Some Comments on my Paper

A Sharp Compactness Theorem for Genus-One Pseudo-Holomorphic Maps

Jingchen Niu (my student) has pointed out that the justification of the two estimates in (4.8),

$$\big\|\mathrm{d}u_\upsilon\big\|_{\upsilon,p}\le C(b)\big\|\mathrm{d}\widetilde{u}_{\upsilon_1}\big\|_{\upsilon_1,p} \quad\text{and}\quad \big\|\bar\partial_{\widetilde{J}}u_\upsilon\big\|_{\upsilon,p}\le C(b)\sum_{h\in I_1}\big\|\mathrm{d}\widetilde{u}_{\upsilon_1}|_{\mathcal{A}^-_{\upsilon,h}}\big\|_{\upsilon_1,p}|\upsilon_h|^{\frac{p-2}{p}}, \tag{4.8}$$

requires more care than implied. The reasoning in [3, Subsection 3.3], which I cite, does not apply directly because the map $\widetilde{u}_{\upsilon_1}$ now depends on the smoothing parameter υ_1. While my reasoning behind these two inequalities 12 years might indeed have been off, both inequalities are correct and not difficult to justify. This is done below.

In (4.8),

$$\widetilde{u}_{\upsilon_1}\colon\Sigma_{\upsilon_1}\longrightarrow X,\qquad u_\upsilon=\widetilde{u}_{\upsilon_1}\circ\widetilde{q}_{\upsilon_0;2}\colon\Sigma_\upsilon\longrightarrow X,\qquad\text{and}\qquad \widetilde{q}_{\upsilon_0;2}\colon\Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_1}$$

are smooth maps, $(X,\omega,\widetilde{J})$ is a compact symplectic manifold with a tame almost complex structure, Σ_{υ_1} and Σ_υ are compact nodal Riemann surfaces endowed with compatible metrics g_{υ_1} and g_υ, respectively, and $p>2$. The norms $\|\cdot\|_{\upsilon_1,p}$ and $\|\cdot\|_{\upsilon,p}$ on

$$\Gamma\big(\Sigma_{\upsilon_1};T^*\Sigma_{\upsilon_1}\otimes_{\mathbb{R}}\widetilde{u}^*_{\upsilon_1}TX\big)\qquad\text{and}\qquad \Gamma\big(\Sigma_\upsilon;T^*\Sigma_\upsilon\otimes_{\mathbb{R}}u^*_\upsilon TX\big)$$

are the modified Sobolev norms of [1, Section 3]. They are the sums of the usual L^p-Sobolev norms with respect to the metrics g_{υ_1} and g_υ and of weighted L^2-norms with respect to these metrics.

The map $\widetilde{u}_{\upsilon_1}$ is $\widetilde{J}$-holomorphic. The map $\widetilde{q}_{\upsilon_0;2}$ is a holomorphic isometry and commutes with the L^2-weights outside of certain annuli $\widetilde{\mathcal{A}}_{b,h}$ with $h\in\aleph$ and $\widetilde{\mathcal{A}}^\pm_{b,h}$ with $h\in I_1$, where $\aleph$ is the set of the nodes of the principal component $\Sigma_{\upsilon_1;P}$ of Σ_{υ_1} and I_1 is the set of the nodes of Σ_{υ_1} shared between $\Sigma_{\upsilon_1;P}$ and other components of Σ_{υ_1} (there is an inconsequential miswording of this statement below (4.6)). Since the map $\widetilde{u}_{\upsilon_1}$ is constant on $\widetilde{q}_{\upsilon_0;2}(\widetilde{\mathcal{A}}_{b,h})$ and $\widetilde{q}_{\upsilon_0;2}(\widetilde{\mathcal{A}}^+_{b,h})$,

$$\big\|\mathrm{d}u_\upsilon\big\|_{\upsilon,p}\le\big\|\mathrm{d}\widetilde{u}_{\upsilon_1}\big\|_{\upsilon_1,p}+\sum_{h\in I_1}\big\|\mathrm{d}u_\upsilon|_{\widetilde{\mathcal{A}}^-_{b,h}}\big\|_{\upsilon,p},\quad \big\|\bar\partial_{\widetilde{J}}u_\upsilon\big\|_{\upsilon,p}\le\sum_{h\in I_1}\big\|\mathrm{d}u_\upsilon|_{\widetilde{\mathcal{A}}^-_{b,h}}\big\|_{\upsilon,p}. \tag{1}$$

For each $h\in I_1$, $\widetilde{\mathcal{A}}^-_{b,h}\subset\Sigma_{\upsilon_1;P}$ is the annulus of radii $\sqrt{\delta_K}/2$ and $\sqrt{\delta_K}$ centered at a nodal point $x_h(\upsilon_\aleph)$ of Σ_{υ_1} and

$$\mathcal{A}^-_{\upsilon,h}\equiv\widetilde{q}_{\upsilon_0;2}(\widetilde{\mathcal{A}}^-_{b,h})\subset\Sigma_{\upsilon_1;h}$$

is the disk of radius $2|\upsilon_h|\le2\delta_K$ centered at the same nodal point $\infty\in\Sigma_{\upsilon_1;h}$ of the other component sharing this node. Let

$$\mathcal{A}^-_{\upsilon,h;0},\mathcal{A}^-_{\upsilon,h;1}\subset\mathcal{A}^-_{\upsilon,h}$$

denote the disk of radius $2|\upsilon_h|$ with the same center and its complement.

By the construction of $\widetilde{q}_{\upsilon_0;2}$, there exists $C\in\mathbb{R}^+$ such that

$$\big|\mathrm{d}_z\widetilde{q}_{\upsilon_0;2}\big|_{g_{\upsilon_1},g_\upsilon}\le C|\upsilon_h|\ \ \forall\,z\in\widetilde{\mathcal{A}}^-_{b,h},\quad \big|\mathrm{Jac}(\mathrm{d}_z\widetilde{q}_{\upsilon_0;2})\big|_{g_{\upsilon_1},g_\upsilon}\ge C^{-p}|\upsilon_h|^2\ \ \forall\,z\in\widetilde{q}^{-1}_{\upsilon_0;2}\big(\mathcal{A}^-_{\upsilon,h;1}\big). \tag{2}$$

The constant C does not depend on the smoothing parameter υ, but does depend on the choice of the cutoff function β and the number δ_K used in constructing $\widetilde{q}_{\upsilon_0;2}$. By (2),

$$\begin{aligned}\bigg(\int_{\widetilde{q}_{\upsilon_0;2}^{-1}(\mathcal{A}^-_{\upsilon,h;1})}\big|\mathrm{d}u_\upsilon\big|^p_{g_\upsilon}\bigg)^{\frac{1}{p}} &= \bigg(\int_{\widetilde{q}_{\upsilon_0;2}^{-1}(\mathcal{A}^-_{\upsilon,h;1})}\big|\mathrm{d}_{\widetilde{q}_{\upsilon_0;2}}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\big|\mathrm{d}\widetilde{q}_{\upsilon_0;2}\big|^p_{g_{\upsilon_1},g_\upsilon}\bigg)^{\frac{1}{p}}\\ &\le C|\upsilon_h|\bigg(\int_{\mathcal{A}^-_{\upsilon,h;1}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\big|\mathrm{Jac}(\mathrm{d}_{\widetilde{q}_{\upsilon_0;2}^{-1}}\widetilde{q}_{\upsilon_0;2})\big|^{-1}\bigg)^{\frac{1}{p}} \le C^2|\upsilon_h|^{\frac{p-2}{p}}\bigg(\int_{\mathcal{A}^-_{\upsilon,h;1}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\bigg)^{\frac{1}{p}}.\end{aligned} \tag{3}$$

By the Mean Value Inequality [2, Lemma 4.3.1(1)] and Hölder's Inequality, there exists $C_X\in\mathbb{R}^+$ such that

$$\big|\mathrm{d}_z\widetilde{u}_{\upsilon_1}\big|_{g_{\upsilon_1}} \le \frac{C_X}{|\upsilon_h|}\bigg(\int_{\mathcal{A}^-_{\upsilon,h}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^2_{g_{\upsilon_1}}\bigg)^{\frac{1}{2}} \le C_X\pi^{\frac{p-2}{2p}}|\upsilon_h|^{-\frac{2}{p}}\bigg(\int_{\mathcal{A}^-_{\upsilon,h}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\bigg)^{\frac{1}{p}} \quad \forall\, z\in\mathcal{A}_{\upsilon,h;0} \tag{4}$$

By (2) and (4),

$$\begin{aligned}\bigg(\int_{\widetilde{q}_{\upsilon_0;2}^{-1}(\mathcal{A}^-_{\upsilon,h;0})}\big|\mathrm{d}u_\upsilon\big|^p_{g_\upsilon}\bigg)^{\frac{1}{p}} &= \bigg(\int_{\widetilde{q}_{\upsilon_0;2}^{-1}(\mathcal{A}^-_{\upsilon,h;0})}\big|\mathrm{d}_{\widetilde{q}_{\upsilon_0;2}}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\big|\mathrm{d}\widetilde{q}_{\upsilon_0;2}\big|^p_{g_{\upsilon_1},g_\upsilon}\bigg)^{\frac{1}{p}}\\ &\le C'|\upsilon_h|^{\frac{p-2}{p}}\bigg(\int_{\mathcal{A}^-_{\upsilon,h}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\bigg)^{\frac{1}{p}}\bigg(\int_{\widetilde{q}_{\upsilon_0;2}^{-1}(\mathcal{A}^-_{\upsilon,h;0})}1\bigg)^{\frac{1}{p}} \le C''|\upsilon_h|^{\frac{p-2}{p}}\bigg(\int_{\mathcal{A}^-_{\upsilon,h}}\big|\mathrm{d}\widetilde{u}_{\upsilon_1}\big|^p_{g_{\upsilon_1}}\bigg)^{\frac{1}{p}}.\end{aligned} \tag{5}$$

Since the L^2-weights of [1, Section 3] are bounded on the annuli $\widetilde{\mathcal{A}}^-_{b,h}$ independently of υ, (3) and (5) give

$$\big\|\mathrm{d}u_\upsilon|_{\widetilde{\mathcal{A}}^-_{b,h}}\big\|_{\upsilon,p} \le C|\upsilon_h|^{\frac{p-2}{p}}\big\|\mathrm{d}\widetilde{u}_{\upsilon_1}|_{\mathcal{A}^-_{\upsilon,h}}\big\|_{\upsilon_1,p} \qquad \forall\; h\in I_1\,.$$

Combined with (1), this establishes (4.8).

Aleksey, May 10, 2016

References

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