

Enumerative Geometry of Calabi-Yau 5-Folds

Appendix

R. Pandharipande and A. Zinger

11 February 2008

A Explicit Formulas for the Local Example

A.1 Recursions

The recursive description of the number $n_1(d)$ is simplified below from the original one using Lemmas 1 and 2 below.¹ The main statement of interest is Conjecture 1, which implies that $n_1(d) = S(d)V(d)$ and thus confirms the BPS integrality prediction for the local Calabi-Yau 5-fold $3\mathcal{O}(-1) \rightarrow \mathbb{P}^2$.

For each d , denote by $\mathcal{P}(d)$ the set of *odd* prime numbers dividing d and by $\ell(d)$ the exponent of the largest power of 2 dividing d . If $d \geq 1$, define

$$A_1(d) = \frac{1}{4} \prod_{x \in \mathcal{P}(d)} (1-x^2) \times \begin{cases} 4, & \text{if } d=1; \\ 1, & \text{if } \ell(d)=0, d \neq 1; \\ -4, & \text{if } \ell(d)=1; \\ 0, & \text{if } \ell(d) \geq 2; \end{cases} \quad (\text{A.1})$$

$$A_2(d) = -\frac{3d}{2} \prod_{x \in \mathcal{P}(d)} (1-x) \cdot \begin{cases} 0, & \text{if } d=1; \\ -1, & \text{if } d=2; \\ 1, & \text{if } \ell(d)=0, d \neq 1; \\ -\frac{3}{2}, & \text{if } \ell(d)=1, d \neq 2; \\ -\frac{1}{2}, & \text{if } \ell(d) \geq 2. \end{cases} \quad (\text{A.2})$$

For $d_1, d_2 \geq 1$, let

$$B_1(d_1, d_2) = -3d_2 \prod_{x \in \mathcal{P}(\langle d_1, d_2 \rangle)} (1-x) \cdot \begin{cases} 1, & \text{if } \ell(\langle d_1, d_2 \rangle) = 0; \\ -\frac{3}{2}, & \text{if } \ell(\langle d_1, d_2 \rangle) = 1; \\ -\frac{1}{2}, & \text{if } \ell(\langle d_1, d_2 \rangle) \geq 2. \end{cases} \quad (\text{A.3})$$

¹The original recursions for $A_2(d) \equiv n_d(\tilde{\psi}^2, H^2)$ and $B_1(d_1, d_2) \equiv n_{d_1 d_2}(|; H^2)$ do not involve any of the other terms. Using Lemma 1, the formulas (A.2) and (A.3) can be shown to satisfy the two recursions. The original formula for $A_1(d) \equiv n_d(\tilde{\psi} H^2)$ expresses $A_1(d)$ in terms of $B_1(d_1, d_2)$. The formula (A.1) follows from (A.3) and Lemma 2. Originally, there are also the terms $n_d(\tilde{\psi} H, H^2)$, but it is not hard to see that

$$n_d(\tilde{\psi} H, H^2) = d n_d(\tilde{\psi} H^2) \quad \forall d \geq 3;$$

so we are able to eliminate these terms.

The following is the initial data for the remaining A -terms:

$$\begin{aligned} A_3(1) &= -3, & A_4(1) &= 6, & A_5(1) &= 6; \\ A_3(2) &= 0, & A_4(2) &= 3, & A_5(2) &= -72. \end{aligned}$$

The following equations apply to $d \geq 3$ (the last two hold for all d); the sums are taken over $d_1, d_2 \geq 1$:

$$\begin{aligned} d^2 A_3(d) &= -d A_1(d) + \sum_{d_1+d_2=d} d_1^2 B_4(d_1, d_2); \\ d^2 A_4(d) &= A_2(d) - 2d A_3(d) + \sum_{d_1+d_2=d} d_1^2 (B_3(d_1, d_2) + B_5(d_1, d_2)); \\ A_5(d) &= d \left(-\frac{9}{2} A_1(d) - 6 A_2(d) + \frac{1}{2} A_4(d) - \sum_{d_1+d_2=d} \left(\frac{5}{2} B_3(d_1, d_2) + 2B_5(d_1, d_2) \right) \right) \\ &\quad + A_3(d) - \sum_{d_1+d_2=d} \left(\frac{1}{2} B_2(d_1, d_2) + 2B_4(d_1, d_2) \right). \end{aligned}$$

The following equations apply to $d_1, d_2 \geq 1$; the sums are taken over $d, d' \geq 1$:

$$\begin{aligned} B_2(d_1, d_2) &= - \begin{cases} B_4(d_2 - d_1, d_1) + \sum_{d < d_1, d_2} d C(d_1 - d, d, d_2 - d) \\ \quad + d_1 \left(B_6(d_2 - d_1, d_1) + \frac{1}{2} \sum_{d+d'=d_2-d_1} C(d, d_1, d') \right), & \text{if } d_2 > d_1; \\ A_5(d_1) + \sum_{d < d_1, d_2} d C(d_1 - d, d, d_2 - d), & \text{if } d_1 = d_2; \\ -B_2(d_2, d_1), & \text{if } d_2 < d_1; \end{cases} \\ d_2^2 B_3(d_1, d_2) &= B_1(d_1, d_2) - 2d_2 B_2(d_1, d_2) + \sum_{d+d'=d_2} d^2 C(d_1, d', d); \\ d_2 B_4(d_1, d_2) &= d_2 B_2(d_1, d_2) - B_1(d_1, d_2) + \sum_{d+d'=d_2} d d' C(d_1, d', d); \\ B_5(d_1, d_2) &= - \sum_{d+d'=d_2} C(d_1, d', d); \\ B_6(d_1, d_2) &= -3B_1(d_1, d_2) + 2B_5(d_1, d_2) + B_3(d_1, d_2) + B_3(d_2, d_1). \end{aligned}$$

The following formulas apply to $d_1, d_2, d_3 \geq 1$:

$$\begin{aligned} \mathfrak{C}^{(1)}(d_1, d_2, d_3) &= \begin{cases} C(d_3 - d_1, d_1, d_2), & \text{if } d_3 > d_1; \\ C(d_1 - d_3, d_3, d_2), & \text{if } d_3 < d_1; \\ B_6(d_2, d_1), & \text{if } d_3 = d_1; \end{cases} \\ \mathfrak{C}^{(2)}(d_1, d_2, d_3) &= \begin{cases} C(d_1, d_2, d_3 - d_2), & \text{if } d_3 > d_2; \\ C(d_1, d_3, d_2 - d_3) + C(d_1, d_2 - d_3, d_3), & \text{if } d_3 < d_2; \\ -3B_1(d_1, d_2) + 2B_5(d_1, d_2), & \text{if } d_3 = d_2; \end{cases} \\ \mathfrak{C}^{(12)}(d_1, d_2, d_3) &= \begin{cases} C(d_3 - d_1 - d_2, d_1, d_2), & \text{if } d_3 > d_1 + d_2; \\ C(d_1 + d_2 - d_3, d_3 - d_2, d_2), & \text{if } d_2 < d_3 < d_1 + d_2; \\ B_6(d_2, d_1), & \text{if } d_3 = d_1 + d_2; \\ 0, & \text{otherwise;} \end{cases} \\ C(d_1, d_2, d_3) &= -\mathfrak{C}^{(1)}(d_1, d_2, d_3) + \mathfrak{C}^{(2)}(d_1, d_2, d_3) + \mathfrak{C}^{(12)}(d_1, d_2, d_3). \end{aligned}$$

A.2 Main Conjecture

For each $d \leq 1$, let

$$\begin{aligned} A_6(d) &= 3A_1(d) - A_4(d) + \sum_{d_1+d_2=d} B_3(d_1, d_2); \\ \sum_{d=1}^{\infty} \sigma(d)q^d &= \sum_{d=1}^{\infty} d \frac{q^d}{1-q^d}; \\ \mu(d) &= \begin{cases} (-1)^r, & \text{if } d \text{ is the product of } r \text{ distinct primes;} \\ 0, & \text{otherwise;} \end{cases} \\ S(d) &= \begin{cases} \mu(d), & \text{if } d \not\equiv 4 \pmod{8}; \\ \mu(d/4), & \text{if } d \equiv 4 \pmod{8}; \end{cases} \\ V(d) &= \frac{k^2-1}{8} \times \begin{cases} \frac{k^2-1}{8}, & \text{if } d=k, 2 \nmid k; \\ \frac{17k^2+7}{8}, & \text{if } d=2k, 2 \nmid k; \\ 2k^2+1, & \text{if } d=4k, 2 \nmid k. \end{cases} \end{aligned}$$

Conjecture 1 (G. Martin) For all $d \geq 1$,

$$\frac{(-1)^d}{8d} = \sum_{r \leq d, r|d} \frac{\sigma(r)}{r} S(d/r) V(d/r) + \frac{1}{24} \sum_{r \leq d, r|d} \frac{1}{r} A_6(d/r).$$

A.3 Some General Identities

For each d , denote by $\mathcal{P}(d)$ the set of *odd* prime numbers dividing d and by $\ell(d)$ the exponent of the largest power of 2 dividing d . If $d \geq 1$, define

$$\eta_1(d) = \prod_{x \in \mathcal{P}(d)} (1-x) \cdot \begin{cases} 1, & \text{if } \ell(d)=0; \\ -3/2, & \text{if } \ell(d)=1; \\ -1/2, & \text{if } \ell(d) \geq 2; \end{cases}$$

$$\eta_2(d) = \prod_{x \in \mathcal{P}(d)} (1-x^2) \times \begin{cases} 1, & \text{if } \ell(d)=0; \\ -4, & \text{if } \ell(d)=1; \\ 0, & \text{if } \ell(d) \geq 2. \end{cases}$$

Lemma 1 *If $d \geq 3$,*

$$\sum_{r=1}^{r=d} \eta_1(\gcd(r, d)) = 0.$$

Lemma 2 *If $d \geq 3$,*

$$d \eta_2(d) = -6 \sum_{\substack{d_1+d_2=d \\ d_1, d_2 > 0}} d_1 d_2 \eta_1(\gcd(d_1, d_2)).$$

Proof of Lemma 1: (1) We first show that Lemma 1 is satisfied if $d = p^a$ for some prime p and $a \geq 1$ ($a \geq 2$ if $p=2$). If p is odd, then

$$\sum_{r=1}^{r=d} \eta_1(\gcd(r, d)) = p^{a-1} \cdot \eta_1(p) + (p^a - p^{a-1}) \cdot \eta_1(1) = p^{a-1} \cdot (1-p) + (p^a - p^{a-1}) \cdot 1 = 0.$$

If $p=2$, then

$$\begin{aligned} \sum_{r=1}^{r=d} \eta_1(\gcd(r, d)) &= 2^{a-2} \cdot \eta_1(4) + (2^{a-1} - 2^{a-2}) \cdot \eta_1(2) + (2^a - 2^{a-1}) \cdot \eta_1(1) \\ &= -\frac{1}{2} \cdot 2^{a-2} - \frac{3}{2} \cdot 2^{a-2} + 2^{a-1} = 0. \end{aligned}$$

(2) Let $\phi: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ denote the Euler ϕ -function:

$$\phi(d) = |\{r \in \mathbb{Z}^+ : r \leq d, \gcd(r, d) = 1\}|.$$

Since $\phi(d)$ is the number of generators of the cyclic group $\mathbb{Z}_d \equiv \mathbb{Z}/d\mathbb{Z}$,

$$\phi(ab) = \phi(a) \cdot \phi(b) \quad \text{if } \gcd(a, b) = 1.$$

The function η_1 satisfies the same identity. Suppose $a, b \in \mathbb{Z}^+$ are relatively prime and Lemma 1 is satisfied for $d=a$: Then,

$$\begin{aligned} \sum_{r=1}^{r=ab} \eta_1(\gcd(r, d)) &= \sum_{c|d} \sum_{\substack{\gcd(r, d)=c \\ 1 \leq r \leq d}} \eta_1(c) = \sum_{c|d} \phi(d/c) \eta_1(c) = \sum_{c_1|a} \sum_{c_2|b} \phi((a/c_1)(b/c_2)) \eta_1(c_1 c_2) \\ &= \left(\sum_{c_1|a} \phi(a/c_1) \eta_1(c_1) \right) \left(\sum_{c_2|b} \phi(b/c_2) \eta_1(c_2) \right) \\ &= \left(\sum_{r=1}^{r=a} \eta_1(\gcd(r, a)) \right) \left(\sum_{r=1}^{r=b} \eta_1(\gcd(r, b)) \right) = 0. \end{aligned}$$

Lemma 1 now follows by induction on the number of factors in d .

Proof of Lemma 2: (1) This will be a direct computation. Note that for any $d \in \mathbb{Z}^+$ and a finite nonempty set \mathcal{S} ,

$$\sum_{r=1}^{r=d} r(d-r) = \frac{d}{6}(d^2-1), \quad \sum_{J \subset \mathcal{S}} (-1)^{|J|} = 0. \quad (\text{A.4})$$

If $d \in \mathbb{Z}^+$, let

$$\bar{\mathcal{P}}(d) = \begin{cases} \mathcal{P}(d), & \text{if } \ell(d)=0; \\ \mathcal{P}(d) \cup \{2\}, & \text{if } \ell(d)=1; \\ \mathcal{P}(d) \cup \{2, 4\}, & \text{if } \ell(d) \geq 2; \end{cases} \quad \begin{aligned} \mathcal{S}(d) &= \{I \subset \bar{\mathcal{P}}(d) : 2 \in I \text{ if } 4 \in I\}; \\ \mathcal{S}^*(d) &= \{I \in \mathcal{S}(d) : 4 \notin I\}. \end{aligned}$$

If $I \subset \bar{\mathcal{P}}(d)$, let

$$x_I = \prod_{x \in I} x \cdot \begin{cases} 1, & \text{if } 4 \notin I; \\ \frac{1}{2}, & \text{if } 4 \in I; \end{cases} \quad d_I = d/x_I, \quad (1-x)_I = \prod_{x \in I \cap \mathcal{P}(d)} (1-x).$$

Note that for any $J \subset \mathcal{P}(d)$,

$$\sum_{I \subset J} (-1)^{|J-I|} (1-x)_I = (-1)^{|J|} x_J, \quad \sum_{J \subset \mathcal{P}(d)} (-1)^{|J|} x_J^2 = \prod_{x \in \mathcal{P}(d)} (1-x^2). \quad (\text{A.5})$$

Furthermore, for any $I \in \mathcal{S}(d)$,

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r, d))=I \\ 0 < r < d}} r(d-r) = \frac{d}{6} x_I \sum_{I \subset J \in \mathcal{S}'(d)} (-1)^{|J-I|} x_{J-I} (d_J^2 - 1), \quad \text{where } \mathcal{S}' = \begin{cases} \mathcal{S}^*(d), & \text{if } 2 \notin I; \\ \mathcal{S}(d), & \text{if } 2 \in I; \end{cases} \quad (\text{A.6})$$

this follows from (A.4).

(2) We break the sum in the statement Lemma 2 into sub-sums with fixed $\bar{\mathcal{P}}(\gcd(r, d)) \in \mathcal{S}(d)$, as $\eta_1(\gcd(r, d))$ is determined by $\bar{\mathcal{P}}(\gcd(r, d))$. If $I \subset \mathcal{P}(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r, d))=I \\ 0 < r < d}} r(d-r) \eta_1(\gcd(r, d)) = \frac{d}{6} x_I (1-x)_I \sum_{I \subset J \in \mathcal{S}^*(d)} (-1)^{|J-I|} x_{J-I} (d_J^2 - 1). \quad (\text{A.7})$$

If $J \subset \mathcal{P}(d)$, the coefficient of $(d_J^2 - 1)$ in the sum of (A.6) over all $I \subset \mathcal{P}(d)$ is

$$\frac{d}{6} x_J \sum_{I \subset J} (-1)^{|J-I|} (1-x)_I = (-1)^{|J|} x_J^2 \frac{d}{6}$$

by the first identity in (A.5). Summing over all $J \subset \mathcal{P}(d)$, we obtain

$$\frac{d}{6} \sum_{J \subset \mathcal{P}(d)} (-1)^{|J|} x_J^2 (d_J^2 - 1) = -\frac{d}{6} \cdot \begin{cases} \prod_{x \in \mathcal{P}(d)} (1-x^2), & \text{if } \mathcal{P}(d) \neq \emptyset; \\ 1-d^2, & \text{if } \mathcal{P}(d) = \emptyset; \end{cases} \quad (\text{A.8})$$

see the second identities in (A.4) and (A.5). This concludes the proof of Lemma 2 in the $\ell(d) = 0$ case.

(3) Suppose $\ell(d) \geq 1$. If $2 \in I \in \mathcal{S}^*(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I \\ 0 < r < d}} r(d-r) \eta_1(\gcd(r,d)) = -\frac{3}{2} \cdot \frac{d}{6} x_I (1-x)_I \sum_{I \subset J \in \mathcal{S}(d)} (-1)^{|J-I|} x_{J-I} (d_J^2 - 1). \quad (\text{A.9})$$

If $J \subset \mathcal{P}(d)$, the coefficient of $(d_J^2/4 - 1)$ in the sum of (A.6) and (A.9) over all possibilities for I is

$$\frac{d}{6} \left(-2 - \frac{3}{2} \cdot 2 \right) \sum_{J \subset \mathcal{P}(d)} (-1)^{|J|} x_J^2 (d_J^2/4 - 1) = 5 \frac{d}{6} \cdot \begin{cases} \prod_{x \in \mathcal{P}(d)} (1-x^2), & \text{if } \mathcal{P}(d) \neq \emptyset; \\ 1-d^2/4, & \text{if } \mathcal{P}(d) = \emptyset. \end{cases} \quad (\text{A.10})$$

Along with (A.8), (A.10) implies Lemma 2 in the $\ell(d) = 1$ case.

(4) Finally, suppose $\ell(d) \geq 2$. If $I \in \mathcal{S}(d) - \mathcal{S}^*(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I \\ 0 < r < d}} r(d-r) \eta_1(\gcd(r,d)) = -\frac{1}{2} \cdot \frac{d}{6} x_I (1-x)_I \sum_{I \subset J \in \mathcal{S}(d)} (-1)^{|J-I|} x_{J-I} (d_J^2 - 1). \quad (\text{A.11})$$

If $J \subset \mathcal{P}(d)$, the coefficient of $(d_J^2/16 - 1)$ in the sum of (A.9) and (A.11) over all possibilities for I is

$$\frac{d}{6} \left(\frac{3}{2} \cdot 4 - \frac{1}{2} \cdot 4 \right) \sum_{J \subset \mathcal{P}(d)} (-1)^{|J|} x_J^2 (d_J^2/16 - 1) = -4 \frac{d}{6} \cdot \begin{cases} \prod_{x \in \mathcal{P}(d)} (1-x^2), & \text{if } \mathcal{P}(d) \neq \emptyset; \\ 1-d^2/16, & \text{if } \mathcal{P}(d) = \emptyset. \end{cases} \quad (\text{A.12})$$

The $\ell(d) = 2$ case follows from (A.8), (A.10), and (A.12).