratum $\mathfrak{X}_T(X)$ has finite-dimensional kernel and cokernel at every point of $\bar{\partial}_J^{-1}(0) \cap \mathfrak{X}_T(X)$. The index of the linearization of $\bar{\partial}_J$ at an element of $\mathfrak{M}_{g,k}^0(X, A; J)$ is the expected dimension of the moduli space $\mathfrak{M}_{g,k}(X, A; J)$,

$$\dim_{g,k}(X, A) = 2((c_1(TX), A) + (1-g)(n-3) + k), \quad \text{where} \quad 2n = \dim gX.$$
This is the dimension of the cycle
\[ \mathcal{M}_{g,k}(X, A; J, \nu) \equiv \{ \bar{\partial}_J + \nu \}^{-1}(0) \]
for a small generic multivalued perturbation
\[ \nu \in \mathcal{G}^{0,1}_{g,k}(X, A; J) \equiv \Gamma(\mathcal{L}_{X, A}(X, A), \mathcal{G}^{0,1}_{g,k}(X, A; J)) \]
of \( \bar{\partial}_J \), where \( \mathcal{G}^{0,1}_{g,k}(X, A; J) \) is the space of all continuous multisections \( \nu \) of \( \mathcal{G}^{0,1}_{g,k}(X, A; J) \) such that the restriction of \( \nu \) to each stratum \( \mathcal{P}_{X, A} \) is smooth. We use the term multisection, or multivalued section, of a vector orbi-bundle as defined in Section 3 of [FuO]. Since the moduli space \( \mathcal{M}_{g,k}(X, A; J) \) is compact, so is \( \mathcal{M}_{g,k}(X, A; J, \nu) \) if \( \nu \) is sufficiently small.

An element \([\Sigma; u]\) of \( \mathfrak{X}_{1,k}(X, A) \) is an equivalence class of pairs consisting of a prestable genus-one Riemann surface \( \Sigma \) and a smooth map \( u : \Sigma \to X \). The prestable surface \( \Sigma \) is a union of the principal component(s) \( \Sigma_P \), which is either a smooth torus or a circle of spheres, and trees of rational bubble components, which together will be denoted by \( \Sigma_B \). Let
\[ \mathfrak{X}_{1,k}^{0}(X, A) = \{ [\Sigma; u] \in \mathfrak{X}_{1,k}(X, A) : u_*[\Sigma_P] \neq 0 \in H_2(X; \mathbb{Z}) \}. \]

Suppose
\[ [\Sigma; u] \in \mathfrak{X}_{1,k}(X, A) - \mathfrak{X}_{1,k}^{0}(X, A), \] i.e. the degree of \( u|_{\Sigma_P} \) is zero. Let \( \chi^0(\Sigma; u) \) be the set of components \( \Sigma_i \) of \( \Sigma \) such that for every bubble component \( \Sigma_h \) that lies between \( \Sigma_i \) and \( \Sigma_P \), including \( \Sigma_i \) itself, the degree of \( u|_{\Sigma_h} \) is zero. The set \( \chi^0(\Sigma; u) \) includes the principal component(s) of \( \Sigma \). We give an example of the set \( \chi^0(\Sigma; u) \) in Figure 1. In this figure, we show the domain \( \Sigma \) of the stable map \( (\Sigma; u) \) and shade the components of the domain on which the degree of the map \( u \) is not zero. Let
\[ \Sigma^{0}_i = \bigcup_{i \in \chi^0(\Sigma; u)} \Sigma_i. \]

Every bubble component \( \Sigma_i \subset \Sigma_B \) is a sphere and has a distinguished singular point, which will be called the attaching node of \( \Sigma_i \). This is the node of \( \Sigma_i \) that lies either on \( \Sigma_P \) or on a bubble \( \Sigma_h \) that lies between \( \Sigma_i \) and \( \Sigma_P \). We denote by \( \chi(\Sigma; u) \) the set of bubble components \( \Sigma_i \) such that the attaching node of \( \Sigma_i \) lies on \( \Sigma^{0}_i \) and the degree of \( u|_{\Sigma_i} \) is not zero.

**Definition 2.1** Suppose \((X, \omega)\) is a compact symplectic manifold and \( J \equiv (J_t)_{t \in [0,1]} \) is a \( C^1 \)-continuous family of \( \omega \)-tamed almost structures on \( X \). A continuous family of multisections \( \nu \equiv (\nu_t)_{t \in [0,1]} \), with \( \nu_t \in \mathcal{G}^{0,1}_{1,k}(X, A; J_t) \) for all \( t \in [0,1] \), is **effectively supported** if for every element
\[ b \equiv [\Sigma; u] \in \mathfrak{X}_{1,k}(X, A) - \mathfrak{X}_{1,k}^{0}(X, A) \]
there exists a neighborhood \( W_b \) of \( \Sigma^{0}_u \) in a semi-universal family of deformations for \( b \) such that
\[ \nu_t(\Sigma'; u')|_{\Sigma' \cap W_b} = 0 \quad \forall \quad [\Sigma'; u'] \in \mathfrak{X}_{1,k}(X, A), \quad t \in [0,1]. \]
If \( b = [\Sigma; u] \) is an element of \( X_{1,k}(X, A) \), a semi-universal universal family of deformations for \( b \) is a fibration

\[
\sigma_b : \tilde{U}_b \longrightarrow \Delta_b
\]
such that \( \Delta_b/\text{Aut}(b) \) is a neighborhood of \( b \) in \( X_{1,k}(X, A) \) and the fiber of \( \sigma_b \) over a point \([\Sigma'; u']\) is \( \Sigma' \). If \( J \equiv (J_t)_{t \in [0,1]} \) is a continuous family of \( \omega \)-tamed almost structures on \( X \), we denote the space of effectively supported families \( \nu \) as in Definition 2.1 by \( \mathcal{G}^{es}_{1,k}(X, A; J) \). Similarly, if \( J \) is an almost complex structure on \( X \), let \( \mathcal{G}^{es}_{1,k}(X, A; J) \) be the subspace of elements \( \nu \) of \( \mathcal{G}^{0,1}_{1,k}(X, A; J) \) such that the family \( \nu_t = \nu \) is effectively supported.

Suppose \( \nu \in \mathcal{G}^{es}_{1,k}(X, A; J) \) and \([\Sigma; u]\) is an element of \( \mathcal{M}_{1,k}(X, A; J, \nu) \) as in Definition 2.1. Since \( \Sigma_i \subset \Sigma_B \) is a sphere, we can represent this element by a pair \([\Sigma; u]\) such that the attaching node of every bubble component \( \Sigma_i \subset \Sigma_B \) is the south pole, or the point \( \infty = (0, 0, -1) \), of \( S^2 \subset \mathbb{R}^3 \). Let \( e_{\infty} = (1, 0, 0) \) be a nonzero tangent vector to \( S^2 \) at the south pole. If \( i \in \chi(\Sigma; u) \), we put

\[
D_i(\Sigma; u) = d\{ u|_{\Sigma_i} \} |_{e_{\infty}} e_{\infty} \in T_{u|_{\Sigma_i}(\infty)} X.
\]

We note that \( u|_{\Sigma_i} \) is a degree-zero holomorphic map and thus constant. Thus, \( u \) maps the attaching nodes of all elements of \( \chi(\Sigma; u) \) to the same point in \( X \).

**Definition 2.2** Suppose \((X, \omega, J)\) is a compact almost Kahler manifold, \( A \in H_2(X; \mathbb{Z})^* \), and \( k \in \mathbb{Z}^+ \). If \( \nu \in \mathcal{G}^{es}_{1,k}(X, A; J) \) is an effectively supported perturbation of the \( \bar{\partial}_J \)-operator, the main component of the space \( \mathcal{M}_{1,k}(X, A; J, \nu) \) is the subset \( \mathcal{M}^{0}_{1,k}(X, A; J, \nu) \) consisting of the elements \([\Sigma; u]\) of \( \mathcal{M}_{1,k}(X, A; J, \nu) \) such that

(a) the degree of \( u|_{\Sigma_p} \) is not zero, or

(b) the degree of \( u|_{\Sigma_p} \) is zero and \( \dim \mathcal{C} \text{Span}(\Sigma, J) \{ D_i(\Sigma; u) : i \in \chi(\Sigma; u) \} < |\chi(\Sigma; u)| \).

We note that \( \mathcal{M}^{0}_{1,k}(X, A; J, \nu) \) is a compact space by Theorem 1.4 of [Z6] if \( \nu \) is effectively supported and sufficiently small. For a generic effectively supported \( \nu \), \( \mathcal{M}^{0}_{1,k}(X, A; J, \nu) \) determines a homology class of the expected dimension in a small neighborhood of \( \mathcal{M}^{0}_{1,k}(X, A; J) \) in \( X_{1,k}(X, A) \) which is independent of \( \nu \) and \( J \); see Theorem 1.5 and Corollary 1.6 in [Z6].

If \( X, A, g, \) and \( k \) are as above and \((\mathcal{L}, \nabla) \longrightarrow X \) is a vector bundle with connection, we denote by

\[
\Gamma_{g,k}(\mathcal{L}, A) \longrightarrow X_{g,k}(X, A)
\]
the cone such that the fiber of \( \Gamma_{g,k}(\mathcal{L}, A) \) over \([b] = [\Sigma; u]\) in \( X_{g,k}(X, A) \) is the Banach space

\[
\Gamma_{g,k}(\mathcal{L}, A)|_{[b]} = \Gamma(b; \mathcal{L})/\text{Aut}(b), \quad \text{where} \quad \Gamma(b; \mathcal{L}) = L^1_l(\Sigma; u^* \mathcal{L}).
\]
The topology on the total space of $\Gamma_{g,k}(\mathcal{L}, A)$ is defined analogously to the topology on $\Gamma_{g,k}(TX, A)$ of Section 3 in [LT1]. Let

$$V_{A}^{g,k} = \{[b, \xi] \in \Gamma_{g,k}(\mathcal{L}, A): [b] \in \mathcal{X}_{g,k}(X, A); \xi \in \ker \partial_{\nabla, b} \subset \Gamma_{g,k}(b; \mathcal{L})\} \subset \Gamma_{g,k}(\mathcal{L}, A).$$

The cone $V_{A}^{g,k} \longrightarrow \mathcal{X}_{g,k}(X, A)$ inherits its topology from $\Gamma_{g,k}(\mathcal{L}, A)$.

### 2.2 Statement and Proof of Hyperplane Property

We will call a cohomology class $\psi$ on $\mathcal{X}_{1,k}(X, A)$ tautological if there exists a vector bundle $\mathcal{W} \longrightarrow \mathcal{X}_{1,k}(X, A)$ such that $\mathcal{W}|_{\mathcal{X}_{T}(X)}$ is smooth for every stratum $\mathcal{X}_{T}(X)$ of $\mathcal{X}_{1,k}(X, A)$ and $\psi = e(\mathcal{W})$.

If $(X, J)$ is an almost complex manifold and $(\mathcal{L}, \nabla) \longrightarrow X$ is a complex vector bundle with connection, we will call a section $s$ of $(\mathcal{L}, \nabla)$-holomorphic if

$$\bar{\partial}_{\nabla}s \equiv \frac{i}{2}(\nabla s + i\nabla s \circ J) = 0.$$

**Theorem 2.3** Suppose $(X, \omega, J)$ is a compact almost Kahler manifold, $A \in H_{2}(X; \mathbb{Z})^{*}$, $k \in \mathbb{Z}^{+}$, $(\mathcal{L}, \nabla) \longrightarrow X$ is a complex vector bundle with connection, and $s$ is a $(\nabla, s)$-holomorphic section of $\mathcal{L}$ such that $J$ is genus-one $A$-regular in the sense of Definition 1.4 in [Z4], $s$ is transverse to the zero set in $\mathcal{L}$, and $(\mathcal{L}, \nabla)$ splits into $(\omega, A)$-positive line bundles. If $V_{1,k}^{A} \longrightarrow \mathcal{X}_{1,k}(X, A)$ is the cone corresponding to $(\mathcal{L}, \nabla)$ and $Y = s^{-1}(0)$,

$$GW_{1,k}^{Y}(A; \psi) = \langle \psi, PD_{\overline{\mathcal{M}}_{1,k}(X, A; J)}e(V_{1,k}^{A}) \rangle \tag{2.2}$$

for every tautological class $\psi$ on $\mathcal{X}_{1,k}(X, A)$.

Since $J$ is genus-one $A$-regular, $\overline{\mathcal{M}}_{1,k}(X, A; J)$ has the expected structure of a topological orbivariety. By the generalization of the proof of the regularity statement of Theorem 1.6 in [Z4] analogous to Subsection 2.5 in [Z6], for all $\nu \in \mathcal{G}^{es}_{1,k}(X, A; J)$ sufficiently small $\overline{\mathcal{M}}_{1,k}(X, A; J, \nu)$ also has the expected structure of a topological orbivariety. In particular, it is stratified by smooth orbifolds of even dimensions as described in Subsection 3.4 below. We will call $\nu \in \mathcal{G}^{es}_{1,k}(X, A; J)$ $(\nabla, s)$-compatible if

$$\nabla s|_{u} \circ \nu(\Sigma; u) = 0 \quad \forall [\Sigma; u] \in \mathcal{X}_{1,k}(X, A).$$

We note that if $\nu$ is $(\nabla, s)$-compatible, then the map

$$(\Sigma; u) \longrightarrow s_{1,k}^{A}(\Sigma; u) = s \circ u \in \Gamma(\Sigma; u^{*}\mathcal{L})$$

defines a continuous section of the cone $V_{1,k}^{A}$ over $\overline{\mathcal{M}}_{1,k}(X, A; J)$.

Since the $(\nabla, s)$-holomorphic section $s$ is transverse to the zero set in $\mathcal{L}$, the $(i, J)$-linear map

$$\nabla s: TX \longrightarrow \mathcal{L}$$
is surjective along $Y = s^{-1}(0)$. Let $U_s$ be a small neighborhood of $Y$ in $X$ such that $\nabla s$ is surjective over $U_s$. The kernel of $\nabla s$ over $U_s$ is then a complex vector bundle of rank $\dim s$.

For every stratum is surjective along $Y$. We denote this subbundle by $\nu$. If $\nu \in \mathfrak{g}_{\mathfrak{h},k}^J(X,A;J)$ is such that for all $[\Sigma; u] \in \mathfrak{X}_{1,k}(X,A)$

$$\nu(\Sigma; u) = \begin{cases} \in \Gamma(\Sigma; \Lambda^J_{\mathfrak{h},k} T^* \Sigma \otimes \tilde{TY}), & \text{if } u(\Sigma) \subset U_s; \\ 0, & \text{otherwise}, \end{cases}$$

then $\nu$ is $(\nabla, s)$-compatible. Thus, every element $\nu_Y \in \mathfrak{g}_{\mathfrak{h},k}^J(Y,A;J)$ can extended to a $(\nabla, s)$-compatible element $\nu$ of $\mathfrak{g}_{\mathfrak{h},k}^J(X,A;J)$. Furthermore, if $\nu_Y$ is a small, then $\nu$ can also be chosen to be small.

For a small generic $\nu_Y \in \mathfrak{g}_{\mathfrak{h},k}^J(Y,A;J)$, $\tilde{M}_{1,k}(Y,A;J,\nu_Y)$ is stratified by smooth orbifolds of even dimensions so that the largest-dimensional stratum is $\prod_{1,k}(Y,A;J,\nu_Y)$ and

$$\dim \prod_{1,k}(Y,A;J,\nu_Y) = \dim_{\mathfrak{h},k}(Y,A).$$

Let $\nu$ be an extension of $\nu_Y$ to a small $(\nabla, s)$-compatible element of $\mathfrak{g}_{\mathfrak{h},k}^J(X,A;J)$. Suppose

$$\mathcal{W} \longrightarrow \mathfrak{X}_{1,k}(X,A)$$

is a complex vector bundle of rank $\dim_{\mathfrak{h},k}(Y,A)/2$ as in the first paragraph of this subsection. Choose a section $f$ of $\mathcal{W}$ over $\tilde{M}_{1,k}(X,A;J,\nu)$ such that $f|_{U_d}$ is transverse to the zero set in $\mathcal{W}|_{U_d}$ for every stratum $U_d$ of $\tilde{M}_{1,k}(X,A;J,\nu)$ and of $\tilde{M}_{1,k}(Y,A;J,\nu_Y)$. Then,

$$f^{-1}(0) \cap \tilde{M}_{1,k}(Y,A;J,\nu_Y) \subset \prod_{1,k}(Y,A;J,\nu_Y) \quad \text{and}$$

$$\mathcal{GW}_{1,k}^Y(A;\psi) = \pm |f^{-1}(0) \cap \tilde{M}_{1,k}(Y,A;J,\nu_Y)|. \quad (2.3)$$

On the other hand, since $\nu$ is $(\nabla, s)$-compatible, $s_{1,k}^A$ is a section of

$$\mathcal{V}_{1,k}^A \longrightarrow \tilde{M}_{1,k}(X,A;J,\nu).$$

Furthermore,

$$\{ s_{1,k}^A \}^{-1}(0) = \tilde{M}_{1,k}(X,A;J,\nu) \cap \mathfrak{X}_{1,k}(Y,A) \equiv \tilde{M}_{1,k}(Y,A;J,\nu_Y);$$

$$f^{-1}(0) \cap \{ s_{1,k}^A \}^{-1}(0) = f^{-1}(0) \cap \tilde{M}_{1,k}(Y,A;J,\nu_Y) \subset \prod_{1,k}^0(X,A;J,\nu). \quad (2.4)$$

Note that if $[b] = [\Sigma; u] \in \prod_{1,k}^0(Y,A;J,\nu_Y)$,

$$\ker ds_{1,k}^A|_b = \{ \xi \in \ker D_{f,v:b}; \nabla s|_u \circ \xi = 0 \}$$

$$= \ker D_{f,v:b \cap \Gamma(\Sigma; u^{*}TY)} = \ker D_{f|_{TV},\nu_Y:b}, \quad (2.5)$$

where $D_{f,v:b}$ and $D_{f|_{TV},\nu_Y:b}$ are the linearizations of $\tilde{\partial} f + \nu$ and $\tilde{\partial} f|_{TV} + \nu_Y$ at $b$. The second equality above is immediate from the transversality of $s$. By (2.5),

$$\dim \ker ds_{1,k}^A|_{[\Sigma; u]} = \dim \ker D_{f,v:b} - \dim \ker D_{f|_{TV},\nu_Y:b}$$

$$= \dim_{1,k}(X,A) - \dim_{1,k}(Y,A) = 2(c_1(\mathcal{L}), A) = \dim \mathcal{V}_{1,k}^A|_{[b]}, \quad (2.6)$$

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The second equality above follows from our assumption that the operators $D_{I,c,h}$ and $D_{|Y,A;v}$ are surjective; the last equality is a consequence of the $(\omega, \mathcal{L})$-positivity assumption. By (2.6), $s_{1,k}$ is transverse to the zero set in $V_{1,k}^A$ along $\mathcal{M}_{1,k}(X, A; J, \nu)$. Since $f$ is transverse to the zero set in $W$ along $\mathcal{M}_{1,k}(X, A; J, \nu)$, it then follows from (2.4) that
\[
\langle \psi, PD_{\mathcal{M}_{1,k}(X,A;J)}(V_{1,k}^A) \rangle = \pm |f^{-1}(0) \cap s_{1,k}^{-1}(0)| = \pm |f^{-1}(0) \cap \mathcal{M}_{1,k}(Y, A; J, \nu)|. \tag{2.7}
\]

Theorem 2.3 follows from (2.3) and (2.7).

3 Ingredients in Proof of Theorem 1.3

3.1 Notation: Genus-Zero Maps

In this subsection we describe our detailed notation for bubble maps from genus-zero Riemann surfaces and for related objects. In general, moduli spaces of stable maps can stratified by the dual graph. However, in the present situation, it is more convenient to make use of linearly ordered sets:

**Definition 3.1** (1) A finite nonempty partially ordered set $I$ is a linearly ordered set if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \leq i_2$ or $i_2 \leq i_1$.

(2) A linearly ordered set $I$ is a rooted tree if $I$ has a unique minimal element, i.e. there exists $0 \in I$ such that $0 \leq i$ for all $i \in I$.

If $I$ is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of $I$. For every $h \in \hat{I}$, denote by $i_h \in I$ the largest element of $I$ which is smaller than $h$, i.e. $i_h = \max \{i \in I : i < h\}$.

We identify $\mathbb{C}$ with $S^2 - \{\infty\}$ via the stereographic projection mapping the origin in $\mathbb{C}$ to the north pole, or the point $(0, 0, 1)$, in $S^2$. Let $M$ be a finite set. A genus-zero $X$-valued bubble map with $M$-marked points is a tuple
\[b = (M, I; x, (j, y), u),\]
where $I$ is a rooted tree, and
\[x: \hat{I} \to \mathbb{C} = S^2 - \{\infty\}, \quad j: M \to I, \quad y: M \to \mathbb{C}, \quad u: I \to C^\infty(S^2; X) \tag{3.1}\]
are maps such that $u_h(\infty) = u_{i_h}(x_h)$ for all $h \in \hat{I}$. We associate such a tuple with Riemann surface
\[\Sigma_b = \left( \bigsqcup_{i \in I} \Sigma_{b,i} \right) / \sim, \quad \text{where} \quad \Sigma_{b,i} = \{i\} \times S^2 \quad \text{and} \quad (h, \infty) \sim (i_h, x_h) \quad \forall h \in \hat{I}, \tag{3.2}\]
with marked points
\[y_i(b) \equiv (j_i, y_i) \in \Sigma_{b, j_i} \quad \text{and} \quad y_0(b) \equiv (0, \infty) \in \Sigma_{b, 0},\]
and continuous map $u_b: \Sigma_b \to X$, given by $u_b|_{\Sigma_{b,i}} = u_i$ for all $i \in I$. The general structure of bubble maps is described by tuples $T = (M, I; j, A)$, where
\[A_i = u_i[S^2] \in H_2(X; \mathbb{Z}) \quad \forall i \in I.\]
We call such tuples bubble types. We denote by \( \tilde{X}_T(X) \) be the space of all bubble maps of type \( T \). For \( l \in \{0\} \sqcup M \), let
\[
ev_l: \tilde{X}_T(X) \longrightarrow X
\]
be the evaluation map corresponding to the marked point \( y_l \).

With notation as above, suppose
\[
b \equiv (M, I; x, (j, y), u) \in \tilde{X}_T(X).
\]
In particular, \( I \) is a linearly ordered set with minimal element \( \hat{0} \) and the special marked point is the point
\[
y_0(b) = (\hat{0}, \infty) \in \Sigma_{b,0}.
\]
Let \( \chi^0(b) \) be the set of components \( \Sigma_{b,i} \) of \( \Sigma_b \) such that for every component \( \Sigma_{b,h} \) that lies between \( \Sigma_i \) and \( \Sigma_{b,0} \), including \( \Sigma_{b,i} \) and \( \Sigma_{b,0} \), the degree of \( u|_{\Sigma_{b,h}} \) is zero. The set \( \chi^0(b) \) is empty if and only if the degree of the restriction of \( u_b \) to the component containing the special marked point is not zero. Let
\[
\Sigma_b^0 = \{(0, \infty)\} \cup \bigcup_{i \in \chi^0(b)} \Sigma_{b,i}.
\]
We denote by
\[
\chi(b) = \chi(T) \subset I
\]
the set of components \( \Sigma_{b,i} \) of \( \Sigma_b \) such that \( \Sigma_{b,i} \) has a point in common with \( \Sigma_b^0 \) and the degree of \( u_b|_{\Sigma_{b,i}} \) is not zero, i.e. \( \Sigma_{b,i} \) is not an element of \( \chi^0(b) \).

If \( b = (\Sigma_b; u_b) \) is a bubble map with a special marked point as above and \( i \in \chi(b) \), we put
\[
D_i b = du_{b,i} \big|_{e_{\infty}} \in T_{u_{b,i}(\infty)} X,
\]
where \( u_{b,i} = u_b|_{\Sigma_{b,i}} \). Similarly, if \( (\mathcal{L}, \nabla) \) is a complex line bundle with connection over \( X \) and \( \xi = (\xi_h)_{h \in I} \) is an element of \( \Gamma(b; \mathcal{L}) \), we put
\[
\mathcal{D}_{b,i} \xi = \nabla_{e_{\infty}} u_{b,i} \big|_{\infty} \in \mathcal{L}_{u_{b,i}(\infty)}.
\]
Note that if \( \xi \in \ker \partial_{\mathcal{V},b} \),
\[
\nabla_{e_{\infty}} u_{b,i} \big|_{\infty} = c \cdot \mathcal{D}_{b,i} \xi \quad \forall c \in \mathbb{C}.
\]
If in addition \( \mathcal{L} \) is \( (\omega, u_b^* [\Sigma_b]) \)-positive, then the linear operator \( \partial_{\mathcal{V},b} \) and the linear map
\[
\partial_{\mathcal{V},b}^0: \ker \partial_{\mathcal{V},b} \longrightarrow \mathcal{L}_{ev_0(b)}, \quad \xi \longrightarrow \xi(y_0(b)),
\]
are surjective. This can be seen by an argument similar to Subsection 6.2 in [Z2].

### 3.2 Notation: Genus-One Maps

We next set up notation for maps from genus-one Riemann surfaces. In this case, in contrast to the genus-zero case, we also need to specify the structure of the principal component. We describe it by enhanced linearly ordered sets:
Definition 3.2 An enhanced linearly ordered set is a pair \((I, \mathcal{N})\), where \(I\) is a linearly ordered set, \(\mathcal{N}\) is a subset of \(I_0 \times I_0\), and \(I_0\) is the subset of minimal elements of \(I\), such that if \(|I_0| > 1\),

\[
\mathcal{N} = \{ (i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n), (i_n, i_1) \}
\]

for some bijection \(i : \{1, \ldots, n\} \rightarrow I_0\).

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 2, the dots denote the elements of \(I\). The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set \(I\). In fact, every directed edge outside of the loop connects a non-minimal element \(h\) of \(I\) with \(i_h\). Inside of the loop, there is a directed edge from \(i_1\) to \(i_2\) if and only if \((i_1, i_2) \in \mathcal{N}\).

The subset \(\mathcal{N}\) of \(I_0 \times I_0\) will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of \(X_{1,M}(X, A)\). If \(\mathcal{N} = \emptyset\), and thus \(|I_0| = 1\), the corresponding principal curve \(\Sigma_P\) is a smooth torus, with some complex structure. If \(\mathcal{N} \neq \emptyset\), the principal components form a circle of spheres:

\[
\Sigma_P = \left( \bigsqcup_{i \in I_0} \{ i \} \times S^2 \right) / \sim, \quad \text{where} \quad (i_1, \infty) \sim (i_2, 0) \text{ if } (i_1, i_2) \in \mathcal{N}.
\]

A genus-one \(X\)-valued bubble map with \(M\)-marked points is a tuple

\[
b = (M, I, \mathcal{N}; S, (j, y), u),
\]

where \(S\) is a smooth Riemann surface of genus one if \(\mathcal{N} = \emptyset\) and the circle of spheres \(\Sigma_P\) otherwise. The objects \(x, j, y, u, \) and \((\Sigma_b, u_b)\) are as in (3.1) and (3.2), except the sphere \(\Sigma_{b,0}\) is replaced by the genus-one curve \(\Sigma_{b,P} \equiv S\). Furthermore, if \(\mathcal{N} = \emptyset\), and thus \(I_0 = \{ \hat{0} \}\) is a single-element set, \(u_0 \in C^\infty(S; X)\) and \(y_0 \in S\) if \(j_0 = 0\). In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form \(T = (M, I, \mathcal{N}; j, A)\). Similarly to the genus-zero case, we denote by \(\mathbar{X}_T(X)\) be the space of all bubble maps of type \(T\). Let

\[
\mathcal{X}_T(X) = \{ [b] \in \mathcal{X}_{1,M}(X, A) : b \in \mathbar{X}_T(X) \}.
\]

If \(\nu\) is an element of \(\mathcal{G}_{1,M}^\text{reg}(X, A)\), we put

\[
\mathcal{U}_{T, \nu}(X; J) = \{ [b] \in \mathcal{X}_T(X) : \{ \partial_T + \nu \}(b) = 0 \}.
\]
If \( T = (M, I, R; j, A) \) is a bubble type such that \( A_i = 0 \) for all minimal elements \( i \) of \( I \) and \( [\Sigma; u] \) is an element of \( \mathcal{U}_{T, \nu}(X; J) \), the map \( u|_{\Sigma_P} \) is constant. Let

\[
\text{ev}_P : \mathcal{U}_{T, \nu}(X; J) \rightarrow X
\]

be the map sending each element \( [\Sigma; u] \) of \( \mathcal{U}_{T, \nu}(X; J) \) to the image of the principal component \( \Sigma_P \) of \( \Sigma \), i.e. the point \( u(\Sigma_P) \) in \( X \). We note that the map

\[
\tilde{X}(X) \rightarrow 2I, \quad b \mapsto \chi(b),
\]

is constant. We denote its value by \( \chi(T) \).

Suppose \( b = (\Sigma_b; u_b) \) is an element of \( \mathcal{X}_{A,k}(X, A) \) as above and \( (\mathcal{L}, \nabla) \) is a complex vector bundle with connection over \( X \). If \( \xi = (\xi_h)_{h \in I} \) is an element of \( \Gamma(b; \mathcal{L}) \) and \( i \in I - I_0 \), similarly to the genus-zero case, we put

\[
D_{b,i} \xi = \nabla u_{b,i}(\xi)_i \bigg|_\infty \in \mathcal{L}_{u_{b,i}(\infty)},
\]

where \( u_{b,i} = u_b|_{\Sigma_{b,i}} \).

Finally, all vector orbi-bundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we implicitly choose a norm once and for all. If \( \pi : \mathfrak{F} \rightarrow X \) is a normed vector bundle and \( \delta : X \rightarrow \mathbb{R} \) is any function, possibly constant, let

\[
\mathfrak{F}_\delta = \{ \psi \in \mathfrak{F} : |\psi| < \delta(\pi_F(\psi)) \}.
\]

If \( \Omega \) is any subset of \( \mathfrak{F} \), we take \( \Omega_\delta = \Omega \cap \mathfrak{F}_\delta \).

### 3.3 Topology

In this subsection we prove a general topological result, Proposition 3.6. For the sake of clarity, we state and prove it only in the manifold/section category, but Proposition 3.6 and its proof carry over easily to the orbifold/multisection category.

Similarly to Subsection 1.2, by a cone \( \pi : W \rightarrow \overline{\mathfrak{M}} \) we mean a continuous map between two topological spaces such that \( W_x = \pi^{-1}(x) \) is vector space for each \( x \in \overline{\mathfrak{M}} \) and the vector space operations induce continuous functions on \( W \times W \) and \( \mathbb{C} \times W \).

**Definition 3.3** (1) A cone \( \pi : W \rightarrow \overline{\mathfrak{M}} \) is **regular** if for every \( b \in \overline{\mathfrak{M}} \), there exist a neighborhood \( U_b \) of \( b \) in \( \overline{\mathfrak{M}} \), \( n_b \in \mathbb{Z}^+ \), and a bundle map

\[
\varphi_b : W|_{U_b} \rightarrow U_b \times \mathbb{C}^{n_b}
\]

over \( U_b \) such that \( \varphi_b \) is a homeomorphism onto its image and the restriction of \( \varphi_b \) to each fiber is linear.

(2) A cone \( \pi : W \rightarrow \overline{\mathfrak{M}} \) is **obstruction-free** if for every \( b \in \overline{\mathfrak{M}} \), \( \xi \in W_b \), and a sequence \( b_r \in \overline{\mathfrak{M}} \) converging to \( b \), there exists a sequence \( \xi_r \in W_{b_r} \) converging to \( \xi \) in \( W \).

If \( W \rightarrow \overline{\mathfrak{M}} \) is a cone, for each \( r \in \mathbb{Z} \), let

\[
\overline{\mathfrak{M}}_r(W) = \{ b \in \overline{\mathfrak{M}} : \text{rk } W_b = r \}.
\]

Note that if \( W \) is obstruction-free, then the set \( \bigcup_{r \leq q} \overline{\mathfrak{M}}_r(W) \) is closed in \( \overline{\mathfrak{M}} \).
Lemma 3.4 Suppose \( \overline{\mathcal{M}} \) is a compact Hausdorff space that has a countable basis at each point, \( A \) is a closed subset of \( \overline{\mathcal{M}} \), \( \mathcal{W} \to \mathcal{M} \) is a regular obstruction-free cone, and \( s \) is a section of \( \mathcal{W} \) over \( A \). If
\[
r_A = \min \{ \text{rk} \mathcal{W}_b : b \in \overline{\mathcal{M}} - A \},
\]
then \( s \) extends to a continuous section \( \tilde{s} \) of \( \mathcal{W} \) over \( A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \).

Remark 1: It is enough to assume that \( \overline{\mathcal{M}} \) is a paracompact (Hausdorff) space that has a countable basis at each point.

Remark 2: An immediate corollary of this lemma is that \( s \) extends to a continuous section of \( \mathcal{W} \) over \( \overline{\mathcal{M}} \).

Let \( \{U_b\}_{b \in A} \) be a finite open cover of the compact set \( A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \) by open subspaces of \( \overline{\mathcal{M}} \) as in (1) of Definition 3.3. Since \( \mathcal{M} \) is normal, we can choose an open cover \( \{U_b'\}_{b \in A} \) of \( A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \) such that \( U_b' \subset U_b \) for all \( b \in A \). Since \( U_b' \) is normal for each \( b \in A \), by Tietze Extension Theorem the continuous section \( f = \varphi_b\circ s \) of \( U_b \times \mathbb{C}^n_b \) over \( A \cap U_b' \) extends to a continuous section \( \tilde{f} \) over \( U_b' \). Let
\[
\pi^-_b : U_b \times \mathbb{C}^n_b \to \text{Im} \varphi_b
\]
be the orthogonal projection map. We will show in the next paragraph that the section \( \pi^-_b \circ \tilde{f} \) is continuous over \( (A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W})) \cap U_b' \). Since \( A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \) is normal, we can choose a partition of unity \( \{\eta_b\}_{b \in A} \) subordinate to \( \{U_b'\}_{b \in A} \). The section
\[
\tilde{s} = \sum_{b \in A} \eta_b \circ (\varphi_b^{-1} \circ \pi^-_b \circ \tilde{f})
\]
is continuous over \( A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \). Since \( f(x) \in \text{Im} \varphi_b \) for all \( x \in A \cap U_b' \),
\[
\pi^-_b \tilde{f}(x) = \pi^-_b f(x) = f(x) \quad \forall x \in A \cap U_b'.
\]
Thus, \( \tilde{s}|_A = s \) as required.

It remains to show that the section \( \pi^-_b \circ \tilde{f} \) is continuous over \( (A \cup \overline{\mathcal{M}}_{r_A}(\mathcal{W})) \cap U_b' \). Since
\[
\pi^-_b \tilde{f}|_{A \cap U_b'} = f|_{A \cap U_b'},
\]
\( \pi^-_b \tilde{f} \) is continuous along the closed subset \( A \cap U_b' \) of \( U_b' \). Thus, we need to show that if \( x_r \in \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \cap U_b' \) is a sequence converging to \( x \in U_b' \), then \( \pi^-_b \tilde{f}(x_r) \) converges to \( \pi^-_b \tilde{f}(x) \). Suppose first
\[
x \in \overline{\mathcal{M}}_{r_A}(\mathcal{W}) \cap U_b'.
\]
We will show that \( \pi^-_b|_{\overline{\mathcal{M}}_{r_A}(\mathcal{W}) \cap U_b'} \) is continuous at \( x \). Let \( \{\xi_i\}_{i \in [r_A]} \) be an orthonormal basis for \( \text{Im} \varphi_b|_x \). By (2) of Definition 3.3, for each \( i \in [r_A] \) there exists a sequence \( \xi_{ir} \in \text{Im} \varphi_b|_x \), converging
to \( \xi_i \). Since \( b, \ell \in \M\mathbb{R}_\mathbb{C}(\mathcal{W}) \), \( \{\xi_i\}_{i \in \mathbb{A}} \) is basis for \( \text{Im} \varphi_b|_{x_r} \) for all \( r \) sufficiently large. Since \( \xi_i \longrightarrow \xi_i \) for all \( i \in \mathbb{A} \) and \( \{\xi_i\}_{i \in \mathbb{A}} \) is an orthonormal basis,

\[
\lim_{r \to \infty} \langle \xi_i, \xi_j \rangle = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}
\]

Thus, applying the Gramm-Schmidt normalization procedure, we can find an orthonormal basis \( \{\xi_i\}_{i \in \mathbb{A}} \) for \( \text{Im} \varphi_b|_{x_r} \) such that \( \xi_i \longrightarrow \xi_i \) for all \( i \in \mathbb{A} \). It follows that \( \pi_{b}^{-1}|_{x_r} \longrightarrow \pi_{b}^{-1}|_{x_r} \) as needed.

On the other hand, suppose that \( x \in A \cap \bar{U}_b \). We will view \( f \), \( \tilde{f} \), and \( \pi_{b,-}^{-1} = \pi_{b,-}^{-1} \) as \( \mathbb{C}^n \)-valued functions. By (2) of Definition 3.3, there exists a sequence \( \xi_r \in \text{Im} \varphi_b|_{x_r} \) converging to \( f(x) \). Since \( \pi_{x, \xi_r} = \xi_r \),

\[
|\pi_{x, \tilde{f}}(x) - \pi_{x, \tilde{f}}(x)| = |\tilde{f}(x) - \pi_{x, \tilde{f}}(x)| \\
\leq |\tilde{f}(x) - \xi_r| + |\pi_{x, \xi_r} - \pi_{x, \tilde{f}}(x)| + |\pi_{x, \tilde{f}}(x) - \pi_{x, \tilde{f}}(x)| \\
\leq 2|\tilde{f}(x) - \xi_r| + |\tilde{f}(x) - \tilde{f}(x)|.
\]

The last two terms above approach 0 by the assumption on \( \xi_r \) and the continuity of \( \tilde{f} \).

**Remark:** The projection \( \pi_{b,-}^{-1} \) is not continuous over \( U_b \) unless the rank of \( \mathcal{W} \) is constant over \( U_b \). Similarly, the section \( \pi_{b,-}^{-1} \circ \tilde{f} \) may not be continuous over \( \bar{U}_b \) unless the rank of \( \mathcal{W} \) is constant over \( \bar{U}_b - A \).

**Definition 3.5** If \( \M \) is a topological space and \((\mathbb{A}, \prec)\) is a finite partially ordered set, a collection \( \{\mathcal{U}_\alpha\}_{\alpha \in \mathbb{A}} \) of subspaces of \( \M \) is a stratification of \( \M \) if \( \mathcal{U}_\alpha \) is a smooth manifold for all \( \alpha \in \mathbb{A} \),

\[
\partial \mathcal{U}_\alpha = \mathcal{U}_{\alpha} - \mathcal{U}_\alpha \subset \bigcup_{\beta \prec \alpha} \mathcal{U}_\beta \quad \forall \alpha \in \mathbb{A}, \quad \text{and} \quad \M = \bigcup_{\alpha \in \mathbb{A}} \mathcal{U}_\alpha.
\]

**Proposition 3.6** Suppose \( \M \) is a compact Hausdorff space that has a countable basis at each point, \( \{\mathcal{U}_\alpha\}_{\alpha \in \mathbb{A}} \) is a stratification of \( \M \), and \( \mathcal{W} \longrightarrow \M \) is a regular obstruction-free cone. If \( \mathcal{W} |_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha \) is a smooth vector bundle for all \( \alpha \in \mathbb{A} \), the cone \( \mathcal{W} \) admits a continuous section \( s \) over \( \M \) such that \( s|_{\mathcal{U}_\alpha} \) is smooth and transverse to the zero set in \( \mathcal{W} |_{\mathcal{U}_\alpha} \) for all \( \alpha \in \mathbb{A} \).

Choose an ordering \( \prec \) on the partially ordered set \((\mathbb{A}, \prec)\) such that for all \( \alpha, \beta \in \mathbb{A} \),

\[
\text{rk} \mathcal{W} |_{\mathcal{U}_\beta} < \text{rk} \mathcal{W} |_{\mathcal{U}_\alpha} \implies \beta \prec \alpha; \quad (3.4)
\]

\[
\text{rk} \mathcal{W} |_{\mathcal{U}_\alpha} = \text{rk} \mathcal{W} |_{\mathcal{U}_\alpha}, \quad \beta \prec \alpha \implies \beta \prec \alpha.
\]

Since \( \mathcal{W} \) is obstruction-free, \( \bigcup_{r \leq q} \M(W) \) is closed in \( \M \), and

\[
\partial \mathcal{U}_\alpha \subset \bigcup_{\beta \prec \alpha} \mathcal{U}_\beta \quad \forall \alpha \in \mathbb{A}, \quad (3.5)
\]

by the closure condition of Definition 3.5. Suppose \( \alpha \in \mathbb{A} \) and we have defined a continuous section \( s \) of \( \mathcal{W} \) over the closed set

\[
A = \bigcup_{\beta \prec \alpha} \mathcal{U}_\beta
\]

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such that $s|_{U\beta}$ is smooth and transverse to the zero set in $W|_{U\beta}$ for all $\beta < \alpha$. By (3.4) and (3.5),

$$r_A \equiv \min \{ \text{rk} W_b : b \in \mathbb{M} - A \} = \text{rk} W|_{U\alpha}.$$ 

Thus, by Lemma 3.4, $s$ extends to a continuous section $\tilde{s}$ over $A \cup U\alpha \subset A \cup \mathbb{M}$. Perturbing $\tilde{s}$ over $U\alpha$, without changing it over $A$, we obtain a continuous section $s$ over $A \cup U\alpha$ such that $s|_{U\beta}$ is smooth and transverse to the zero set in $W|_{U\beta}$ for all $\beta \leq \alpha$. This construction implies Proposition 3.6.

Remark 1: In the orbifold/multisection category as needed for the purposes of this paper, a stratum $U\alpha$ locally is a union of finitely many smooth suborbifolds of a smooth orbifold $X\alpha$. We will still call such unions smooth orbifolds. The bundle $W|_{U\alpha}$ is the restriction of a smooth orbibundle over $X\alpha$.

Remark 2: The cone $V_{A,1,k}$ is not obstruction-free, but is regular. In Subsection 3.5, we describe a subcone $W_{A,1,k} \subset V_{A,1,k}$ which is obstruction-free and sufficiently large for the purposes of Theorem 1.3.

3.4 The Structure of the Moduli Space $\mathcal{M}^0_{1,k}(X, A; J, \nu)$

In this subsection, we describe the strata of the moduli space $\mathcal{M}^0_{1,k}(X, A; J, \nu)$ for a small generic element $\nu$ of $\mathfrak{G}^{\nu}_{1,k}(X, A; J)$. If $k \in \mathbb{Z}$, we denote by $[k]$ the set of positive integers that do not exceed $k$. Let $2n = \dim_{\mathbb{R}} X$.

Lemma 3.7 Suppose $(X, \omega, J), A, k,$ and $\nu$ are as in Theorem 1.3. If

$$T = ([k], I, \aleph; j, A)$$

is a bubble type such that $\sum_{i \in I} A_i = A$ and $A_i \neq 0$ for some minimal element $i$ of $I$, then $U_{T,\nu}(X; J)$ is a smooth orbifold and

$$\dim U_{T,\nu}(X; J) = \dim_{1,k}(X, A) - 2(|\aleph| + |\bar{i}|).$$

The statement that $U_{T,\nu}(X; J)$ is smooth should be interpreted as in Remark 1 at the end of the previous subsection. The branches of $U_{T,\nu}(X; J)$ correspond to the branches of $\nu$. For a generic $\nu$, the linearization $D_{J,\nu;b}$ of the bundle section $\partial J + \nu$ at $[b]$ is surjective for every element $b$ in $\mathcal{M}^0_{1,k}(X, A; J, \nu)$ such that $u_b|_{\Sigma_{a,p}}$ is not constant. Thus, Lemma 3.7 is obtained by a standard Contraction Principle argument, such as in Chapter 3 of [McSa].

Lemma 3.8 Suppose $(X, \omega, J), A, k,$ and $\nu$ are as in Theorem 1.3. If

$$T = ([k], I, \aleph; j, A)$$

...
is a bubble type such that $\sum_{i \in I} A_i = A$ and $A_i = 0$ for all minimal elements $i$ of $I$, then for each $m \in \mathbb{Z}^+$,

$$U^m_{T,\nu;1}(X; J) = \{ [\nu] \in U_{T,\nu}(X; J) : \dim \text{Span}_{\mathbb{C}}(\mathcal{D}_i : i \in \chi(T)) = |\chi(T)| - m \}$$

is a smooth orbifold and

$$\dim U^m_{T,\nu;1}(X; J) = \dim_{1,k}(X, A) - 2(|N| + |\hat{I}|) - n + (m + n - |\chi(T)|) m).$$

If $T$ is as in Lemma 3.8 and $b \equiv (\Sigma_b; u_b) \in U_{T,\nu}(X; J)$, $u_b|\Sigma_{b,\nu}$ is constant. Let

$$\Gamma_B(b) = \{ \zeta \in \Gamma(\Sigma_b; u^*_b TX) : \zeta|_{\Sigma_{b,\nu}} = 0 \};$$

$$\Gamma^0_{B,1}(b; J) = \{ \eta \in \Gamma(\Sigma_b; \Lambda^1_{\nu,j} T^* \Sigma_b \otimes u^*_b TX) : \eta|_{\Sigma_{b,\nu}} = 0 \}.$$

If $\nu$ is genetic, the operator $D^B_{J,\nu;b}: \Gamma_B(b) \rightarrow \Gamma^0_{B,1}(b; J)$ induced by $D_{J,\nu;b}$ is surjective. Thus, the space $U_{T,\nu}(X; J)$ is a smooth orbifold of dimension

$$\dim U^m_{T,\nu;1}(X; J) = \dim_{1,k}(X, A) - 2(|N| + |\hat{I}|) + 2n.$$
If $T \in A^*$ and $(T', m') \in A^0$, we define
\[ T \prec (T', m') \iff T \prec T' \land (T', m') \prec T \iff T' \prec T. \]

Finally, if $(T, m), (T', m') \in A_0$, we define
\[ (T', m') \prec (T, m) \iff T' \prec T \lor T' = T, m' > m. \]

By definition of the stable-map topology,
\[ \partial \bar{U}_{T, \nu}(X; J) \cap U_{T', \nu}(X; J) \neq \emptyset \implies T' \prec T. \]

Thus, the closure requirement of Definition 3.5 follows from the continuity of the maps $D_i$ on $\tilde{X}_T(X)$ with $T$ as in Lemma 3.8.

3.5 The Structure of the Cone $\mathcal{V}_{1,k}^A$

In this subsection, we describe an obstruction-free subcone $W_{1,k}^A$ of the cone $\mathcal{V}_{1,k}^A \to M_{0,1,k}(X, A; J, \nu)$.

The cone $\mathcal{V}_{1,k}^A$ over $X_{1,k}$ can be shown to be regular by standard arguments; see Remark 2 at the end of Subsection 4.3. Thus, $W_{1,k}^A$ is regular as well.

If $T$ is a bubble type as in Lemma 3.8 and $[b] \in [\Sigma_b, u_b] \in \mathcal{U}_{T, \nu}(X; J)$, let
\[ F_b^1 = \{(w_i)_{i \in \chi(T)} \in C^X(b) : \sum_{i \in \chi(T)} w_i \cdot J D_i b = 0\}; \]
\[ \Gamma_-(b; \mathcal{L}) = \{ \xi \in \ker \partial_{\mathcal{B}, b} : \sum_{i \in \chi(T)} w_i D_{b,i} \xi = 0 \land (w_i)_{i \in \chi(T)} \in F_b^1\}; \]
\[ W_{1,k}^A|_{[b]} = \{ [(b, \xi)] \in \mathcal{V}_{1,k}^A|_{[b]} : \xi \in \Gamma_-(b; \mathcal{L})\}. \]

By (3.3), the subspace $W_{1,k}^A|_{[b]}$ of $\mathcal{V}_{1,k}^A|_{[b]}$ is well-defined. If $T$ is a bubble type as in Lemma 3.7 and $[b] \in \mathcal{U}_{T, \nu}(X; J)$, let
\[ W_{1,k}^A|_{[b]} = \mathcal{V}_{1,k}^A|_{[b]} |_{\mathcal{V}_{1,k}^A}^A. \]

We take
\[ W_{1,k}^A = \bigcup_{[b] \in \mathcal{M}_{1,k}^0(X, A; J, \nu)} W_{1,k}^A|_{[b]} \subset \mathcal{V}_{1,k}^A. \]

**Proposition 3.9** If $(X, \omega, J)$, $(\mathcal{L}, \nabla)$, $A$, $k$, and $\nu$ are as in Theorem 1.3, the cone $\mathcal{V}_{1,k}^A \to M_{0,1,k}(X, A; J, \nu)$ is regular and obstruction-free. If $T$ is a bubble type as in Lemma 3.7, then $W_{1,k}^A|_{\mathcal{U}_{T, \nu}(X; J)}$ is a smooth vector orbibundle and
\[ \text{rk} W_{1,k}^A|_{\mathcal{U}_{T, \nu}(X; J)} = \langle c_1(\mathcal{L}), A \rangle. \]
If $T$ is a bubble type as in Lemma 3.8 and $m$ is an integer as in (3.6), then $W^{A}_{1,k}|_{U^{m}_{T,\nu},1}(X;J)$ is a smooth vector orbibundle and

$$\text{rk} \, W^{A}_{1,k}|_{U^{m}_{T,\nu},1}(X;J) = \langle c_1(\mathcal{L}), A \rangle - (m-1) > \frac{1}{2} \dim U^{m}_{T,\nu},1(X;J) - \frac{1}{2} \dim_{1,k}(X, A; \mathcal{L}).$$

If $T$ is a bubble type as in Lemma 3.7 and $[b] \in U_{T,\nu}(X; J)$, the operator $\tilde{\partial}_{\mathcal{V},b}$ is surjective, by the positivity assumption on the bundle $\mathcal{L}$ and the same argument as in Subsection 6.2 of [Z2]. In particular,

$$\dim V^{A}_{1,k}|_{[b]} = \text{ind} \, \tilde{\partial}_{\mathcal{V},b} = \langle c_1(\mathcal{L}), A \rangle \quad \forall [b] \in U_{T,\nu}(X; J).$$

By standard arguments, the surjectivity of $\tilde{\partial}_{\mathcal{V},b}$ for every $[b] \in U_{T,\nu}(X; J)$ implies that

$$W^{A}_{1,k}|_{U_{T,\nu}(X; J)} = V^{A}_{1,k}|_{U_{T,\nu}(X; J)}$$

is a smooth vector bundle and that the restriction of $V^{A}_{1,k}$ to a neighborhood of $U_{T,\nu}(X; J)$ in $X_{1,k}(X, A)$ is a vector bundle. Local trivializations can be constructed using the homomorphisms $R_{\nu,\mathcal{L}}$ as in Subsection 4.2. In particular, the cone $W^{A}_{1,k}$ satisfies the requirements of (1) and (2) of Definition 3.3 for every $[b] \in U_{T,\nu}(X; J)$.

If $T$ is a bubble type as in Lemma 3.8 and $[b] \equiv [\Sigma_b; u_b] \in U_{T,\nu}(X; J)$, $u_b|_{\Sigma_b; p}$ is constant. Let

$$\Gamma_B(b; \mathcal{L}) = \{ \xi \in \Gamma(\Sigma_b; u_b^*\mathcal{L}) : \xi|_{\Sigma_b; p} = \text{const} \};$$

$$\Gamma^{0,1}_{B}(b; \mathcal{L}) = \{ \eta \in \Gamma(\Sigma_b; \Lambda^{0,1}_{i,j} T^*\Sigma_b \otimes u_b^*\mathcal{L}) : \eta|_{\Sigma_b; p} = 0 \}.$$

By the positivity assumption on the bundle $\mathcal{L}$ and the same argument as in Subsection 6.2 of [Z2], the operator

$$\tilde{\partial}_{\mathcal{V},b} : \Gamma_B(b; \mathcal{L}) \longrightarrow \Gamma^{0,1}_{B}(b; \mathcal{L})$$

induced by $\tilde{\partial}_{\mathcal{V},b}$ is surjective. In particular,

$$\dim V^{A}_{1,k}|_{[b]} = \dim \ker \, \tilde{\partial}_{\mathcal{V},b} = \dim \ker \, \tilde{\partial}_{\mathcal{V},b} = \text{ind} \, \tilde{\partial}_{\mathcal{V},b} = \langle c_1(\mathcal{L}), A \rangle + 1 \quad \forall [b] \in U_{T,\nu}(X; J).$$

Thus, $V^{A}_{1,k}|_{U_{T,\nu}(X; J)}$ is a smooth vector bundle. Similarly to Subsection 3.3 in [Z5], for every $m$ as in (3.6) we can construct a vector bundle $F^1 \longrightarrow U^{m}_{T,\nu},1(X; J)$ of rank $m$ and a surjective bundle homomorphism

$$\mathcal{D} : V^{A}_{1,k} \longrightarrow \text{Hom}(F^1, \text{ev}_p^*\mathcal{L})$$

over $U^{m}_{T,\nu},1(X; J)$ such that the kernel of $\mathcal{D}$ is $W^{A}_{1,k}|_{U^{m}_{T,\nu},1}(X; J)$. Thus, $W^{A}_{1,k}|_{U^{m}_{T,\nu},1}(X; J)$ is a smooth vector bundle of the claimed rank.

4 Proof of Proposition 3.9

4.1 Outline

In this section we prove a generalization of Proposition 3.9. It implies that the Poincare dual of the euler class of $V^{A}_{1,k}$ defined as the zero set of a generic section of $V^{A}_{1,k}$ over $F^{\text{dim}W^{A}_{1,k}}_{1,k}(X, A; J, \nu)$ is
independent of $J$ and $\nu$.

Suppose $(X, \omega)$ is a compact symplectic manifold, $(\mathcal{L}, \nabla)$ is an $(\omega, A)$-positive line bundle with connection over $X$, $A \in H_2(X; \mathbb{Z})^*$, $k \in \mathbb{Z}^+$, $J \equiv (J_t)_{t \in [0, 1]}$ is a continuous family of $\omega$-tamed almost complex structures on $X$, and

\[ \mathcal{L} \equiv (\nu_t)_{t \in [0, 1]} \in \mathcal{G}^e_{(X, A; J)} \]

is a family of sufficiently small perturbations of the $\bar{\partial}_{J_t}$-operators on $X$. Let $t_r$ and $[b_r]$ be sequences of elements in $[0, 1]$ and in $\mathcal{M}_{I, k}(X, A; J_t, J_r, \nu_r)$ such that

\[ \lim_{r \to \infty} t_r = 0 \quad \text{and} \quad \lim_{r \to \infty} [b_r] = [b] \in \mathcal{M}_{I, k}(X, A; J_0, \nu_0). \]

We need to show that for every $\xi, \nu \in \mathcal{W}_{I, k}\{0\}$ there exists a sequence $\xi_r \in \mathcal{W}_{I, k}\{b_r\}$ converging to $\xi$. By the paragraph following Proposition 3.9, it is sufficient to assume that $[b]$ is an element of $\mathcal{U}_{T, \nu_0}(X; J_0)$ for a bubble type

\[ T = ([k], I, R; j, A) \]

such that $A_i = 0$ for all minimal elements $i \in I$.

We can also assume that for some bubble type

\[ T' = ([k], I', \nu'; j', A') \]

$[b_r] \in \mathcal{U}_{T', \nu_r}(X; J_r)$ for all $r$. We note that by Definition 1.2, for every map $u: \mathbb{P}^1 \to X$ such that

\[ \langle \omega, u_*[\mathbb{P}^1] \rangle < \langle \omega, A \rangle \]

the linear operators

\[ \bar{\partial}_{\mathcal{G}, u}: \Gamma(\mathbb{P}^1; u^*\mathcal{L}) \to \Gamma(\mathbb{P}^1; \Lambda^{0,1}_{\mathcal{G}, u} T^*\mathbb{P}^1 \otimes u^*\mathcal{L}) \quad \text{and} \quad \ker \bar{\partial}_{\mathcal{G}, u} \to \mathcal{L}_{u(\infty)}, \quad \xi \to \xi(\infty), \]

are surjective. Thus, it is sufficient to consider two possibilities for $T'$:

1. $A_i' = 0$ for all $i \in I_0'$ and $\{i \in I': A_i' \neq 0\} = \chi(T')$;
2. $A_i' \neq 0$ for some $i \in I_0'$ and $I' = \emptyset$,

where $I_0'$ is the subset of minimal elements of $I'$. In the first case, for every $[b_r] \in \mathcal{U}_{T', \nu_r}(X; J_r)$, the map $u_{b_r}$ is constant on the principal component $\Sigma_{b_r, p}$ of $\Sigma_{b_r}$, and thus so is every element $\xi \in \ker \bar{\partial}_{\mathcal{G}, b_r}$. In this case, the question of existence of a sequence $\xi_r$ as above is an issue concerning the behavior of holomorphic bundle sections for genus-zero $(J, \nu)$-holomorphic maps, for a certain class of perturbations $\nu$ of the $\bar{\partial}_J$-operator. This class is induced from the class of effectively supported perturbations of Definition 2.1 and is described in Definition 4.3 at the end of this subsection. The existence of a desired sequence in case (1) follows from Lemma 4.1 below. In the second case, $\Sigma_{b_r, p} = \Sigma_{b_r}$ is either a smooth torus or a circle of spheres, depending on whether $\nu'$ is empty or not. There are no bubble components. In this case, the desired result follows from Lemma 4.2.

**Lemma 4.1** Suppose $(X, \omega)$ is a compact symplectic manifold, $(\mathcal{L}, \nabla)$ is an $(\omega, A)$-positive line bundle with connection over $X$, $A \in H_2(X; \mathbb{Z})^*$, $M$ is a finite set, $J \equiv (J_t)_{t \in [0, 1]}$ is a continuous family of $\omega$-tamed almost complex structures on $X$, and

\[ \mathcal{L} \equiv (\nu_t)_{t \in [0, 1]} \in \mathcal{G}^e_{0,\{0\} \cup M}(X, A; J) \]
is a family of sufficiently small perturbations of the $\tilde{\partial}_I$-operators on $X_{0,\{0\} \sqcup M}(X, A)$. Let $t_r$ and $[b_r]$ be sequences of elements in $[0, 1]$ and in $\mathfrak{M}^0_{0,\{0\} \sqcup M}(X, A; J_{t_r}, \nu_{t_r})$ such that

$$\lim_{r \to -\infty} t_r = 0 \quad \text{and} \quad \lim_{r \to -\infty} [b_r] = [b] \in \mathfrak{M}^0_{0,\{0\} \sqcup M}(X, A; J_0, \nu_0).$$

Then there exist $(c_{r,i})_{i \in \chi(b)} \in (\mathbb{C}^\times)^{\chi(b)}$, $\epsilon_r \in \mathbb{R}^+$, and isomorphisms

$$R_{b_r, b} : \ker \tilde{\partial}_{\mathcal{G}, b} \longrightarrow \ker \tilde{\partial}_{\mathcal{G}, b_r}$$

such that

$$\left| D_{0,b} - \sum_{i \in \chi(b)} c_{r,i} \cdot J_0 D_{i,b} \right| \leq \epsilon_r \sum_{i \in \chi(b)} |c_{r,i}| \quad \forall r \in \mathbb{Z}^+; \quad (4.1)$$

$$\left| D_{b_r, b \xi} - \sum_{i \in \chi(b)} c_{r,i} D_{b_r,i} \xi \right| \leq \epsilon_r \sum_{i \in \chi(b)} |c_{r,i}| \cdot \|\xi\| \quad \forall \xi \in \ker \tilde{\partial}_{\mathcal{G}, b_r}, r \in \mathbb{Z}^+; \quad (4.2)$$

$$\lim_{r \to -\infty} [R_{b_r, b} \xi] = [\xi] \in V^A_{0, \{0\}}[b] \quad \forall \xi \in \ker \tilde{\partial}_{\mathcal{G}, b}, \lim_{r \to -\infty} \epsilon_r = 0, \quad |c_{r,i}| \leq 1 \quad \forall i \in \chi(b). \quad (4.3)$$

**Lemma 4.2** Suppose $(X, \omega)$, $(\Sigma, \nabla)$, $J$, $A$, and $M$ are as in Lemma 4.1 and

$$\nu := (\nu_t)_{t \in [0, 1]} \in \mathfrak{S}^\nu_{1,M}(X, A; J)$$

is a family of sufficiently small perturbations of the $\tilde{\partial}_J$-operators on $X_{1,M}(X, A)$. Let $t_r$ and $[b_r]$ be sequences of elements in $[0, 1]$ and in $\mathfrak{M}^0_{1,M}(X, A; J_{t_r}, \nu_{t_r})$ such that

$$\Sigma_{b_r, P} = \Sigma_{b_r} \quad \forall r, \quad \lim_{r \to -\infty} t_r = 0, \quad \text{and} \quad \lim_{r \to -\infty} [b_r] = [b] \in \mathfrak{M}^0_{1,M}(X, A; J_0, \nu_0).$$

If $b = (\Sigma_{b}; u_b)$ is such that the degree of $u_b|\Sigma_{b, P}$ is zero, there exist $(w_i)_{i \in \chi(b)} \in (\mathbb{C}^\times)^{\chi(b)}$ such that

$$\sum_{i \in \chi(b)} w_i \cdot J_0 D_{i,b} = 0$$

and a subsequence of $\{b_r\}$, which we still denote by $\{b_r\}$, such that

$$\xi_r \in \ker \tilde{\partial}_{\mathcal{G}, b_r}, \quad \lim_{r \to -\infty} [\xi_r] = [\xi] \in V^A_{1,\{0\}}[b] \quad \Rightarrow \quad \sum_{i \in \chi(b)} w_i D_{b,i} \xi = 0.$$

Suppose $\{b_r\}$ and $b$ are as in Lemma 4.2, $\xi \in \tilde{\Gamma}_-(b; \Sigma)$, and there exists no sequence

$$\xi_r \in \ker \tilde{\partial}_{\mathcal{G}, b_r} \quad \text{s.t.} \quad \lim_{r \to -\infty} [\xi_r] = [\xi]. \quad (4.4)$$

After passing to a subsequence of $\{b_r\}$ if necessary, we can assume that

$$[\xi] \not\in \bigcup_{r=1}\infty V^A_{1,M}[b_r] \subset V^A_{1,M}. \quad (4.5)$$
Choose \( w \equiv (w_i)_{i \in \chi(b)} \in (\mathbb{C}^a)^{\chi(b)} \) and a subsequence of \( \{b_r\} \), which we still denote by \( \{b_r\} \), as at the end of Lemma 4.2. Let \( \{\xi_j^r\} \) be a basis for \( \ker \partial_{\Gamma,b_r} \) which is orthonormal, with respect to a regularization at \( b \) as in (1) of Definition 3.3 for example. After passing to a subsequence if necessary, for some linearly independent \( \xi_j^r \in \ker \partial_{\Gamma,b_r} \),

\[
\lim_{r \to \infty} [\xi_j^r] = [\xi_j] \in \mathcal{V}_{1,M}^A \|b\| \quad \forall \ j.
\]

By Lemma 4.2,

\[
\xi_j \in \Gamma_-(b; \mathcal{L}; [w]) \equiv \{ \xi \in \ker \partial_{\Gamma,b} : \sum_{i \in \chi(b)} w_i \mathcal{D}_{b,i} \xi = 0 \}.
\]

By the positivity assumption of Definition 1.2,

\[
\dim \ker \partial_{\Gamma,b_r} = \text{ind} \partial_{\Gamma,b_r} = \langle c_1(\mathcal{L}), A \rangle \quad \text{and}
\]

\[
\dim \Gamma_-(b; \mathcal{L}; [w]) = \dim \ker \partial_{\Gamma,b} - 1 = \text{ind} \partial_{\Gamma,b}^B_{\mathcal{L}} - 1 = \langle c_1(\mathcal{L}), A \rangle.
\]

Thus, \( \{\xi_j\} \) is a basis for \( \Gamma_-(b; \mathcal{L}; [w]) \). Since \( \tilde{\Gamma}_-(b; \mathcal{L}) \subset \Gamma_-(b; \mathcal{L}; [w]) \), there exists a sequence \( \xi_r \in \ker \partial_{\Gamma,b_r} \) as in (4.4), with \( \{b_r\} \) replaced by a subsequence. However, this contradicts (4.5).

An element \( [b_r] \in \mathcal{U}_{T',\nu_{T'}}(X; J_{r_r}) \), with \( T' \) as in the first case above, corresponds to the genus-one curve \( \Sigma_{b_r,0} = \Sigma_{b_r} \) and genus-zero maps \( \{b_r,h\}_{h \in \chi(T')} \) such that \( \Sigma_{b_r,h} = \mathbb{P}^1 \). If \( [b] \) is the limit of the sequence \( \{b_r\} \), \( b \) corresponds to a genus-one curve \( \Sigma_{b,0} \subset \Sigma_b \) and genus-zero maps \( \{b_h\}_{h \in \chi(T')} \) such that

\[
\lim_{r \to \infty} \Sigma_{b_r,0} = \Sigma_{b,0} \quad \text{and} \quad \lim_{r \to \infty} [b_r,h] = [b] \quad \forall \ h \in \chi(T').
\]

Similarly, \( \xi_r \in \ker \partial_{\Gamma,b_r} \) and \( \xi \in \ker \partial_{\Gamma,b_r} \) correspond to \( \xi_r,h \in \ker \partial_{\Gamma,b_r,h} \) and \( \xi \in \ker \partial_{\Gamma,b_h,n} \) with \( h \in \chi(T') \), such that

\[
\xi_r,h(g_0(b_r,h)) = \xi_r,h'(g_0(b_r,h')) \quad \text{and} \quad \xi_h(g_0(b_h)) = \xi_h'(g_0(b_h')) \quad \forall \ h, h' \in \chi(T').
\]

Furthermore,

\[
\lim_{r \to \infty} [\xi_j^r] = [\xi_j] \in \mathcal{V}_{1,k}^A \|b\| \quad \iff \quad \lim_{r \to \infty} [\xi_r,h] = [\xi_h] \in \mathcal{V}_{1,M}^A \|b\| \quad \forall \ h \in \chi(T'),
\]

where \( M_h \) is the index set for the marked points of \( b_{r,h} \) and \( b_h \). We will assume that \( [b_r] \in \mathcal{U}_{T',\nu_{T'}}(X; J_r) \) for some \( m' \in \mathbb{Z}^+ \) and for all \( r \).

With \( (c_{r,h})_{i \in \chi(b_h)} \in (\mathbb{C}^a)^{\chi(b_h)} \) for each \( h \in \chi(T') \) as in Lemma 4.1, let

\[
\alpha_r : \mathbb{C}^{\chi(T')} \longrightarrow \mathbb{C}^{\chi(b)} \equiv \prod_{h \in \chi(T')} \mathbb{C}^{\chi(b_h)}
\]

be the injective homomorphism defined by

\[
\alpha_r ((w_h)_{h \in \chi(T')}) = (c_{r,h} w_h)_{i \in \chi(b_h), h \in \chi(T')}.\]

We denote by \( F_{b_r}^1 \subset \mathbb{C}^{\chi(b)} \) the image of

\[
F_{b_r}^1 = \{ (w_h)_{h \in \chi(T')} \in \mathbb{C}^{\chi(T')} : \sum_{h \in \chi(T')} w_h \cdot J_h b_r = 0 \}.
\]
under $\alpha_r$. By our assumption on $b_r$, $\dim \tilde{F}^1_r = m'$ for all $r$. Let
\[
\{(v^l_{r,i})_{i \in \chi(b)}\}_{l \in [m']} \quad \text{and} \quad \{(\xi^j_r)_{h \in \chi(T')}\}
\]
be orthonormal bases for $\tilde{F}^1_r$ and for
\[
\tilde{\Gamma}_-(b_r; \mathcal{L}) \equiv \{\xi \equiv (\xi_h)_{h \in \chi(T')} \in \ker \tilde{\partial}_{\chi,b_r}: \sum_{h \in \chi(T')} w_h \partial D_{b,r,h} \xi_h = 0 \ \forall (w_h)_{h \in \chi(T')} \in F^1_{b_r}\},
\]
respectively. After passing to a subsequence if necessary, we can find
\[
w^l \equiv (w^l_i)_{i \in \chi(b)} \in \mathbb{C}^{\chi(b)} \quad \text{and} \quad \xi^j \equiv (\xi^j_h)_{h \in \chi(T')} \in \ker \tilde{\partial}_{\chi,b}
\]
such that
\[
\lim_{r \to -\infty} (v^l_{r,i})_{i \in \chi(b)} = w^l \in \mathbb{C}^{\chi(b)} \quad \forall \ l \quad \text{and} \quad \lim_{r \to -\infty} \xi^j_r = [\xi^j] \in V^A_{1,k}|_b \quad \forall \ j.
\]
Each of the sets $\{w^l\}$ and $\{\xi^j\}$ is orthonormal and thus linearly independent. By Lemma 4.1,
\[
\left| \sum_{i \in \chi(b)} v^l_{r,i} \cdot J_0 \partial_{b,i} \right| \equiv \left| \sum_{h \in \chi(T')} \sum_{i \in \chi(b)} v^l_{r,i} \cdot J_0 \partial_{b,h} \right| \leq \tilde{\varepsilon}_r \quad \forall \ l \quad \text{and} \quad \left| \sum_{i \in \chi(b)} v^l_{r,i} \partial_{b,i} \xi^j_r \right| \equiv \left| \sum_{h \in \chi(T')} \sum_{i \in \chi(b)} v^l_{r,i} \partial_{b,h} \xi^j_h \right| \leq \tilde{\varepsilon}_r \quad \forall \ j,
\]
for some sequence $\tilde{\varepsilon}_r$ converging to 0. Thus,
\[
w^l \in \tilde{F}^1_{b} \equiv \{(w_i)_{i \in \chi(b)} \in \mathbb{C}^{\chi(b)} : \sum_{i \in \chi(b)} w_i \cdot J_0 \partial_{b} = 0\} \quad \forall \ l \quad \text{and} \quad \xi^j \in \Gamma_-(b; \mathcal{L}; \{(w^l_{i})_{i \in [m']}\}) \equiv \{\xi \in \ker \tilde{\partial}_{\chi,b} : \sum_{i \in \chi(b)} w^l_i \partial_{b,i} \xi = 0 \ \forall \ l \in [m']\} \quad \forall \ j.
\]
By Definition 1.2,
\[
\dim \Gamma_-(b; \mathcal{L}; \{(w^l_{i})_{i \in [m']}\}) = \dim \ker \tilde{\partial}_{\chi,b} - m' = \dim \ker \tilde{\partial}_{\chi,b,r} - m' = \dim \tilde{\Gamma}_-(b_r; \mathcal{L}).
\]
Thus, $\{\xi^j\}$ is a basis for $\Gamma_-(b; \mathcal{L}; \{(w^l_{i})_{i \in [m']}\})$. Since $w^l \in \tilde{F}^1_{b}$ for all $l$,
\[
\tilde{\Gamma}_-(b; \mathcal{L}) \subset \Gamma_-(b; \mathcal{L}; \{(w^l_{i})_{i \in [m']}\}) \quad \text{and} \quad \mathcal{W}^A_{1,k}|_b \subset \bigcup_{r=1}^{\infty} V^A_{1,M}|_{b_r} \subset V^A_{1,M}.
\]
As in the first paragraph after Lemma 4.2, this implies that for every $\xi \in \tilde{\Gamma}_-(b; \mathcal{L})$ there exists a sequence $\xi_r \in \Gamma_-(b_r; \mathcal{L})$ such that
\[
\lim_{r \to -\infty} [\xi_r] = [\xi] \in V^A_{1,k}|_b.
\]
Remark: In (4.1) and (4.2), the differences are taken via a parallel transport along the shortest geodesic, with respect to a metric on $X$, between $\text{ev}_0(b_r)$ and $\text{ev}_0(h)$. 

Lemmas 4.1-4.2 are proved in the next two subsections by extending the gluing constructions of Subsections 2.4 and 2.5 in [Z6] from $J$-holomorphic maps to holomorphic bundle sections. These extensions parallel constructions in Subsections 4.2 and 4.3 in [Z5]. In the rest of this subsection we recall the definition of the type of perturbations $\nu$ of the $\bar{\partial}_J$-operator on space of genus-zero stable maps that appears in Lemma 4.1; see Subsection 2.1 in [Z6] for details.

Definition 4.3 Suppose $(X, \omega)$ is a compact symplectic manifold, $J \equiv (J_t)_{t \in [0,1]}$ is a continuous family of $\omega$-tamed almost structures on $X$, $A \in H_2(X; \mathbb{Z})^*$, and $M$ is a finite set. A continuous family of multisections $\nu \equiv (\nu_t)_{t \in [0,1]}$, with $\nu_t \in G_{0,\{0\} \sqcup M}(X, A; J_t)$ for all $t \in [0,1]$, is effectively supported if for every element $b$ of $X_0$, $\{0\} \sqcup M(X, A)$ there exists a neighborhood $W_b$ of $\Sigma_b$ in a semi-universal family of deformations for $b$ such that

$$
\nu_t(b')|_{\Sigma_b \cap W_b} = 0 \quad \forall \ [b'] \in X_{0,\{0\} \sqcup M}(X, A), \ t \in [0,1].
$$

We denote the space of effectively supported families $\nu$ as in Definition 4.3 by $G_{es,0,\{0\} \sqcup M}(X, A; J)$. If $\nu \in G_{es,0,\{0\} \sqcup M}(X, A; J)$, $t \in [0,1]$, $[b]$ is an element of

$$
\overline{\mathcal{M}}_{0,\{0\} \sqcup M}(X, A; J_t, \nu_t) \equiv \{\bar{\partial}_J + \nu_t\}^{-1}(0),
$$

and $i \in \chi(b)$, then $u_b|_{\Sigma_{b,i}}$ is $J_t$-holomorphic on a neighborhood of $\infty$ in $\Sigma_{b,i}$ and $\mathcal{C}_{J_t}\mathcal{D}_ib$ is determined by $b$, just as in Subsection 2.1. Furthermore, in this case $u_b|_{\Sigma^0_b}$ is a degree-zero holomorphic map and thus is constant. Thus, $u_b$ maps the attaching nodes of all elements of $\chi(b)$ to the same point in $X$, as in the genus-one case of Subsection 2.1.

4.2 Proof of Lemma 4.1

In this subsection we review the genus-zero gluing construction of Subsection 2.4 in [Z6] and extend it to holomorphic bundle sections in a manner similar to Subsection 4.2 in [Z5]. This construction essentially constitutes the first step of the two-step gluing construction described in Subsection 4.3. Throughout this subsection we assume that $M$ is a finite set, $A \in H_2(X; \mathbb{Z})$, and $T = (M, I; j, A)$ is a bubble such that $0$ is the minimal element of $I$,

$$
\sum_{i \in I} A_i = A \quad \text{and} \quad \langle \omega, A_i \rangle \geq 0 \quad \forall \ i \in I.
$$

Let $(\mathfrak{L}, \nabla) \longrightarrow X$ be an $(\omega, A)$-positive line bundle with connection.

We put

$$
\hat{\mathcal{X}}_{T,B}(X) = \{(\Sigma_b; u_b) \in \hat{\mathcal{X}}_T(X) : u_b|_{\Sigma^0_b} = \text{const}\}.
$$
We denote by
\[ \tilde{\mathcal{F}} \equiv \tilde{X}_{T,B}(X) \times \mathbb{C}^i \]
the bundle of smoothing parameters and by \( \tilde{\mathcal{F}}^0 \) the subset of \( \tilde{\mathcal{F}} \) consisting of the elements with all components nonzero. For each
\[ b \in \tilde{X}_{T,B}(X), \quad i \in \chi(T) \equiv \chi(b), \quad v \equiv (b, v) = (b, (v_h)_{h \in I}) \in \tilde{\mathcal{F}}, \]
we put
\[ \rho_i(v) = \prod_{\delta < h \leq i} v_h \in \mathbb{C} \quad \text{and} \quad x_i(v) = \sum_{\delta < h' \leq i} \left( x_{i'}(b) \prod_{\delta < h < h'} v_h \right) \in \mathbb{C}, \]
where \( x_i(b) \) is the point of \( \Sigma_{b, i} \) to which the bubble \( \Sigma_{b,i} \) is attached; see (3.2) and Figure 3.

For each sufficiently small element \( v = (b, v) \) of \( \tilde{\mathcal{F}}^0 \), let
\[ q_v: \Sigma_v \to \Sigma_b \]
be the basic gluing map constructed in Subsection 2.2 of [Z3]. In this case, \( \Sigma_v \) is the projective line \( \mathbb{P}^1 \) with \( |M|+1 \) marked points. The map \( q_v \) collapses \( |I| \) circles on \( \Sigma_v \). It induces a metric \( g_v \) on \( \Sigma_v \) such that \( (\Sigma_v, g_v) \) is obtained from \( \Sigma_b \) by replacing the \( |I| \) nodes of \( \Sigma_b \) by thin necks.

We put
\[ u_v = u_b \circ q_v, \quad b(v) = (\Sigma_v; u_v), \quad \text{and} \quad \bar{\partial}_v = \bar{\partial}_{v,b(v)}. \]
Fix a metric \( g \) on \( X \) and denote the corresponding Levi-Civita connection by \( \nabla^X \). By the same construction as in Subsection 3.3 of [Z3], the map \( q_v \) induces weighted \( L^p \)-norms \( \| \cdot \|_{v,p,1} \) on the spaces
\[ \{ \xi \in \Gamma(\Sigma_v; u_v^*TX): \xi(\infty) = 0 \} \quad \text{and} \quad \Gamma(\Sigma_v; u_v^*\mathcal{L}) \]
and a weighted \( L^p \)-norm \( \| \cdot \|_{v,p} \) on the space \( \Gamma(\Sigma_v; \Lambda_{y_j}^{0,1}T^*\Sigma_v \otimes u_v^*\mathcal{L}) \). We denote the corresponding completions by \( \Gamma(v), \Gamma(v; \mathcal{L}), \) and \( \Gamma^{0,1}(v; \mathcal{L}) \). The norms \( \| \cdot \|_{v,p,1} \) and \( \| \cdot \|_{v,p} \) are analogous to the ones used in Section 3 of [LT1] for the bundle \( TX \). We put
\[ \Gamma_-(v; \mathcal{L}) = \{ \xi \in \Gamma(v; \mathcal{L}): \xi(\infty) = 0 \} \quad \text{and} \quad \Gamma_+(v; \mathcal{L}) = \{ \xi \in \Gamma(v; \mathcal{L}): \xi(\infty) = 0 \} \]
and for all \( v = (b, v) \in \tilde{\mathcal{F}}^0 \),
\[ \| \bar{\partial}_v \xi \|_{v,p} \leq C(b)|v|^{1/p} \| \xi \|_{v,p,1} \quad \forall \xi \in \Gamma_-(v; \mathcal{L}). \]

By the construction of the map \( q_v \) in Subsection 2.2 of [Z3],
\[ \| \bar{\partial}_v \xi \|_{v,p} \leq C(b)|v|^{1/p} \| \xi \|_{v,p,1} \quad \forall \xi \in \Gamma_+(v; \mathcal{L}). \]

On the other hand, for the same reasons as in Section 3 of [LT1], for some \( \delta, C \in C(\tilde{X}_{T,B}(X); \mathbb{R}^+) \) and for all \( v = (b, v) \in \tilde{\mathcal{F}}^0 \),
\[ C(b)^{-1} \delta \| \xi \|_{v,p,1} \leq \| \bar{\partial}_v \xi \|_{v,p} \leq C(b) \xi \|_{v,p,1} \quad \forall \xi \in \Gamma_+(v; \mathcal{L}); \]
see Subsections 5.3 and 5.4 in [Z3]. In particular, the operators \( \bar{\partial}_v \) are surjective, since \( \bar{\partial}_v \) and \( \bar{\partial}_v(b) \) are.
If $\zeta \in \Gamma(v)$, we set
\[ b(v, \zeta) = (\Sigma_v; \exp_{u_v} \zeta) \quad \text{and} \quad u_{v, \zeta} = \exp_{u_v} \zeta, \]
where $\exp$ is the exponential map with respect to the connection $\nabla^X$. Let
\[ \Pi_{v, \zeta}: \Gamma(\Sigma_v; u_v^* \mathcal{L}) \longrightarrow \Gamma(\Sigma_v; u_{v, \zeta}^* \mathcal{L}) \]
be the isomorphism induced by the $\nabla$-parallel transport along the $\nabla^X$-geodesics $\tau \rightarrow \exp_\tau \zeta$ with $\tau \in [0, 1]$. By a direct computation,
\[ \|\Pi_{v, \zeta}^{-1} \partial_{\nabla, b(v, \zeta)} \Pi_{v, \zeta} \xi - \partial_{\nabla, v, \zeta} \xi\|_{v, p} \leq C(b) \|\zeta\|^2_{v, p, 1} \|\xi\|_{v, p, 1} \quad \forall \xi \in \Gamma(v; \mathcal{L}), \ z \in \Gamma(v); \tag{4.9} \]
see the proof of Corollary 2.3 in [Z1]. By (4.7)-(4.9), for every $\zeta \in \Gamma(v)$ sufficiently small and $\xi \in \ker \partial_{\nabla, b}$, there exists a unique
\[ \xi_{v, \zeta}(\xi) \in \Gamma_+(v; \mathcal{L}) \quad \text{s.t.} \quad \Pi_{v, \zeta}(\xi \circ q_v + \xi_{v, \zeta}(\xi)) \in \ker \partial_{\nabla, b(v, \zeta)}. \]
Furthermore,
\[ \|\xi_{v, \zeta}(\xi)\|_{v, p, 1} \leq C(b)(|\xi|^{1/p} + \|\xi\|^2_{v, p, 1}) \|\xi\|_{v, p, 1} \quad \forall \xi \in \ker \partial_{\nabla, b}. \tag{4.10} \]
We define the isomorphism
\[ R_{v, \zeta}: \ker \partial_{\nabla, b} \longrightarrow \ker \partial_{\nabla, b(v, \zeta)} \quad \text{by} \quad R_{v, \zeta} \xi = \Pi_{v, \zeta}(\xi \circ q_v + \xi_{v, \zeta}(\xi)). \]

We will use a convenient family of connections in the vector bundles $u^* \mathcal{L} \longrightarrow \Sigma$, which is provided by Lemma 4.4 below. First, if $b = (\Sigma_b; u_b)$ is a stable bubble map, $g_b$ is a Hermitian metric in the vector bundle $u_b^* \mathcal{L} \longrightarrow \Sigma_b$, and $\nabla^b$ is a connection in $u_b^* \mathcal{L}$, we will call the pair $(g, \nabla)$-admissible if $(g \nabla 1)$ $g_b$-compatible and $\partial_{\nabla, b}$-compatible;
$(g \nabla 2)$ $g_b = g_{ab}$ and $\nabla^b = \nabla^{u_b}$ on $\Sigma_b^0$,
where $g_{ab}$ is the Hermitian metric in $u_b^* \mathcal{L}$ induced from the standard metric in $\mathcal{L}$. The second condition in $(g \nabla 1)$ means that
\[ \partial_{\nabla, b} = \frac{1}{2}(\nabla^{u_b} + i\nabla^{u_b} \circ j) = \frac{1}{2}(\nabla^b + i\nabla^b \circ j), \]
where $\nabla^{u_b}$ is as in Subsection 1.2.

If $b$ is any genus-zero bubble map and $\delta \in \mathbb{R}^+$, we put
\[ \Sigma^0_b(\delta) = \Sigma^0_b \cup \bigcup_{i \in \chi(b)} A_{b, i}(\delta), \quad \text{where} \quad A_{b, i}(\delta) = \{(i, z): |z| \geq \delta^{-1/2}/2\} \subset \Sigma_{b, i} \approx S^2. \tag{4.11} \]
If $v = (b, v)$ is a gluing parameter such that the map $q_v$ is defined, let
\[ \Sigma^0_v(\delta) = q_v^{-1}(\Sigma^0_b(\delta)). \tag{4.12} \]
Lemma 4.4 If \((X, \omega), (\mathcal{L}, \nabla), A, M, T, \tilde{X}_{T,B}(X), \tilde{F}\) are as above, there exist
\[
\delta, C \in C(\tilde{X}_{T,B}(X); \mathbb{R}^+) \]
with the following property. For every
\[
b \in \tilde{X}_{T,B}(X), \quad v = (b, v) \in \tilde{F}_0^q, \quad \text{and} \quad \zeta \in \Gamma(v) \ \text{s.t.} \quad \|\zeta\|_{v, p, 1} \leq \delta(b) \]
there exist metrics \(g_b\) and \(g_{(v, \zeta)}\) and connections \(\nabla^b\) and \(\nabla^{(v, \zeta)}\) in the vector bundles
\[
u_b^* \mathcal{L} \rightarrow \Sigma_b \quad \text{and} \quad 
u_{v, \zeta}^* \mathcal{L} \rightarrow \Sigma_v \]
such that
1. all pairs \((g_b, \nabla^b)\) and \((g_{(v, \zeta)}, \nabla^{(v, \zeta)})\) are admissible;
2. the curvatures of \(\nabla^b\) and \(\nabla^{(v, \zeta)}\) vanish on \(\Sigma_0^0(2\delta)\) and \(\Sigma_0^0(2\delta)\), respectively;
3. \(\|\Pi_{v, \zeta}\nabla^{(v, \zeta)}(\Pi_{v, \zeta} \xi \circ b_v) - \nabla^b \xi \circ d_q v \|_{v, p} \leq C\|v\|^{1/p} + \|\xi\|_{v, p, 1}\|\xi\|_{b, p, 1}\) for all \(\xi \in \Gamma(b; \Sigma)\);
4. the map \(b \mapsto (g_b, \nabla^b)\) is continuous.

This lemma is proved by exactly the same argument as Lemma 4.1 in \([Z5]\).

If \(b \in \tilde{X}_{T,B}(X)\) and \(i \in \chi(b)\), let \(w_i\) be the standard holomorphic coordinate centered at the point \(\infty\) in \(\Sigma_{b,i} = S^2\). If \(m \in \mathbb{Z}^+\) and \(\xi = (\xi_i)_{i \in I} \in \Gamma(b; \mathcal{L})\), we put
\[
\mathcal{D}^{(m)}_{b,i} \xi = \frac{1}{m!} \frac{d^m}{dw_i^m} \xi_i(w_i)\bigg|_{w_i = 0} \in \mathcal{L}_{evo(b)},
\]
where the derivatives are taken with respect to the connection \(\nabla^b\). Similarly, for all
\[
v = (b, v) \in \tilde{F}_0^q, \quad \zeta \in \Gamma(v) \ \text{s.t.} \quad \|\zeta\|_{v, p, 1} \leq \delta(b), \quad \text{and} \quad \xi \in \Gamma(\Sigma_v; \nu_{v, \zeta}^* \mathcal{L}),
\]
let
\[
\mathcal{D}^{(m)}_{(v, \zeta), 0} \xi = \frac{1}{m!} \frac{d^m}{dw^m} \xi(w)\bigg|_{w = 0} \in \mathcal{L}_{evo(b)},
\]
where \(w\) is the standard holomorphic coordinate centered at the point \(\infty\) in \(\Sigma_v \approx S^2\) and the derivatives are taken with to the connection \(\nabla^{(v, \zeta)}\). We note that
\[
\mathcal{D}^{(1)}_{b,i} \xi = \mathcal{D}_{b,i} \xi \quad \forall \ i \in \chi(b), \ \xi \in \Gamma(b; \mathcal{L}), \quad \text{and} \quad \mathcal{D}^{(1)}_{(v, \zeta), 0} \delta \xi = \mathcal{D}_{(v, \zeta), 0} \delta \xi \quad \forall \ \xi \in \Gamma(\Sigma_v; \nu_{v, \zeta}^* \mathcal{L}), \quad (4.13)
\]
by the second condition in \((g \nabla 2)\) above.

A key step in understanding the obstruction to extending holomorphic bundle sections from singular to smooth domains is the following power series expansion. For every
\[
i \in \chi(b), \quad k \in \mathbb{Z}^+, \quad v = (b, v) \in \tilde{F}_0^q, \quad \text{and} \quad \zeta \in \Gamma(v),
\]
there exist
\[
\varepsilon_i^{(k)}(v, \zeta) \in \text{Hom}(\ker \partial \mathcal{D}_{b,i}, \mathcal{L}_{evo(b)})
\]
by the second condition in \((g \nabla 2)\) above.
For the purposes of Lemma 4.1, we take 
\[ \epsilon \] to be the composition of
\[
\mathcal{D}_{b(v,\zeta),0}^{(1)} v, \zeta \equiv v_{h_1}(\mathcal{D}_{b,h_1}^{(1)} \zeta) + v_{h_3} v_{h_4} (\mathcal{D}_{b,h_4}^{(1)} \zeta) + v_{h_5} v_{h_5} (\mathcal{D}_{b,h_5}^{(1)} \zeta)
\]

such that for all \( \zeta \in \ker \tilde{\partial}_{\nabla, b} \)
\[
\mathcal{D}_{b(v,\zeta),0}^{(m)} v, \zeta = \sum_{k=1}^{\infty} \sum_{i \in \chi(b)} x_i^m \rho_i(v) (\mathcal{D}_{b_i}^{(k)} \zeta + \epsilon_i^{(k)}(v, \zeta) \zeta)
\]

and
\[
|\epsilon_i^{(k)}(v, \zeta)| \leq C \delta^{-k/2} (|v|^{1/p} + \|\zeta\|_{v,p,1}) \|\zeta\|_{b,p,1}.
\]

The expansion (4.14) is obtained by exactly the same integration-by-parts argument as the expansion in Theorem 2.8 of [Z3]; see also the paragraph following Lemma 4.2 in [Z5]. We point out that \( \epsilon_i^{(k)} \) is independent of \( m \). The \( m = 1 \) case of the estimates (4.14) and (4.15) is illustrated in Figure 3.

Let \( t_r \) and \( b_r \) be as in Lemma 4.1. Since the sequence \([b_r] \) converges to \([b] \), for all \( r \) sufficiently large there exist
\[
b'_r \in \tilde{X}_{T;B}(X), \quad v_r = (b'_r, v_r) \in \tilde{F}_0, \quad \text{and} \quad \zeta_r \in \Gamma(v_r)
\]
such that
\[
\lim_{r \to \infty} b'_r = b, \quad \lim_{r \to \infty} |v_r| = 0, \quad \lim_{r \to \infty} \|\zeta_r\|_{v_r,p,1} = 0,
\]
and
\[
b_r \equiv (\Sigma_{b_r}; u_{b_r}) = (\Sigma_{v_r}; \exp_{u_{v_r}} \zeta_r).
\]

The last equality holds for a representative \( b_r \) for \([b_r] \). By the estimate (2.12) in [Z6] and (4.16),
\[
|\mathcal{D}_0 b_r - \sum_{i \in \chi(b)} \rho_i(v_r) (\mathcal{D}_i b'_r)| \leq \epsilon_r \sum_{i \in \chi(b)} |\rho_i(v_r)|
\]
for a subsequence \( \epsilon_r \) converging to 0. Furthermore, by the \( m = 1 \) case of (4.14) and (4.15),
\[
|\mathcal{D}_{b,0} R_{v_r, \zeta_r, \xi} - \sum_{i \in \chi(b)} \rho_i(v_r) \mathcal{D}_{b'_r,i} \xi| \leq \epsilon_r \sum_{i \in \chi(b)} |\rho_i(v_r)| \cdot \|\xi\|_{b,p,1} \quad \forall \xi \in \ker \tilde{\partial}_{\nabla, b'_r}.
\]

For the purposes of Lemma 4.1, we take
\[
R_{b_r, b} : \ker \tilde{\partial}_{\nabla, b} \longrightarrow \ker \tilde{\partial}_{\nabla, b_r}
\]
to be the composition of
\[
R_{v_r, \zeta_r} : \ker \tilde{\partial}_{\nabla, b'_r} \longrightarrow \ker \tilde{\partial}_{\nabla, b_r}
\]
with an isomorphism
\[ R_{b',b} : \ker \bar{\partial}_{b'} \longrightarrow \ker \bar{\partial}_{b}, \quad \text{s.t.} \quad \lim_{r \to \infty} R_{b',b} \xi = \xi \quad \forall \xi \in \ker \bar{\partial}_{b}. \]

We take \( c_{r,i} = \rho_i(\nu_r) \). It is immediate that the requirements (4.3) of Lemma 4.1 are satisfied. Since
\[ \lim_{r \to \infty} D_{b'} D_{b'} = D_b \quad \text{and} \quad \lim_{r \to \infty} D_{b',i} R_{b',b} \xi = D_{b,i} \xi \quad \forall \xi \in \ker \bar{\partial}_{b}, i \in \chi(b), \]
the requirements (4.1) and (4.2) are satisfied as well. This concludes the proof of Lemma 4.1.

**Remark:** A regularization \( \varphi_b \) of the cone \( \mathcal{V}_{0,M}^A \) near \( b \), as in Definition 3.3, can be constructed using the isomorphisms \( R_{\nu,\zeta} \) as above and a description of open subsets in \( \mathcal{X}_{0,M}(X,A) \) as in Section 3 of [LT1]. In this case, \( \varphi_b \) is a homeomorphism.

### 4.3 Proof of Lemma 4.2

In this subsection, we review the two-step gluing construction used in Subsection 2.5 of [Z6] and extend it to holomorphic bundle sections in a way similar to Subsection 4.2 in [Z5]. We assume that
\[ T = (M, I, \mathbb{R}; j, A) \]
is a bubble type such that
\[ \sum_{i \in I} A_i = A, \quad \langle \omega, A_i \rangle \geq 0 \quad \forall i \in I, \quad A_i = 0 \quad \forall i \in I_0, \quad \text{and} \quad I_0 \subsetneq I, \]
where \( I_0 \) is the subset of minimal elements of \( I \) as before. Let \( (\mathfrak{E}, \nabla) \longrightarrow X \) be an \( (\omega, A) \)-positive vector bundle with connection. Throughout this subsection we focus on the case
\[ [b_r] \in \mathfrak{M}_{1,M}^0(X, A; J, \nu), \]
i.e. \( \Sigma_{b_r} = \Sigma_b \) is a smooth torus for all \( r \).

Similarly to Subsection 4.2, we put
\[ \mathfrak{X}_{T,B}(X) = \{ (\Sigma_b; u_b) \in \mathfrak{X}_T(X) : u_b|_{\Sigma_b} = \text{const} \}. \]

Let
\[ I_1 = \{ h \in \tilde{I} : \nu_h \in I_0 \} \quad \text{and} \quad k_0 = |I_1| + |\{ l \in M : j_l \in I_0 \}|. \]

We denote by \( \mathcal{M}_{1,k_0} \) the moduli space of genus-one curves with \( k_0 \) marked points and by
\[ \pi_P : \mathfrak{X}_{T,B}(X) \longrightarrow \mathcal{M}_{1,k_0}, \quad b \longrightarrow [\Sigma_{b,P}], \]
\[ \text{ev}_P : \mathfrak{X}_{T,B}(X) \longrightarrow X, \quad b \longrightarrow u_b(\Sigma_{b,P}), \]
the maps sending each element \( b \) of \( \mathfrak{X}_{T,B}(X) \) to the equivalence class of the principal component(s) \( \Sigma_{b,P} \) of its domain and to the image of \( \Sigma_{b,P} \) in \( X \). Let
\[ E \longrightarrow \mathcal{M}_{1,k_0} \]
be the Hodge line bundle, i.e. the line bundle of holomorphic differentials.

Let

$$\tilde{F} \longrightarrow \tilde{X}_{T,B}(X)$$

be the bundle of gluing parameters. It has three distinguished components:

$$\tilde{F} = \tilde{F}_R \oplus \tilde{F}_0 \oplus \tilde{F}_1,$$

where

$$\tilde{F}_R = \tilde{X}_{T,B}(X) \times \mathbb{C}^N, \quad \tilde{F}_1 = \tilde{X}_{T,B}(X) \times \mathbb{C}^{l-I_1},$$

$$\tilde{F}_0 = \bigoplus_{h \in I_1} \tilde{F}_h,$$

and

$$\tilde{F}_h|_b = T_{x_h(b)} \Sigma_{b,P} \quad \forall \ b \in \tilde{X}_{T,B}(X).$$

The total space of \( \tilde{F}_h \) has a natural topology; see Subsection 2.2 in [Z4]. We denote by \( \tilde{F}^0 \) the subset of \( \tilde{F} \) consisting of the elements with all components nonzero. If \( i \in I \), let \( h(i) \in I_1 \) be the unique element such that \( h(i) \leq i \). For each \( v = (b, v) \), where \( b \in \tilde{X}_{T,B}(X) \) and \( v = (v_i)_{i \in \mathbb{N} \cup I} \), and \( i \in \chi(b) \), we put

$$v_0 = (b, (v_i)_{i \in \mathbb{N} \cup I_1}), \quad v_1 = (b, (v_i)_{i \in I_1}),$$

$$\tilde{\rho}_i(v) = \prod_{h(i) < h \leq i} v_h \in \mathbb{C}, \quad \rho_i(v) = \tilde{\rho}_i(v) \cdot v_{h(i)} \in T_{x_h(b)} \Sigma_{b,P},$$

and

$$\rho(v) = (b, (\rho_i(v))_{i \in \chi(b)}) \in \tilde{F} \equiv \bigoplus_{i \in \chi(b)} F_{h(i)}.$$

The component \( v_1 \) of \( v \) consists of the smoothings of the nodes of \( \Sigma_b \) that lie away from the principal component. In the case of Figure 4, these are the attaching nodes of the bubbles \( h_2, h_4, \) and \( h_5 \).

For each element \( \tilde{v} = (b, (\tilde{v}_i)_{i \in \chi(b)}) \) of \( \tilde{F} \), we define the linear map

$$\mathcal{D}_b : \ker \tilde{\nabla}_b \longrightarrow \mathbb{E}^*_{\pi_{\Sigma}(b)} \otimes \mathcal{L}_{\pi_{\Sigma}(b)}$$

by

$$\{ \mathcal{D}_b \xi \}(\psi) = \sum_{i \in \chi(b)} \psi_{x_{h_i}(b)}(\tilde{v}_i) \cdot \mathcal{D}_b \xi,$$

if \( \psi \in \mathbb{E}^*_{\pi_{\Sigma}(b)} \).

For each sufficiently small element \( v = (b, v) \) of \( \tilde{F}^0 \), let

$$q_{v_1} : \Sigma_{v_1} \longrightarrow \Sigma_b$$

be the basic gluing map constructed in Subsection 2.2 of [Z3]. In this case, the principal component \( \Sigma_{v_1;P} \) of \( \Sigma_{v_1} \) is the same as principal component \( \Sigma_{b;P} \) of \( \Sigma_b \), and \( \Sigma_{v_1} \) has \( |I_1| \) bubble components.
\( \Sigma_{v_1,h} \), with \( h \in I_1 \), attached directly to \( \Sigma_{v_1,P} \). The map \( q_{v_1} \) collapses \( |\tilde{I} - I_1| \) circles on the bubble components of \( \Sigma_{v_1} \). It induces a metric \( g_{v_1} \) on \( \Sigma_{v_1} \) such that \( (\Sigma_{v_1}, g_{v_1}) \) is obtained from \( \Sigma_b \) by replacing \( |\tilde{I} - I_1| \) nodes by thin necks. Let

\[
u_{v_1} = u_b \circ q_{v_1}, \quad b(v_1) = (\Sigma_{v_1}; u_{v_1}), \quad \text{and} \quad \bar{\partial}_{\nu,v_1} = \bar{\partial}_{\nu,b(v_1)}.
\]

The map \( q_{v_1} \) induces weighted \( L^p \)-norms \( \| \cdot \|_{v_1,p,1} \) on the spaces

\[
\{ \zeta \in \Gamma(\Sigma_{v_1}; u_{v_1}^* TX): \zeta|_{\Sigma_{v_1,P}} = 0 \} \quad \text{and} \quad \{ \xi \in \Gamma(\Sigma_{v_1}; u_{v_1}^* \xi): \xi|_{\Sigma_{v_1,P}} = 0 \}
\]

and a weighted \( L^p \)-norm \( \| \cdot \|_{v_1,p} \) on

\[
\{ \eta \in \Gamma(\Sigma_{v_1}; \Lambda_{i,j}^{0,1} T^* \Sigma_{v_1} \otimes u_{v_1}^* \xi); \eta|_{\Sigma_{v_1,P}} = 0 \};
\]

see Subsection 3.3 of [Z3] and the first remark in Subsection 2.5 in [Z6]. We denote the corresponding completions by \( \Gamma_B(v_1), \Gamma_B(v_1; \xi), \text{ and } \Gamma_B(v_1; \xi) \).

For each \( \zeta \in \Gamma_B(v_1) \), let

\[
u_{v_1,\zeta} = \exp_{u_{v_1}} \zeta \quad \text{and} \quad b(v_1, \zeta) = (\Sigma_{v_1}; u_{v_1,\zeta}).
\]

For \( \delta \in \mathcal{C}(\tilde{X}_{T,B}(X); \mathbb{R}^+) \) sufficiently small,

\[
u \equiv (b, v) \in \tilde{X}_\delta, \quad \text{and} \quad \zeta \in \Gamma_B(v_1) \quad \text{s.t.} \quad \| \zeta \|_{v_1,p,1} \leq \delta(b),
\]

the isomorphisms \( R_{i,\zeta} \) of Subsection 4.2 corresponding to the restriction of \( b(v_1, \zeta) \) to \( \Sigma_{v_1,h} \), with \( h \in I_1 \), induce an isomorphism

\[
R_{v_1,\zeta}: \ker \bar{\partial}_{\nu,b} \longrightarrow \ker \bar{\partial}_{\nu,b(v_1,\zeta)}
\]

such that

\[
\| R_{v_1,\zeta} \xi \|_{v_1,p,1} \leq 2 \| \xi \|_{b,p,1}.
\]

Furthermore, by the \( m=1 \) case of (4.14) and (4.15),

\[
\left| \mathcal{D}_{b(v_1,\zeta),h} R_{v_1,\zeta} \xi - \sum_{i \in \chi(h), h(i)=h} \tilde{\rho}_{i}(v) \mathcal{D}_{b,h} \xi \right| \leq C(b) \left( \| v \|^{1/p} + \| \xi \|_{v_1,p,1} \right) \sum_{i \in \chi(h), h(i)=h} |\tilde{\rho}_{i}(v)| \cdot \| \xi \|_{b,p,1}, \quad (4.19)
\]

for all \( h \in I_1 \) and \( \xi \in \ker \bar{\partial}_{\nu,b} \). Let \( \nabla_{v_1,\zeta} \) and \( g_{v_1,\zeta} \) be the connection and metric in the line bundle \( u_{v_1,\zeta}^* \xi \) induced by the connections and metrics of Lemma 4.4. For each \( h \in I_1 \) and \( \delta \in \mathbb{R}^+ \), let

\[
A_{v_1,h}(\delta) = q_{v}^{-1}(A_{b,h}(\delta)).
\]

From the estimates (4.14) and (4.15), we find that

\[
\| \nabla_{v_1,\zeta} R_{v_1,\zeta} \xi \|_{C^0(A_{v_1,h}(\delta(h)))} \leq C(b) \left( \| v \|^{1/p} + \| \xi \|_{v_1,p,1} \right) \sum_{i \in \chi(h), h(i)=h} |\tilde{\rho}_{i}(v)| \cdot \| \xi \|_{b,p,1}, \quad (4.20)
\]

for all \( h \in I_1 \) and \( \xi \in \ker \bar{\partial}_{\nu,b} \).
Fix a smooth function \( \epsilon : \widetilde{X}_{T,B}(X) \longrightarrow \mathbb{R}^+ \) such that for every \( b \in \widetilde{X}_{T,B}(X) \) and \( h \in I_1 \) the disk of radius of \( 8\epsilon_b \) in \( \Sigma_{b,P} \) around the node \( x_h(b) \) contains no other special, i.e. singular or marked, point of \( \Sigma_b \). For each

\[
\nu \equiv (b, \nu) \equiv (b, (\nu_h)_{h \in \mathbb{N} \cup I}) \in \widetilde{F}^0
\]
sufficiently small, let

\[
q_{\nu_0;2} : \Sigma_{\nu} \longrightarrow \Sigma_{\nu_1} \quad \text{and} \quad \tilde{q}_{\nu_0;2} : \Sigma_{\nu} \longrightarrow \Sigma_{\nu_1}
\]
be the basic gluing map of Subsection 2.2 in [Z3] corresponding to the gluing parameter \( \nu_0 \) and the modified basic gluing map defined in the middle of Subsection 4.2 in [Z4] with the collapsing radius \( \epsilon_b \). In this case, \( \Sigma_{\nu} \) is a smooth genus-one curve. For each \( h \in I_1 \), the maps \( q_{\nu_0;2} \) and \( \tilde{q}_{\nu_0;2} \) collapse the circles of radii \( |\nu_h|^{1/2} \) and \( \epsilon_b \), respectively, around the point \( x_h(b) \in \Sigma_{v;1,P} \). As before, the map

\[
q_0 \equiv q_{\nu_0;2} \circ q_{\nu_1} : \Sigma_{\nu} \longrightarrow \Sigma_b
\]
induces a metric \( g_0 \) on \( \Sigma_{\nu} \) such that \( (\Sigma_{\nu}, g_0) \) is obtained from \( \Sigma_b \) by replacing all nodes by thin necks. Let

\[
I_1^* = I_1 - \{ h \in I_1 : A_i = 0 \ \forall \ i \geq h \}.
\]
The map \( \tilde{q}_{\nu_0;2} \) is biholomorphic outside \( \mathcal{N} \) thin necks \( A_{v,h} \), with \( h \in \mathcal{N} \), of \( (\Sigma_{\nu}, g_0) \) and the \( |I_1^*| \) annuli

\[
\tilde{A}_{b,h} \equiv \tilde{A}_{b,h}^+ \cup \tilde{A}_{b,h}^-
\]
with \( h \in I_1 \), where

\[
\tilde{A}_{b,h}^\pm = \tilde{A}_{b,h}^\pm(\delta(b)) \subset \Sigma_{b,P} \approx \Sigma_{\nu}
\]
are annuli independent of \( v \). In addition,

\[
u_{v_1,\zeta} \big|_{\tilde{q}_{\nu_0;2}(A_{b,h})} = \text{const} \quad \forall h \in \mathbb{N},
\]

\[
u_{v_1,\zeta} \big|_{\tilde{q}_{\nu_0;2}(A_{b,h})} = \text{const} \quad \forall h \in I_1 - I_1^* \quad \text{and} \quad \nu_{v_1,\zeta} \big|_{\tilde{q}_{\nu_0;2}(\tilde{A}_{b,h}^\pm)} = \text{const} \quad \forall h \in I_1^*;
\]

\[
\tilde{q}_{\nu_0;2} \big( \tilde{A}_{b,h}^- \big) \subset A_{v_1,h} \big( |\nu_h|^2/\delta(b) \big) \quad \text{and} \quad \|d\tilde{q}_{\nu_0;2}\|_{C^0(\tilde{A}_{b,h}^-)} \leq C(b)|\nu_h| \quad \forall h \in I_1^*
\]

(4.22)

if the \( C^0 \)-norm of \( d\tilde{q}_{\nu_0;2} \) is computed with respect to the metrics \( g_0 \) on \( \Sigma_{\nu} \) and \( g_{v_1} \) on \( \Sigma_{v_1} \). Furthermore,

\[
\|d\tilde{q}_{\nu_0;2}\|_{C^0} \leq C(b).
\]

(4.23)

If \( v = (b, \nu) \in \widetilde{F}^0 \) is sufficiently small and \( \zeta \in \Gamma_B(v_1) \), we put

\[
u_{v,\zeta} = \nu_{v_1,\zeta} \circ \tilde{q}_{\nu_0;2} \quad \text{and} \quad \nu(b, \zeta) = (\Sigma_{v}; u_{v,\zeta}).
\]
The map \( q_{\nu_0;2} \) induces weighted \( L^1 \)-norms \( \| \cdot \|_{v,p,1} \) on the spaces

\[
\Gamma(\Sigma_{v}; u_{v,\zeta}^* TX) \quad \text{and} \quad \Gamma(\Sigma_{v}; u_{v,\zeta}^* \mathcal{L})
\]
and a weighted \( L^p \)-norm \( \| \cdot \|_{v,p} \) on the space

\[
\Gamma(\Sigma_{v}; \Lambda_{i,j}^{0,1} \Sigma_{v} \otimes u_{v,\zeta}^* TX).
\]
Let $\Gamma(v, \zeta), \Gamma(v, \zeta; \mathcal{L})$, and $\Gamma^{0,1}(v, \zeta; \mathcal{L})$ be the corresponding completions. We put

$$
\Gamma^{-}(v, \zeta; \mathcal{L}) = \{ R_{v,\zeta} \xi \equiv R_{v_1,\zeta} \xi \circ \tilde{q}_{v_0} : \xi \in \ker \partial_{\nabla, b} \} \subset \Gamma(v, \zeta; \mathcal{L}).
$$

By (4.20)-(4.23), for all $\xi \in \ker \partial_{\nabla, b}$,

$$
\| \partial_{\nabla, b(v, \zeta)} R_{v,\zeta} \xi \|_{v, p} \leq C(b) \sum_{h \in I_1} |v_h| \sum_{i \in \chi(h), h(i) = h} |\tilde{p}_i(v)| \cdot \| \xi \|_{b, p, 1} = C(b) |\rho(v)| \cdot \| \xi \|_{b, p, 1}.
$$

Let $\Gamma^{0,1}(v, \zeta; \mathcal{L})$ be the $L^2$-orthogonal complement of $\Gamma^{-}(v, \zeta; \mathcal{L})$ in $\Gamma(v, \zeta; \mathcal{L})$. We denote by

$$
\pi_{v,\zeta, -} : \Gamma(v, \zeta; \mathcal{L}) \longrightarrow \Gamma^{-}(v, \zeta; \mathcal{L})
$$

the $L^2$-projection map. For the same reasons as before,

$$
C(b)^{-1} \| \xi \|_{v, p, 1} \leq \| \partial_{\nabla, b(v, \zeta)} \xi \|_{v, p} \leq C(b) \| \xi \|_{v, p, 1} \quad \forall \xi \in \Gamma^{0,1}(v, \zeta; \mathcal{L}),
$$

if $v = (b, v) \in \mathcal{F}^0$ and $\zeta \in \Gamma(v_1)$ are sufficiently small. Let $\Gamma^{0,1}_{+}(v, \zeta; \mathcal{L})$ be the image of $\Gamma^{0,1}(v, \zeta; \mathcal{L})$ under $\partial_{\nabla, b(v, \zeta)}$.

The operator $\partial_{\nabla, b(v, \zeta)}$ is not surjective. We next describe its cokernel. Since the operator $\partial_{\nabla, b}^{0,1}$ is surjective, the cokernel of $\partial_{\nabla, b}$ can be identified with the vector space

$$
\mathcal{H}_{b, P} \otimes \mathcal{L}_{ev, P}(b) \approx \mathbb{E}_{\pi_{b, P}} \otimes \mathcal{L}_{ev, P}(b),
$$

where $\mathcal{H}_{b, P}$ is the space of harmonic antilinear differentials on the main component $\Sigma_{b, P}$ of $\Sigma_b$. As in Subsection 4.3 in [Z5], there exist isomorphisms

$$
R_{v, P}^{0,1} : \mathcal{H}_{b, P} \longrightarrow \mathcal{H}_{v, P} \equiv \mathcal{H}_{b(v, \zeta); P}, \quad v = (b, v) \in \tilde{\mathcal{F}},
$$

such that the family of induced homomorphisms

$$
\mathcal{H}_{b, P} \longrightarrow \Gamma^{0,1}(v; \mathbb{C})^*, \quad \{ R_{v, P}^{0,1} \eta \}(\eta') = \langle R_{v, P}^{0,1} \eta, \eta' \rangle_2 \quad \forall \eta \in \mathcal{H}_{b, P}, \eta' \in \Gamma^{0,1}(v; \mathbb{C}),
$$

is continuous on $\tilde{\mathcal{F}}$, and

$$
R_{v, P}^{0,1}|_b = \text{id} \quad \forall b \in \tilde{\mathcal{F}}_{T, B}(X).
$$

Let $\beta : \mathbb{R}^+ \longrightarrow [0, 1]$ be a smooth function such that

$$
\beta(t) \in \begin{cases} 
0, & \text{if } t \leq 1;
1, & \text{if } t \geq 2.
\end{cases}
$$

If $r \in \mathbb{R}^+$, let $\beta_r(t) = \beta(t/\sqrt{r})$. We define $\beta_b \in C^\infty(\Sigma_b; \mathbb{R})$ by

$$
\beta_b(z) = \begin{cases} 
1, & \text{if } z \in \Sigma_{b, i}, i \in \chi^0(b); \\
1 - \beta_{\delta(b)}(r(z)), & \text{if } z \in \Sigma_{b, i}, i \in \chi(b); \\
0, & \text{otherwise},
\end{cases}
$$

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where \( r(z) = |g^{-1}(z)| \) if \( q_8 : \mathbb{C} \to S^2 \) is the stereographic projection mapping the origin to the south pole of \( S^2 \). In other words, \( \beta_0 = 1 \) on \( \Sigma_0^0(\delta(b)/2) \) and vanishes outside of \( \Sigma_0^0(2\delta(b)) \subset \Sigma_b \). Let \( \beta_0 = \beta_b \circ q_v \). If \( z \in \Sigma_0^0(2\delta(b)) \), we denote by \( \Pi_{\nu,\zeta}^{(v)} \) the parallel transport in the line bundle \( u_{v,\zeta}^* \mathfrak{L} \) along a path from \( x \in \tilde{q}_{v,0}^{-1}(\Sigma_{v_1,P}) \) to \( z \in \Sigma_0^0(2\delta(b)) \) with respect to the connection \( \tilde{q}_{v,0}^* \nabla^{(v,\zeta)} \). For each 
\[
v = (b, v) \in J_3^0 \quad \text{and} \quad \eta \in \Gamma_+^0(b; \mathfrak{L}),
\]
let \( R_{v,\zeta}^0 \eta \in \Gamma_+^0(v, \zeta; \mathfrak{L}) \) be given by 
\[
\{ R_{v,\zeta}^0 \eta \}_+ w = \beta_v(z) \Pi_{\nu,\zeta}^{(v)} \eta_{\nu}(w) \in \mathfrak{L}_{\nu,\zeta}(z) \quad z \in \Sigma_v, \ w \in T_z \Sigma_v.
\]
Since the curvature of \( \tilde{q}_{v,0}^* \nabla^{(v,\zeta)} \) vanishes over \( \Sigma_0^0(2\delta(b)) \), \( \{ R_{v,\zeta}^0 \eta \}_+ w \) is independent of the choice of \( x \) and path from \( x \) to \( z \) above.

If \( \eta \in \Gamma_+^0(b; \mathfrak{L}) \), we put 
\[
|| \eta || = \sum_{k \in I^1} || \eta ||_{x_k(b)}
\]
where \( || \eta ||_{x_k(b)} \) is the norm of \( \eta \) with respect to the metric \( g_{\pi P}(b) \) on \( \Sigma_{b,P} \). If \( v \) and \( \eta \) are as in (4.27) and \( || \eta || = 1 \), we define by 
\[
\pi_{v,\zeta}^{(0,1)} : \Gamma_+^0(v, \zeta; \mathfrak{L}) \to \Gamma_+^0(b; \mathfrak{L}) \quad \text{by} \quad \pi_{v,\zeta}^{(0,1)}(\eta') = \langle \eta', R_{v,\zeta}^0 \eta \rangle_2 \eta' \forall \eta' \in \Gamma_+^0(v, \zeta; \mathfrak{L}).
\]
Since the space \( \Gamma_+^0(b; \mathfrak{L}) \) is one-dimensional, \( \pi_{v,\zeta}^{(0,1)} \) is independent of the choice of \( \eta \). We note that since \( p > 2 \), by Holder’s inequality 
\[
|| \pi_{v,\zeta}^{(0,1)} \eta' || \leq C(b)|| \eta' ||_{v,p} \forall \eta' \in \Gamma_+^0(v, \zeta; \mathfrak{L}).
\]
Furthermore, by the proof of Lemma 2.2 in [Z3], 
\[
|| \pi_{v,\zeta}^{(0,1)} \partial_{\nabla_b(v,\zeta)} \xi || \leq C(b)|| \rho(v)||_{v,p,1} || \xi ||_{v,p} \forall \xi \in \Gamma(v, \zeta; \mathfrak{L}).
\]
With the same restriction on the homomorphisms \( R_{v,P}^{0,1} \) and identification of gluing parameters as described in Subsection 4.2 of [Z4], we also have 
\[
\pi_{v,\zeta}^{(0,1)} \partial_{\nabla_b(v,\zeta)} R_{v,\zeta} \xi = -2\pi i \partial_{\rho(v)} \xi \forall \xi \in \ker \partial_{\nabla_b},
\]
by the proof of Proposition 4.4 in [Z3].

For each \( v = (b, v) \in J_3^0 \), \( \zeta \in \Gamma_B(v_1) \), and \( \zeta' \in \Gamma(v, \zeta) \), we put 
\[
u_{v,\zeta,\zeta'} = \exp_{u_\nu,\zeta} \zeta' \quad \text{and} \quad b(v, \zeta, \zeta') = (\Sigma_v; u_{v,\zeta,\zeta'}).
\]
We denote by \( \Pi_{\nu} \) the isomorphisms 
\[
\Gamma(\Sigma_v; U_{v,\zeta}^*) \to \Gamma(\Sigma_v; U_{v,\zeta,\zeta'}^*) \quad \text{and} \quad \Gamma(\Sigma_v; A_{\nu,j}^{(0,1)} T^* \Sigma_v \otimes u_{v,\zeta}^*) \to \Gamma(\Sigma_v; A_{\nu,j}^{(0,1)} T^* \Sigma_v \otimes u_{v,\zeta,\zeta'}^*)
\]
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induced by the $\nabla$-parallel transport along the geodesics $\tau \mapsto \exp_{u,\xi} \tau \zeta'$ with $\tau \in [0, 1]$. Let

$$L_{v,\zeta,\zeta'} = \Pi^{-1}_{\zeta'} \partial_{\nabla_{b(v,\zeta,\zeta')}} \Pi_{\zeta'} - \partial_{\nabla_{b(v,\zeta)}'; \Gamma(v,\zeta; \mathcal{L})} \to \Gamma^{0,1}(v,\zeta; \mathcal{L}).$$

Similarly to (4.9),

$$\|L_{v,\zeta,\zeta'} \xi\|_{v,p} \leq C(b)\|\zeta\|^2_{v,p,1}\|\xi\|_{v,p,1} \quad \forall \xi \in \Gamma(v,\zeta; \mathcal{L}). \tag{4.31}$$

Let $J$ be an almost complex structure on $X$. With notation as at the beginning of this subsection and in Subsection 3.2, we define the linear bundle map

$$D_{J,T} : \tilde{\mathcal{E}} \to \pi_p^* E^* \otimes \Theta(TX, J)$$

over $\tilde{\mathfrak{X}}_{T,B}(X)$ by

$$\{D_{J,T}(b, (\tilde{v}_i)_{i \in \chi(b)})\}(\psi) = \sum_{i \in \chi(b)} \psi_{x_{b(i)}(b)}(\tilde{v}_i) \cdot J D_i b \in T_{v,p,1}(b) X.$$

Suppose $t_r$ and $b_r$ are as in Lemma 4.2 and $\Sigma_{b_r} : = \Sigma_{b_r}$ is a smooth torus. Since the sequence $[b_r]$ converges to $[b]$, by Subsection 2.5 in [Z6] there exist $C \in \mathbb{R}^+$ and for all $r$ sufficiently large

$$b'_r \in \tilde{\mathfrak{X}}_{T,B}(X), \quad v_r = (b'_r, v_r) \in \tilde{\mathfrak{X}}_d, \quad \zeta_r \in \Gamma(v_r), \quad \zeta'_r \in \Gamma(v_r, \zeta_r), \quad \text{and} \quad \epsilon_r \in \mathbb{R}^+ \tag{4.32}$$

such that

$$\lim_{r \to \infty} b'_r = b, \quad \lim_{r \to \infty} |v_r| = 0, \quad \lim_{r \to \infty} \|\zeta'_r\|_{v_r,1,p,1} = 0, \quad \|\zeta'_r\|_{v_r,1,p,1} \leq C|\rho(v)|, \tag{4.33}$$

$$\lim_{r \to \infty} \epsilon_r = 0, \quad |D_{b_0;T} \rho(v)| \leq \epsilon_r |\rho(v)|, \tag{4.34}$$

and

$$b_r \equiv (\Sigma_{b_r} ; u_{b_r}) = (\Sigma_{v_r}, \exp_{u_{v_r,\zeta'}}, \zeta'_r).$$

The last equality holds for a representative $b_r$ for $[b_r]$.

By (4.31) and the last inequality in (4.33), for some $C \in \mathbb{R}^+$ and for all $r$ sufficiently large

$$\|L_{v_r,\zeta_r,\zeta'_r} \xi\|_{v_r,p} \leq C|\rho(v_r)| \xi\|_{v_r,p,1} \quad \forall \xi \in \Gamma(v_r, \zeta_r; \mathcal{L}). \tag{4.35}$$

Thus, by (4.24) and (4.25),

$$\|\pi_{v_r,\zeta_r,\zeta'_r}^{-1} \xi - \Pi^{-1}_{\zeta'_r} \xi\|_{v_r,p,1} \leq C|\rho(v_r)| \xi\|_{v_r,p,1} \quad \forall \xi \in \ker \partial_{\nabla_{b_r}}. \tag{4.36}$$

Since

$$\{\partial_{\nabla_{b(v,\zeta_r)}}, + L_{v_r,\zeta_r,\zeta'_r}\} \Pi^{-1}_{\zeta'_r} \xi = 0 \quad \forall \xi \in \ker \partial_{\nabla_{b_r}},$$

by (4.28)-(4.30), (4.35), and (4.36),

$$|D_{\rho(v)}\xi| \leq C|\rho(v_r)|^2 \cdot \xi\|_{b'_r,p,1} \quad \forall R_{v_r,\zeta_r} \xi \in \pi_{v_r,\zeta_r,\zeta'_r}^{-1}\ker \partial_{\nabla_{b_r}}. \tag{4.37}$$
After passing to a subsequence if necessary, let
\[
\bar{u}' \equiv (u'_i)_{i \in \chi(b)} \in \widetilde{\delta}_b - \{0\}
\]
be such that
\[
\lim_{r \to \infty} \left[ \rho(u_r) \right] = [\bar{u}'] \in \mathbb{P}\widetilde{\delta}_b.
\]
Since \(D_i b_r \longrightarrow D_i b\) for all \(i \in \chi(b)\),
\[
\left\{ D_{b_r} \tau \left( b, (w'_i)_{i \in \chi(b)} \right) \right\}(\psi) = \sum_{i \in \chi(b)} \psi_{x_h(i)(b)}(w'_i) \cdot D_{b_i} b = 0 \in T_{ev_P(b)}X \quad \forall \psi \in \mathbb{E}_{\pi_P(b)},
\]
by (4.34). If \(\xi_r \in \ker \partial_{\psi, b_r}\) and \(\xi \in \ker \partial_{\psi, b}\) are such that
\[
\lim_{r \to \infty} [\xi_r] = [\xi] \in \mathcal{V}_{1,M}^A[b],
\]
then by (4.36) and (4.37),
\[
|\{ \mathcal{D}_w \xi \} \psi| \equiv \left| \sum_{i \in \chi(b)} \psi_{x_h(i)(b)}(w'_i) \cdot D_{b_i} \xi \right| \leq \hat{c}_r |\bar{u}'| \|\psi\| \quad \forall \psi \in \mathbb{E}_{\pi_P(b)}(4.39)
\]
for a sequence \(\hat{c}_r\) converging to zero. Thus, by (4.38) and (4.39), for the purposes of Lemma 4.2 we can take
\[
w_i = \psi_{x_h(i)(b)}(w'_i) \in \mathbb{C}^*.
\]
where \(\psi \in \mathbb{E}_{\pi_P(b)}(b)\) is any nonzero element.

Remark 1: If \(\Sigma_{b_r,P} = \Sigma_{b_c}\) is a circle of spheres, i.e. \(N' \neq \emptyset\) in the notation of Subsection 4.1, the proof of Lemma 4.2 is formally the same, but some details change in a way analogous to Subsection 3.9 in [Z3]. In particular, in (4.32),
\[
\forall \in \widetilde{\mathcal{F}^0} \equiv \left\{ (b, (v_h)_{h \in \mathcal{N}_{\mathcal{E}}}) \in \widetilde{\mathcal{F}} : v_h = 0 \iff h \in \mathcal{N}_0 \right\},
\]
for a nonempty subset \(\mathcal{N}_0\) of \(\mathcal{N}\). If \(v \in \widetilde{\mathcal{F}^0}, \Sigma_v\) is a circle of spheres with nodes \(\mathcal{N}_0\). If in addition \(\zeta \in \Gamma(v_1), \Gamma(v, \zeta)\) consists of the vector fields on the \(|\mathcal{N}_0|\) components of \(\Sigma_v\) that agree at the nodes of \(\Sigma_v\). Similarly, \(\Gamma(v, \zeta; \mathcal{E})\) consists of the sections of \(u_{v, \zeta}^* \mathcal{E}\) over the components of \(\Sigma_v\) that agree at the nodes. If \(\eta \in \Gamma_{0,1}^0(b, \mathcal{E}), \) the \(u^*_{v, \zeta} \mathcal{E}\)-valued \((0,1)\)-form \(R_{v, \zeta, \eta}^0\) has poles at the nodes of \(\Sigma_v\) with residues that add up to zero at each node. In particular, \(R_{v, \zeta, \eta}^0\) is not an element of \(\Gamma^0_{0,1}(v, \zeta; \mathcal{E})\), but the homomorphism \(\pi_{v, \zeta, -}^0\) is well defined and still satisfies (4.28)-(4.30). Finally, the argument of Subsection 2.5 in [Z6] easily generalizes to show that \((v_r, \zeta_r, \zeta'_r, \epsilon_r)\) as in (4.32)-(4.34) exist in this situation.

Remark 2: A regularization \(\varphi_b\) of the cone \(\mathcal{V}_{1,M}^A\) near \(b\), as in Definition 3.3, can be constructed using the description of open subsets in \(\mathcal{X}_{1,M}(X, A)\) of Section 3 in [LT1] and the corresponding analogues of the isomorphisms \(R_{v, \zeta}\) and the injective homomorphisms \(\pi_{v, \zeta, -}^0\) as above.
References


