The moduli space of maps with crosscaps: the relative signs of the natural automorphisms

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Abstract

Just as a symmetric surface with separating fixed locus halves into two oriented bordered surfaces, an arbitrary symmetric surface halves into two oriented symmetric half-surfaces, i.e. surfaces with crosscaps. Motivated in part by the string theory view of real Gromov-Witten invariants, we previously introduced moduli spaces of maps from surfaces with crosscaps, developed the relevant Fredholm theory, and resolved the orientability problem in this setting. In this paper, we determine the relative signs of the automorphisms of these moduli spaces induced by interchanges of boundary components of the domain and by the anti-symplectic involution on the target manifold, without any global assumptions on the latter. As immediate applications, we describe sufficient conditions for the moduli spaces of real genus 1 maps and for real maps with separating fixed locus to be orientable; we treat the general genus 2+ case in a separate paper. Our sign computations also lead to an extension of recent Floer-theoretic applications of anti-symplectic involutions and to a related reformulation of these results in a more natural way.

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1 Introduction

The theory of J-holomorphic maps plays a prominent role in symplectic topology, algebraic geometry, and string theory. The foundational work of [17, 37, 22, 30, 9, 20] has established the theory of (closed) Gromov-Witten invariants, i.e. counts of J-holomorphic maps from closed Riemann surfaces to symplectic manifolds. In contrast, the theory of open and real Gromov-Witten invariants, i.e. counts of J-holomorphic maps from bordered Riemann surfaces with boundary mapping to a Lagrangian submanifold and of J-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target, has been under development over the past 10-15 years and still is today.

The two main obstacles to defining the open invariants are the potential non-orientability of the moduli space and the existence of real codimension-one boundary strata. The orientability problem in open Gromov-Witten theory is studied in [10, 33] and is fully addressed in [13], by specifying a procedure for locally orienting moduli spaces of open maps and conditions sufficient for the existence of a global orientation. Some approaches [21, 29, 7] to dealing with the codimension-one boundary have raised the issue of orientability in real Gromov-Witten theory. Symmetric Riemann surfaces, however, have convoluted degenerations, making the orientability of their moduli spaces difficult to study. Physical considerations [32, 2, 34] suggest that oriented surfaces with crosscaps provide a suitable replacement for symmetric Riemann surfaces in real Gromov-Witten theory. In [15], we introduced moduli spaces of J-holomorphic maps from oriented surfaces with crosscaps, developed the necessary Fredholm theory, and fully addressed the orientability problem for these moduli spaces, by specifying a procedure for locally orienting them and conditions sufficient for the existence of a global orientation. The last problem is related to the orientability problem in real Gromov-Witten theory via the automorphisms of these moduli spaces induced by interchanges of boundary components of the domain and by the anti-symplectic involution on the target manifold. The effect of the only such automorphism on the local system of orientations of the disk moduli space is computed in [14], without assuming the existence of any global orienting structure on X, as a step in defining a count of disk maps in a much wider collection of cases than in [5, 33]; these counts correspond to real invariants of the target manifold for the standard anti-holomorphic involution on the sphere as defined in [35, 36].

In this paper, we apply the relative sign idea of [14] to compute the change in local orientations of the moduli space of real J-holomorphic maps from an oriented symmetric half-surface (Σ, c) under an interchange of boundary components of Σ and the action of an anti-symplectic involution ϕ on the target manifold X, without any global assumptions on the latter; see Propositions 4.1 and 4.2.[1] The special case of these propositions with the diffeomorphism of Σ fixing the components of $\partial\Sigma$

[1] Proposition 4.2 also applies to the action of orientation-preserving diffeomorphisms of Σ on moduli spaces of open maps to a pair (X, L) consisting of a symplectic manifold and a Lagrangian submanifold; an anti-symplectic involution ϕ on X with $L = X^\phi$ is needed only for the action of orientation-reversing diffeomorphisms of Σ.

includes orientability results of [12, 21, 33, 7]; see Remark 4.6. Applying Propositions 4.1 and 4.2, we obtain sufficient conditions for the moduli spaces of real J-holomorphic maps from a symmetric surface (Σ,σ) of genus 0 and 1, with any given involution on the domain, or from a symmetric surface of arbitrary genus with an involution with a separating fixed locus to be orientable; see Theorems 1.1, 1.2, and 1.4. In [16], we treat the orientability question for arbitrary symmetric surfaces. In a future paper, we will study compactifications of the moduli spaces of maps with crosscaps and use them to define real Gromov-Witten invariants in the style of [34]. The disk case of Propositions 4.1 and 4.2 immediately extends and reformulates, in a more natural way, the arguments of [12] utilizing anti-symplectic involutions on the target for Floer-theoretic applications; see Section 5.4. In addition to leading to orientability results in more general settings, the relative sign approach of [14] does not involve a global orienting structure coming from the ambient manifold and thus reduces the likelihood of computational mistakes of the kind that did appear in earlier approaches to the orientability problem in real settings; see the beginning of Section 4 for details.

An involution on a topological space (resp. smooth manifold) M is a homeomorphism (resp. diffeomorphism) $c\colon M\longrightarrow M$ such that $c\circ c=\mathrm{id}_M$; in particular, the identity map on M is an involution. Let

$$M^c=\big\{x\in M:\ c(x)=x\big\}$$

denote the fixed locus. An involution c determines an action of $\mathbb{Z}_2$ on M; we denote by $H_c^*(M)$ the $\mathbb{Z}_2$-equivariant cohomology with $\mathbb{Z}_2$-coefficients; see Section 3.1. A conjugation on a complex vector bundle $V\longrightarrow M$ lifting an involution c is a vector bundle homomorphism $\tilde{c}\colon V\longrightarrow V$ covering c (or equivalently a vector bundle homomorphism $\tilde{c}\colon V\longrightarrow c^*V$ covering id_M) such that the restriction of $\tilde{c}$ to each fiber is anti-complex linear and $\tilde{c}\circ\tilde{c}=\mathrm{id}_V$. We denote by

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(V,\tilde{c})=(\Lambda_{\mathbb{C}}^{\mathrm{top}}V,\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{c})$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation and by

$$w_i^{\tilde{c}}(V)\in H_c^i(M)$$

the i-th $\mathbb{Z}_2$-equivariant Stiefel-Whitney class of V.

A symmetric surface $(\hat{\Sigma},\sigma)$ consists of a closed connected oriented smooth surface $\hat{\Sigma}$ (manifold of real dimension 2) and an orientation-reversing involution $\sigma:\hat{\Sigma}\longrightarrow\hat{\Sigma}$. There are two equivalence classes of orientation-reversing involutions on $S^2=\mathbb{P}^1$,

$$\tau,\eta\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1,\qquad \tau\big([u,v]\big)=[\bar{v},\bar{u}],\quad \eta\big([u,v]\big)=[-\bar{v},\bar{u}],\tag{1.1}$$

and three equivalence classes of such involutions on $\mathbb{T}=S^1\times S^1\subset\mathbb{C}\times\mathbb{C}$,

$$\mathfrak{t}_0,\mathfrak{t}_1,\mathfrak{t}_2\colon\mathbb{T}\longrightarrow\mathbb{T},\qquad \mathfrak{t}_0(u,v)=(u,\bar{v}),\quad \mathfrak{t}_1(u,v)=(v,u),\quad \mathfrak{t}_2(u,v)=(-u,\bar{v});\tag{1.2}$$

see [3, Section 9], for example. The fixed loci of τ and $\mathfrak{t}_0$ are a circle separating $\mathbb{P}^1$ into two disks interchanged by τ and a pair of disjoint circles separating $\mathbb{T}$ into two annuli interchanged by $\mathfrak{t}_0$, respectively. The fixed loci of η and $\mathfrak{t}_2$ are empty, with the quotients $\hat{\Sigma}/\sigma$ being $\mathbb{RP}^2$ and the Klein bottle $\mathbb{K}$, respectively, while the fixed locus of $\mathfrak{t}_1$ is one circle.

Let (X,ϕ) be a smooth manifold with an involution. If $(\hat{\Sigma},\sigma)$ is a symmetric surface, a real map

$$F\colon (\hat{\Sigma},\sigma)\longrightarrow (X,\phi)$$

is a map $F\colon \hat{\Sigma}\longrightarrow X$ such that $F\circ\sigma=\phi\circ F$. We denote the space of smooth real maps from $(\hat{\Sigma},\sigma)$ to (X,ϕ) by $\mathfrak{B}(X)^{\phi,\sigma}$. Since σ is orientation-reversing,

$$\phi_*\big(F_*[\hat{\Sigma}]_{\mathbb{Z}}\big)=-F_*[\hat{\Sigma}]_{\mathbb{Z}}\in H_2(X;\mathbb{Z})\qquad\forall\ F\in\mathfrak{B}(X)^{\phi,\sigma},$$

where $[\hat{\Sigma}]_{\mathbb{Z}}\in H_2(\hat{\Sigma};\mathbb{Z})$ is the fundamental homology class of $\hat{\Sigma}$. For each $B\in H_2(X;\mathbb{Z})$, let

$$\mathfrak{B}(X,B)^{\phi,\sigma}=\big\{F\in\mathfrak{B}(X)^{\phi,\sigma}:\ F_*[\Sigma]_{\mathbb{Z}}=B\big\};$$

this space is empty unless $\phi_*[\hat{\Sigma}]_{\mathbb{Z}}=-[\hat{\Sigma}]_{\mathbb{Z}}$.

For a symplectic manifold (X,ω), we denote by $\mathcal{J}_\omega$ the space of ω-compatible almost complex structures on X. If ϕ is an anti-symplectic involution on (X,ω), let

$$\mathcal{J}_\phi=\big\{J\in\mathcal{J}_\omega:\ \phi^*J=-J\big\}.$$

For a genus $\hat{g}$ symmetric surface $(\hat{\Sigma},\sigma)$, we similarly denote by $\mathcal{J}_\sigma$ the space of complex structures on $\hat{\Sigma}$ compatible with the orientation such that $\sigma^*\mathfrak{j}=-\mathfrak{j}$. For $J\in\mathcal{J}_\phi$, $\mathfrak{j}\in\mathcal{J}_\sigma$, and $F\in\mathfrak{B}(X)^{\phi,\sigma}$, let

$$\bar{\partial}_{J,\mathfrak{j}}F=\frac{1}{2}\big(\mathrm{d}F+J\circ\mathrm{d}F\circ\mathfrak{j}\big).$$

If $J\in\mathcal{J}_\phi$ and $B\in H_2(X;\mathbb{Z})$, let

$$\mathfrak{M}(X,B;J)^{\phi,\sigma}=\big\{(F,\mathfrak{j})\in\mathfrak{B}(X,B)^{\phi,\sigma}\times\mathcal{J}_\sigma\ :\ \bar{\partial}_{J,\mathfrak{j}}F=0\big\}/\!\sim$$

be the moduli space of equivalence classes of degree B real J-holomorphic maps from $(\hat{\Sigma},\sigma)$ to (X,ϕ); two J-holomorphic maps are equivalent in this space if they differ by a diffeomorphism of $\hat{\Sigma}$ commuting with σ. By [9, 20] and [33, Section 7], this moduli space comes with a natural Kuranishi structure; we call the former orientable if the latter is.

Mathematical considerations, such as [19, Section 4], [31, Section 5], [21, Sections 3,4], [29, Section 1.5], and [7, Section 3], suggest that the map counts arising from involutions σ of different topological types should be combined to get well-defined invariants, as there is a path through one-nodal degenerations between any two involutions of different topological types on the same closed oriented surface $\hat{\Sigma}$. For this reason, for each $\hat{g}\in\mathbb{Z}^{\ge0}$ we define

$$\mathfrak{M}_{\hat{g}}(X,B;J)^{\phi}=\bigsqcup_{\sigma}\mathfrak{M}(X,B;J)^{\phi,\sigma}, \tag{1.3}$$

where the disjoint union is taken over representatives for the equivalence classes of orientation-reversing involutions on a genus $\hat{g}$ closed connected oriented smooth surface $\hat{\Sigma}$. There are $\left\lfloor\frac{3\hat{g}+4}{2}\right\rfloor$ such classes; see [28, Corollary 1.4]. In order to define real invariants, it is essential to study the orientability of these moduli spaces. The next two theorems, which are specializations of Corollaries 5.9 and 5.10 of Propositions 4.1 and 4.2, describe topological conditions on (X,ω,ϕ) insuring that the moduli spaces (1.3) with $\hat{g}=0,1$ are orientable. The $\hat{g}\ge2$ case is treated in [16].

Theorem 1.1. *Let (X,ω) be a symplectic manifold with an anti-symplectic involution ϕ, $J\in\mathcal{J}_\phi$, and $B\in H_2(X;\mathbb{Z})$. If $X^\phi\subset X$ is orientable, there exists $\varpi\in H^2(X;\mathbb{Z}_2)$ such that*

$$w_2(TX^\phi)=\varpi|_{X^\phi} \qquad \textit{and} \qquad \frac{1}{2}\langle c_1(TX),B\rangle+\langle\varpi,B\rangle\in 2\mathbb{Z}\,, \tag{1.4}$$

and $w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\mathrm{d}\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}TX)=\kappa_\phi^2$ for some $\kappa_\phi\in H^1_\phi(X)$, then the moduli space $\mathfrak{M}_0(X,B;J)^\phi$ is orientable.

Theorem 1.2. *Let (X,ω) be a symplectic $2n$-manifold with an anti-symplectic involution ϕ, $J\in\mathcal{J}_\phi$, and $B\in H_2(X;\mathbb{Z})$. If n is odd, $X^\phi\subset X$ is orientable, there exists a real bundle pair $(E,\tilde\phi)\longrightarrow(X,\phi)$ such that*

$$w_2(TX^\phi)=w_1(E^{\tilde\phi})^2 \qquad \textit{and} \qquad \frac{1}{2}\langle c_1(TX),B\rangle+\langle c_1(E),B\rangle\in 2\mathbb{Z}\,, \tag{1.5}$$

and $w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\mathrm{d}\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}TX)=0$, then the moduli space $\mathfrak{M}_1(X,B;J)^\phi$ is orientable.

The first two conditions in Theorem 1.1 insure that the moduli space $\mathfrak{M}(X,B;J)^{\phi,\tau}$ is orientable. A special case of this result, which is also captured by [33, Proposition 5.1], can be seen as corresponding to [12, Theorem 1.3], which involves a less readily checkable condition. In fact, this part of Theorem 1.1 immediately establishes [12, Proposition 3.14]; see Corollary 5.12. The $\Sigma=D^2$ case of Corollary 5.9, which implies the τ-orientability part of Theorem 1.1, relaxes the second condition beyond what the assumptions of [33, Proposition 5.1] allow. The last condition in Theorem 1.1 insures that the moduli space $\mathfrak{M}(X,B;J)^{\phi,\eta}$ is orientable. By Corollary 3.2, it is satisfied if either $\pi_1(X)=0$ and $w_2(TX)=0$ or $\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)$ admits a real square root, i.e. there exist a rank 1 real bundle pair $(L,\tilde\phi)\longrightarrow(X,\phi)$ and an isomorphism of real bundle pairs

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)\approx(L,\tilde\phi)^{\otimes 2}\,.$$

As explained in [15, Section 1], this result on the orientability of the η-moduli space is readily implied by [15, Theorem 1.1] and contains [7, Theorem 1.3].

The first three conditions in Theorem 1.2, which are relaxed in Corollary 5.9, insure that the moduli space $\mathfrak{M}(X,B;J)^{\phi,\mathfrak{t}_0}$ is orientable. The last three conditions insure that the moduli space $\mathfrak{M}(X,B;J)^{\phi,\mathfrak{t}_1}$ is orientable; they are relaxed in Corollary 5.10. The first and last conditions in Theorem 1.2, which are also relaxed in Corollary 5.10, imply that the moduli space $\mathfrak{M}(X,B;J)^{\phi,\mathfrak{t}_2}$ is orientable. By Corollary 3.2, the condition on the equivariant Stiefel-Whitney class in Theorem 1.2 is satisfied if $\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)$ admits a real square root. Special cases of Theorem 1.2 have been obtained independently in [6, 8] using different methods.

The paradigmatic example of a symplectic manifold with an anti-symplectic involution is the complex projective space $\mathbb{P}^{n-1}$ with the standard Fubini-Study symplectic forms ω_n and the standard conjugation

$$\tau_n\colon\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}, \qquad [Z_1,\ldots,Z_n]\longrightarrow[\bar Z_1,\ldots,\bar Z_n];$$

the fixed point locus of this involution is $\mathbb{RP}^{n-1}$. An odd-dimensional projective space also admits an involution without fixed points,

$$\eta_{2m}\colon\mathbb{P}^{2m-1}\longrightarrow\mathbb{P}^{2m-1}, \qquad [Z_1,Z_2,\ldots,Z_{2m-1},Z_{2m}]\longrightarrow\big[-\bar Z_2,\bar Z_1,\ldots,-\bar Z_{2m},\bar Z_{2m-1}\big].$$

If $k\geq 0$, $\mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k$, and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n, $\tau_{n;\mathbf{a}}\equiv\tau_n|_{X_{n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{n;\mathbf{a}}$ with respect to the symplectic form $\omega_{n;\mathbf{a}}\equiv\omega_n|_{X_{n;\mathbf{a}}}$. Similarly, if $X_{2m;\mathbf{a}}\subset\mathbb{P}^{2m-1}$ is preserved by η_{2m}, $\eta_{2m;\mathbf{a}}\equiv\eta_{2m}|_{X_{2m;\mathbf{a}}}$ is an anti-symplectic involution on $X_{2m;\mathbf{a}}$ with respect to the symplectic form $\omega_{2m;\mathbf{a}}\equiv\omega_{2m}|_{X_{2m;\mathbf{a}}}$.

Corollary 1.3. *Let $n\in\mathbb{Z}^+$, $k\in\mathbb{Z}^{\geq 0}$, $\mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k$, and $B\in H_2(X_{n;\mathbf{a}};\mathbb{Z})$.*

(1) If $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n,

$$\sum_{i=1}^k a_i\equiv n \mod 2, \qquad \textit{and} \qquad \sum_{i=1}^k a_i^2\equiv\sum_{i=1}^k a_i \mod 4,$$

then the moduli space $\mathfrak{M}_0(X_{n;\mathbf{a}},B;J)^{\tau_{n;\mathbf{a}}}$ is orientable for every $J\in\mathcal{J}_{\tau_{n;\mathbf{a}}}$. If in addition $n-k$ is even, then the moduli space $\mathfrak{M}_1(X_{n;\mathbf{a}},B;J)^{\tau_{n;\mathbf{a}}}$ is also orientable.

(2) If $n=2m$, $X_{n;\mathbf{a}}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by η_{2m}, and

$$a_1+\ldots+a_k\equiv n \mod 2,$$

then the moduli space $\mathfrak{M}_0(X_{n;\mathbf{a}},B;J)^{\eta_{n;\mathbf{a}}}$ is orientable for every $J\in\mathcal{J}_{\eta_{n;\mathbf{a}}}$. If in addition k is even and $a_1+\ldots+a_k\equiv n \mod 4$, the moduli space $\mathfrak{M}_1(X_{n;\mathbf{a}},B;J)^{\eta_{n;\mathbf{a}}}$ is also orientable.

This corollary follows immediately from Theorems 1.1 and 1.2. If $a_1+\ldots+a_k\equiv n \mod 4$, we take $\varpi=0$ and E to be the trivial rank 0 vector bundle for the purposes of applying these two theorems. Otherwise, we take

$$\varpi=c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))|_{X_{n;\mathbf{a}}} \qquad \text{and} \qquad (E,\tilde{\phi})=\big(\mathcal{O}_{\mathbb{P}^{n-1}}(1),\tilde{\tau}_{n-1}\big)\big|_{X_{n;\mathbf{a}}},$$

where $\mathcal{O}_{\mathbb{P}^{n-1}}(1)\longrightarrow\mathbb{P}^{n-1}$ is the hyperplane line bundle with the involution $\tilde{\tau}_n$ canonically induced by τ_n. The idea of introducing an additional bundle $(E,\tilde{\phi})$ in Theorem 1.2 is directly motivated by [14] and provides a way of bypassing the requirements that X^ϕ be spin and $\langle c_1(TX),B\rangle$ be divisible by 4, which appear in the genus 0 orientability results in [12] and the genus 1 orientability results of [6, 8]. It can also be applied in Theorem 1.1, instead of using ϖ, but the latter is the more customary relative spin condition on TX^ϕ. In [16], we extend Corollary 1.3 to arbitrary genus.

Theorems 1.1 and 1.2 are obtained by reducing the orientability problem for the moduli spaces $\mathfrak{M}(X,B;J)^{\phi,\sigma}$ of maps from $(\hat{\Sigma},\sigma)$ to orientability questions about the moduli spaces $\mathfrak{M}(X,B;J)^{\phi,c}$ of maps from a corresponding oriented sh-surface (Σ,c), where Σ is a bordered oriented smooth surface and $c\colon\partial\Sigma\longrightarrow\partial\Sigma$ is an orientation-preserving involution; see Section 2.1. The former moduli space is the quotient of the latter moduli space by automorphisms, which include the involution

$$[\mathfrak{j},f]\longrightarrow[-\mathfrak{c}_\Sigma^*\mathfrak{j},\phi\circ f\circ\mathfrak{c}_\Sigma], \tag{1.6}$$

where $\mathfrak{c}_\Sigma:\Sigma\longrightarrow\Sigma$ is an orientation-reversing involution. The remaining automorphisms involve diffeomorphisms of Σ that interchange its boundary components. The second restrictions in (1.4) and in (1.5) and the dimensional condition in Theorem 1.2 imply that the relevant automorphisms on the moduli spaces $\mathfrak{M}(X,B;J)^{\phi,c}$ are orientation-preserving; the remaining conditions insure that these moduli spaces of maps with crosscaps are orientable.

The orientability of the moduli spaces $\mathfrak{M}(X,B;J)^{\phi,c}$ of maps with crosscaps is studied in [15]. In this paper, we compute the effect of the actions of the natural automorphisms of these spaces on local orientations; see Propositions 4.1 and 4.2. If $\Sigma=\mathbb{P}^1$, (1.6) is the only relevant automorphism and is equivalent to the automorphism

$$[\mathfrak{j}_0,f]\longrightarrow[\mathfrak{j}_0,\phi\circ f\circ\mathfrak{c}_{D^2}], \tag{1.7}$$

where $\mathfrak{j}_0$ is the standard complex structure on D^2 and $\mathfrak{c}_{D^2}$ is the standard anti-holomorphic involution (conjugation) on D^2. By [7, Section 2.1] and the proof of Corollary 5.10, this automorphism is orientation-preserving in the case $\sigma=\eta$ if the appropriate moduli space $\mathfrak{M}(X,B;J)^{\phi,c}$ is orientable and has no effect on the relative sign of the automorphism (1.7) in general. In the case $\sigma=\tau$, the purely cohomological condition (1.4) in Theorem 1.1 replaces the rather artificial case-by-case conditions in [12]; see Section 5.4 for more details.

Theorems 1.1 and 1.2 can be extended to the corresponding moduli spaces $\mathfrak{M}_{k,l}(X,B;J)^{\phi,\sigma}$ of real maps with k boundary and l interior marked points, as the effect of adding marked points on the sign of the relevant automorphisms can be easily determined. The introduction of decorated marked points on oriented sh-surfaces in [14] in fact removes the need to consider the effect of conjugate pairs of marked points, while making the resulting invariants precisely agree with counts of real curves in sufficiently positive symplectic manifolds.[2]

The moduli spaces $\mathfrak{M}_{k,l}(X,B;J)^{\phi,\sigma}$ typically have codimension-one boundary and often of more than one type. The codimension-one boundary stratum consisting of maps from $\hat{\Sigma}$ with a bubble attached at a real point of the domain can be eliminated by the gluing of procedure of [5, 33], which is adapted to maps with decorated marked points in [14, Section 3]. By [14, Theorems 1.3], the proof of [14, Corollary 6.1], Propositions 4.1 and 4.2, and Corollary 4.5, the glued moduli spaces $\widetilde{\mathfrak{M}}_{\hat{g},0,l}(X,B;J)^{\phi}$, with $\hat{g}=0,1$, are still orientable under the conditions of Theorems 1.1 and 1.2. For $k>0$, $\widetilde{\mathfrak{M}}_{\hat{g},k,l}(X,B;J)^{\phi}$ is typically not orientable, but it may still be possible to define invariants in some cases, as done in [14, Section 7.1].

The remaining types of codimension-one boundary strata of $\mathfrak{M}_{\hat{g},0,l}(X,B;J)^{\phi}$ correspond to one-nodal degenerations of $\hat{\Sigma}$ passing between involutions on $\hat{\Sigma}$ of different topological types, as described in detail in [19, Section 4], [31, Section 5], and [21, Sections 3,4]. As suggested in [29, Section 1.5] and carried out in [7, Section 3] in the case $\hat{\Sigma}=\mathbb{P}^1$, the moduli spaces $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\sigma}$ with different types of involutions σ on $\hat{\Sigma}$ should in general be combined to get well-defined invariants by gluing along codimension-one boundaries. However, in positive genus, it is simpler to consider such transitions in the setting of oriented sh-surfaces. We intend to pursue this approach to real Gromov-Witten invariants in a future paper, making use of the preliminaries developed in [15] and of the signs of the natural automorphisms provided by Propositions 4.1 and 4.2.

We conclude this section by describing topological conditions on (X,ω,ϕ) insuring that the actions of the natural automorphisms on topological components of the moduli space

$$\mathfrak{M}_{\Sigma}(X,X^{\phi},B;J)\equiv\mathfrak{M}(X,B;J)^{\phi,\mathrm{id}_{\Sigma}}$$

[2] In contrast, the invariants of [33], for example, differ by $\pm 2^{l-1}$.

of J-holomorphic maps $(\Sigma,\partial\Sigma)\longrightarrow(X,X^\phi)$ are orientation-preserving. On one hand, these automorphisms being orientation-preserving is equivalent to the moduli space $\mathfrak{M}(X,B;J)^{\phi,\sigma}$, where σ is the involution on $\hat\Sigma$ with $\hat\Sigma^\sigma=\partial\Sigma$, being orientable. On the other hand, notable Floer-theoretic applications of this property in the $\Sigma=D^2$ case are demonstrated in [12]; in light of recent work on higher-genus generalizations of Floer homology, it is reasonable to expect similar applications of this property in other cases as well.

Let $(\partial\Sigma)_1,\ldots,(\partial\Sigma)_m$ be the boundary components of Σ. For each

$$\mathbf{b}\equiv(\beta,b_1,\ldots,b_m)\in H_2(X,X^\phi;\mathbb{Z})\oplus H_1(X^\phi;\mathbb{Z})^{\oplus m}\,, \tag{1.8}$$

we denote by $\mathfrak{M}(X,X^\phi,\mathbf{b};J)$ the space of equivalence classes of pairs $(f,\mathfrak{j})$, where $\mathfrak{j}$ is a complex structure on Σ compatible with the orientation and

$$f\colon(\Sigma,\partial\Sigma)\longrightarrow(X,X^\phi)\qquad\text{s.t.}\quad\bar\partial_{J,\mathfrak{j}}f=0,\quad f_*[\Sigma,\partial\Sigma]_{\mathbb{Z}}=\beta,\quad f_*[(\partial\Sigma)_i]=b_i\ \ \forall\,i.$$

Every diffeomorphism of Σ, orientation-preserving or orientation-reversing, induces an automorphism of this moduli space. For β as in (1.8), let $\mathfrak{d}(\beta)\in H_2(X;\mathbb{Z})$ denote the natural ϕ-double of β; see [14, Section 3].

Theorem 1.4. *Let (X,ω) be a symplectic $2n$-manifold with an anti-symplectic involution ϕ, $J\in\mathcal{J}_\phi$, Σ be an oriented bordered surface, and $\mathbf{b}$ be as in (1.8). If $b_i=b_j$ for some $i\neq j$, assume also that n is odd. If $X^\phi\subset X$ is orientable and there exists $\varpi\in H^2(X;\mathbb{Z}_2)$ satisfying (1.4) with $B=\mathfrak{d}(\beta)$, then $\mathfrak{M}(X,X^\phi,\mathbf{b};J)$ can be oriented by a relative spin structure so that the natural automorphisms of this moduli space induced by diffeomorphisms of Σ are orientation-preserving.*

This theorem is implied by Corollary 5.9 of Propositions 4.1 and 4.2, which establishes the same conclusion under weaker assumptions. In the case of the natural automorphisms induced by diffeomorphisms preserving the boundary components, Theorem 1.4 recovers the orientable case of [33, Proposition 5.1]; the full statement of the latter is recovered in Remark 4.6. In the case of the natural automorphisms induced by diffeomorphisms preserving the orientation, Theorem 1.4 applies to arbitrary Lagrangians, as the involution ϕ plays no role then.

This paper is organized as follows. Section 2.1 reviews the analytic setup and Fredholm theory for oriented sh-surfaces. In Section 2.2, we determine the signs of natural automorphisms on the Deligne-Mumford moduli spaces of bordered Riemann surfaces, establishing one of the three key statements in this paper. In Section 2.3, we make a number of simple, but useful, topological observations that naturally fit with the orientability problem in Gromov-Witten theory. Section 3 reviews basic notions in equivariant cohomology and investigates in detail properties of the equivariant w_2 of real bundle pairs over surfaces. Section 4 establishes the two remaining key statements of this paper, Propositions 4.1 and 4.2, which determine the effects of the conjugation and interchanges of boundary components on local orientations of index bundles of real Cauchy-Riemann operators. We introduce a weaker version of spin structures in Section 5.1 and show in Section 5.2 that relative spin sub-structures induce orientations on moduli spaces of open maps. In Section 5.3, we combine Proposition 2.5 with Propositions 4.1 and 4.2 to show that relative spin sub-structures compatible with an involution ϕ induce orientations preserved by the natural automorphisms of the moduli spaces of maps from surfaces with crosscaps and establish more general versions of Theorems 1.1, 1.2, and 1.4. In Section 5.4, we apply Propositions 4.1 and 4.2 to Floer homology, following the principles laid out in [12].

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2 Preliminaries

We begin with some background material. We first review the analytic setup for maps from oriented sh-surfaces introduced in [15] and recall some of the related results obtained in [15]. We then determine the signs of natural automorphisms on the Deligne-Mumford moduli spaces of bordered marked Riemann surfaces; this is one of the key ingredients in the proof of Theorem 1.4. We conclude with a number of purely topological observations that naturally fit with the orientability conditions discovered in [13] in open Gromov-Witten theory and adapted in [15] to real Gromov-Witten theory.

2.1 Review of analytic setup

An oriented symmetric half-surface (or simply oriented sh-surface) is a pair (Σ, c) consisting of an oriented bordered smooth surface Σ and an involution $c\colon \partial\Sigma \longrightarrow \partial\Sigma$ preserving each component and the orientation of $\partial\Sigma$. The restriction of such an involution c to a boundary component $(\partial\Sigma)_i$ is either the identity or the antipodal map

$$\mathfrak{a}\colon S^1 \longrightarrow S^1, \qquad z \longrightarrow -z, \tag{2.1}$$

for a suitable identification of $(\partial\Sigma)_i$ with $S^1 \subset \mathbb{C}$; the latter type of boundary structure is called crosscap in the string theory literature. We define

$$c_i = c|_{(\partial\Sigma)_i}, \qquad |c_i| = \begin{cases} 0, & \text{if } c_i = \text{id}; \\ 1, & \text{otherwise}; \end{cases} \qquad |c|_k = \big|\{(\partial\Sigma)_i \subset \Sigma\colon |c_i| = k\}\big| \quad k = 0, 1.$$

Thus, $|c|_0$ is the number of standard boundary components of $(\Sigma, \partial\Sigma)$ and $|c|_1$ is the number of crosscaps. We order the boundary components $(\partial\Sigma)_i$ of Σ so that $|c_i| = 0$ for $i = 1, \ldots, |c|_0$.

An oriented symmetric half-surface (Σ, c) doubles to the topological symmetric surface

$$\begin{aligned} &\hat{\Sigma} \equiv \big(\Sigma^+ \sqcup \Sigma^-\big)/\sim \;\equiv \{+,-\}\times\Sigma/\sim, \qquad && (+,z) \sim \big(-, c(z)\big) \quad \forall\, z \in \partial\Sigma, \\ &\hat{c}\colon \hat{\Sigma} \longrightarrow \hat{\Sigma}, && \hat{c}\big([\pm, z]\big) = [\mp, z] \quad \forall\, z \in \Sigma, \end{aligned}$$

with Σ^+ having the same orientation as Σ (and Σ^- having the opposite orientation to Σ). By [28, Theorems 1.1, 1.2], every symmetric surface $(\hat{\Sigma}, \sigma)$ can be obtained in this way. If (X, ϕ) is a manifold with an involution, a real map $f : (\Sigma, c) \longrightarrow (X, \phi)$ is a map $f : \Sigma \longrightarrow X$ such that $f \circ c = \phi \circ f$ on $\partial\Sigma$. Such a map doubles to a real map $\hat{u}\colon (\hat{\Sigma}, \hat{c}) \longrightarrow (X, \phi)$ such that $\hat{u}|_{\Sigma^+} = u$.

Given an oriented sh-surface (Σ,c), let $\mathcal{D}_\Sigma$ be the group of diffeomorphisms of Σ preserving the orientation and the boundary components and $\mathcal{D}_c\subset\mathcal{D}_\Sigma$ be the subgroup of diffeomorphisms that commute with the involution c on $\partial\Sigma$. Let $\mathcal{J}_\Sigma$ be the space of all complex structures on Σ compatible with the orientation. For each $\mathfrak{j}\in\mathcal{J}_\Sigma$, Σ can be covered by coordinate charts

$$\psi\colon (U,U\cap\partial\Sigma)\longrightarrow(\mathbb{H},\mathbb{R}) \qquad \text{s.t.}\quad \mathfrak{j}=\psi^*\mathfrak{j}_0,$$

where $\mathbb{H}$ is the closed upper-half plane and $\mathfrak{j}_0$ is the standard complex structure on $\mathbb{C}$, and the overlap maps between such charts are holomorphic; see [15, Corollary A.2]. We call such charts $\mathfrak{j}$-holomorphic. Let $\mathcal{J}_c\subset\mathcal{J}_\Sigma$ denote the subspace of complex structures $\mathfrak{j}$ such that c is real-analytic with respect to $\mathfrak{j}$, i.e. for every $z\in\partial\Sigma$ there exist $\mathfrak{j}$-holomorphic charts

$$\psi_z\colon U_z\longrightarrow U_z' \qquad\text{and}\qquad \psi_{c(z)}\colon U_{c(z)}\longrightarrow U_{c(z)}',$$

where U_z and $U_{c(z)}$ are open subsets of Σ containing z and $c(z)$, respectively, and U_z' and $U_{c(z)}'$ are open subsets of $\mathbb{H}$, such that

$$\psi_{c(z)}\circ c\circ\psi_z^{-1}\colon\ \psi_z\big(U_z\cap c(U_{c(z)}\cap\partial\Sigma)\big)\longrightarrow\mathbb{R}$$

is a real-analytic function on an open subset of $\mathbb{R}\subset\mathbb{C}$. By [15, Lemma 3.1], the subspace $\mathcal{J}_c$ of $\mathcal{J}_\Sigma$ is preserved by $\mathcal{D}_c$. By [15, Corollary 3.3], each $\mathfrak{j}\in\mathcal{J}_c$ doubles to a complex structure $\hat{\mathfrak{j}}$ on $\hat\Sigma$ so that $\hat{\mathfrak{j}}|_\Sigma=\mathfrak{j}$, $\hat c^*\hat{\mathfrak{j}}=-\hat{\mathfrak{j}}$, and $\partial\Sigma$ is a real-analytic curve in $\hat\Sigma$; in particular, $\hat{\mathfrak{j}}$ determines a smooth structure on $\hat\Sigma$. If (X,ϕ) is a manifold with an involution and $u\colon(\Sigma,c)\longrightarrow(X,\phi)$ is a real $(J,\mathfrak{j})$-holomorphic map, then $\hat u$ is a $(J,\hat{\mathfrak{j}})$-holomorphic map.

For applications to moduli problems, it is convenient to introduce subspaces $\mathcal{J}_c^*$ and $\mathcal{D}_c^*$ of $\mathcal{J}_c$ and $\mathcal{D}_c$ so that the natural map

$$\mathcal{J}_c^*/\mathcal{D}_c^*\longrightarrow\mathcal{J}_\Sigma/\mathcal{D}_\Sigma \tag{2.2}$$

induced by the inclusions $\mathcal{J}_c^*\longrightarrow\mathcal{J}_\Sigma$ and $\mathcal{D}_c^*\longrightarrow\mathcal{D}_\Sigma$ is an isomorphism (as Artin stacks) whenever (Σ,c) is not a disk with an involution different from the identity. We take $\mathcal{J}_c^*=\{\mathfrak{j}_0\}$ if $\Sigma=D^2$, $\mathcal{D}_c^*=\mathrm{PGL}_2^0\mathbb{R}$ to be the group of holomorphic automorphisms of D^2 if $(\Sigma,c)=(D^2,\mathrm{id}_{S^1})$, and $\mathcal{D}_c^*=S^1$ to be the group of standard rotations of D^2 if $(\Sigma,c)=(D^2,\mathfrak{a})$.

Suppose next that Σ is a cylinder with ordered boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$. Let $\mathring{\mathbb{I}}=(0,1)$ and

$$h_{\mathbb{C}^*}\colon\mathbb{C}^*\longrightarrow\mathbb{C}^*,\qquad z\longrightarrow 1/z\,.$$

For each $r\in\mathring{\mathbb{I}}$, we define

$$A_r=\big\{z\in\mathbb{C}\colon\ (|z|-r)(r|z|-1)\le0\big\},\quad (\partial A_r)_1=\big\{z\in\mathbb{C}\colon\ |z|=r\big\},\quad (\partial A_r)_2=\big\{z\in\mathbb{C}\colon\ r|z|=1\big\}.$$

Choose a smooth map

$$\Psi\colon\mathring{\mathbb{I}}\times\Sigma\longrightarrow\mathbb{C}^*,\qquad \Psi(r,z)\longrightarrow\Psi_r(z),$$

such that each map

$$\Psi_r\colon\big(\Sigma,(\partial\Sigma)_1,(\partial\Sigma)_2\big)\longrightarrow\big(A_r,(\partial A_r)_1,(\partial A_r)_2\big),\qquad r\in\mathring{\mathbb{I}},$$

is a diffeomorphism so that $\mathfrak{a}\circ\Psi_r=\Psi_r\circ c_i$ on $(\partial\Sigma)_i$ if $|c_i|=1$, $i=1,2$, and the diffeomorphisms

$$\Psi_r\circ\Psi_{r'}^{-1}\colon A_{r'}\longrightarrow A_r,\qquad r,r'\in\mathring{\mathbb{I}},$$

commute with $h_{\mathbb{C}^*}$, the standard conjugation $\mathfrak{c}_{\mathbb{C}}$ on $\mathbb{C}$, and the standard action of $S^1\subset\mathbb{C}^*$ on $\mathbb{C}$. The last condition implies that the diffeomorphisms

$$\begin{aligned} h_\Sigma\colon \Sigma \longrightarrow \Sigma, &\qquad h_\Sigma(z) = \Psi_r^{-1}\big(h_{\mathbb{C}^*}(\Psi_r(z))\big), \qquad \text{and}\\ \mathfrak{c}_\Sigma\colon \Sigma \longrightarrow \Sigma, &\qquad \mathfrak{c}_\Sigma(z) \longrightarrow \Psi_r^{-1}\big(\mathfrak{c}_{\mathbb{C}}(\Psi_r(z))\big), \end{aligned}$$

and the S^1-action on Σ given by

$$S^1\times\Sigma \longrightarrow \Sigma, \qquad \theta\cdot z = \Psi_r^{-1}\big(\theta\Psi_r(z)\big) \quad \forall\, z\in\Sigma,\, \theta\in S^1\subset\mathbb{C}, \tag{2.3}$$

are independent of $r\in\mathring{\mathbb{I}}$. In this case, we take

$$\mathcal{J}_c^* = \big\{\Psi_r^*\mathfrak{j}_0\colon\, r\in\mathring{\mathbb{I}}\big\}$$

and $\mathcal{D}_c^*\subset\mathcal{D}_c$ to be the subgroup corresponding to the action (2.3). The latter is the group of automorphisms of each complex structure in $\mathcal{J}_c^*$ that preserve each boundary component of Σ. By the classification of complex structures on the cylinder [3, Section 9], for every $\mathfrak{j}\in\mathcal{J}_\Sigma$, there exist a unique $r\in\mathring{\mathbb{I}}$ and a diffeomorphism h of Σ preserving the orientation and the boundary components such that $\mathfrak{j}=h^*\Psi_r^*\mathfrak{j}_0$. It follows that the map (2.2) is an isomorphism.

If Σ is not a disk or a cylinder, i.e. the genus of its double is at least 2, we identify each boundary component $(\partial\Sigma)_i$ of $\partial\Sigma$ with S^1 in such a way that $c_i\equiv c|_{(\partial\Sigma)_i}$ corresponds to either the identity or the antipodal map on S^1 and denote by $\mathcal{D}_i$ the subgroup of diffeomorphisms of $(\partial\Sigma)_i$ corresponding to the rotations of S^1 under this identification. For each $\mathfrak{j}\in\mathcal{J}_\Sigma$, there exists a unique metric $\hat{g}_{\mathfrak{j}}$ on the double $(\hat{\Sigma}',\hat{\mathfrak{j}}')$ of $(\Sigma,\mathfrak{j})$ with respect to the involution $\mathrm{id}_{\partial\Sigma}$ so that $\hat{g}_{\mathfrak{j}}$ has constant scalar curvature -1 and is compatible with $\hat{\mathfrak{j}}'$. Each boundary component $(\partial\Sigma)_i$ is a geodesic with respect to $\hat{g}_{\mathfrak{j}}$, and each isometry of $(\partial\Sigma)_i$ with respect to $\hat{g}_{\mathfrak{j}}$ is real-analytic with respect to $\mathfrak{j}$. We denote by $\mathcal{J}_c^*\subset\mathcal{J}_\Sigma$ the subspace of complex structures $\mathfrak{j}$ so that each $\mathcal{D}_i$ is the group of isometries of $(\partial\Sigma)_i$ with respect to $\hat{g}_{\mathfrak{j}}$ and by $\mathcal{D}_c^*$ the subgroup of diffeomorphisms of Σ that preserve the orientation and the boundary components and restrict to elements of $\mathcal{D}_i$ on each boundary component $(\partial\Sigma)_i$ of Σ. Since $c_i\in\mathcal{D}_i$ for each i, $\mathcal{J}_c^*\subset\mathcal{J}_c$ and $\mathcal{D}_c^*\subset\mathcal{D}_c$. By [15, Lemma 6.1], the map (2.2) is an isomorphism in this case as well.

Let (Σ,c) be an oriented sh-surface with orderings

$$(\partial\Sigma)_1,\ldots,(\partial\Sigma)_{|c|_0} \qquad \text{and} \qquad (\partial\Sigma)_{|c|_0+1},\ldots,(\partial\Sigma)_{|c|_0+|c|_1}$$

of the boundary components with $|c_i|=0$ and with $|c_i|=1$ and (X,ϕ) be a smooth manifold with an involution. Given

$$\mathbf{b} = (B,b_1,\ldots,b_{|c|_0+|c|_1}) \in H_2(X;\mathbb{Z})\oplus H_1(X^\phi;\mathbb{Z})^{\oplus|c|_0}\oplus H_1^\phi(X;\mathbb{Z})^{\oplus|c|_1}, \tag{2.4}$$

let $\mathfrak{B}(X,\mathbf{b})^{\phi,c}$ denote the space of smooth maps $u\colon\Sigma\longrightarrow X$ such that

- $u\circ c=\phi\circ u$ on $\partial\Sigma$,
- $\hat{u}_*[\hat{\Sigma}]=B$, $u_*[(\partial\Sigma)_i]=b_i$ for $i=1,\ldots,|c|_0$, and
- $[u|_{(\partial\Sigma)_i}]_{\mathbb{Z}_2}^{c_i}=b_i$ for $i=|c|_0+1,\ldots,|c|_0+|c|_1$, where $[u|_{(\partial\Sigma)_i}]_{\mathbb{Z}_2}^{c_i}$ is the equivariant pushforward of $[(\partial\Sigma)_i]_{\mathbb{Z}_2}^{c_i}$ by $u|_{(\partial\Sigma)_i}$, as in Section 3.1.

We denote by $\mathfrak{B}(X)^{\phi,c}$ the disjoint union of the spaces $\mathfrak{B}(X,\mathbf{b})^{\phi,c}$ over all tuples $\mathbf{b}$ as in (2.4). If in addition $\mathbf{k}=(k_1,\ldots,k_{|c|_0+|c|_1})$ is a tuple of nonnegative integers, let

$$\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}=\mathfrak{B}(X,\mathbf{b})^{\phi,c}\times\prod_{i=1}^{|c|_0+|c|_1}\big((\partial\Sigma)_i^{k_i}-\Delta_{i,k_i}\big),$$

where

$$\Delta_{i,k_i}=\big\{(x_{i,1},\ldots,x_{i,k_i})\in(\partial\Sigma)_i^{k_i}:\ x_{i,j'}\in\{x_{i,j},c(x_{i,j})\}\text{ for some }j,j'=1\ldots,k_i,\ j\neq j'\big\}$$

is the big c-symmetrized diagonal. Let

$$\begin{aligned}\mathcal{H}^*_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}&=\big(\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}\times\mathcal{J}^*_c\big)/\mathcal{D}^*_c,\\ \mathfrak{M}_{\mathbf{k}}(X,J,\mathbf{b})^{\phi,c}&=\big\{[u,\mathbf{x}_1,\ldots,\mathbf{x}_{|c|_0+|c|_1},\mathfrak{j}]\in\mathcal{H}^*_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}:\ \bar\partial_{J,\mathfrak{j}}u=0\big\},\end{aligned}\tag{2.5}$$

where $\bar\partial_{J,\mathfrak{j}}$ is the usual Cauchy-Riemann operator with respect to the complex structures J on X and $\mathfrak{j}$ on Σ. If X is a point and $\mathbf{b}$ is the zero tuple, we denote $\mathcal{H}^*_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}$ by $\mathcal{M}^c_{\Sigma,\mathbf{k}}$; by [15, Lemma 6.1], this is the usual Deligne-Mumford moduli space

$$\mathcal{M}_{\Sigma,\mathbf{k}}=\mathcal{M}^{\mathrm{id}_\Sigma}_{\Sigma,\mathbf{k}}$$

of stable bordered Riemann surfaces with ordered boundary components with boundary if Σ is not a disk with $k_1<3$ or a cylinder with $k_1,k_2=0$ (for stability reasons).

If M is a manifold, possibly with boundary, or a (possibly nodal) surface, and c is an involution on a submanifold $M'\subset M$, a real bundle pair $(V,\tilde c)\longrightarrow(M,c)$ consists of a complex vector bundle $V\longrightarrow M$ and a conjugation $\tilde c$ on $V|_{M'}$ lifting c. A real bundle pair $(V,\tilde c)\longrightarrow(\Sigma,c)$, where (Σ,c) is an oriented sh-surface, doubles to a real bundle pair over $(\hat\Sigma,\hat c)$,

$$\begin{aligned}&\hat V\equiv\big(\{+\}\times V\sqcup\{-\}\times\bar V\big)/\sim, &&(+,v)\sim\big(-,\tilde c(v)\big)\quad\forall\,v\in V|_{\partial\Sigma},\\ &\check c\colon\hat V\longrightarrow\hat V, &&\check c\big([\pm,v]\big)=[\mp,v]\quad\forall\,v\in V,\end{aligned}$$

where $\bar V$ denotes the same real vector bundle over Σ as V, but with the opposite complex structure on the fibers. We define the Maslov index of $(V,\tilde c)$ by

$$\mu(V,\tilde c)=\langle c_1(\hat V),[\hat\Sigma]\rangle.\tag{2.6}$$

By [23, Theorem C.3.5 and (C.3.4)], this agrees with the usual definition of the Maslov index of $(V,V^{\tilde c})$ if $c=\mathrm{id}_{\partial\Sigma}$. By [4, Propositions 4.1, 4.2], real bundle pairs $(V,\tilde c)\longrightarrow(\Sigma,c)$ are classified by their rank, the Maslov index, and the orientability of $V^{\tilde c}$ over each boundary component $(\partial\Sigma)_i$ with $|c_i|=0$.

A real Cauchy-Riemann operator on a real bundle pair $(V,\tilde c)\longrightarrow(\Sigma,c)$, where (Σ,c) is an oriented sh-surface, is a linear map of the form

$$\begin{aligned}D=\bar\partial+A\colon\Gamma(\Sigma;V)^{\tilde c}&\equiv\big\{\xi\in\Gamma(\Sigma;V):\ \xi\circ c=\tilde c\circ\xi|_{\partial\Sigma}\big\}\\ &\longrightarrow\Gamma^{0,1}_{\mathfrak{j}}(\Sigma;V)\equiv\Gamma\big(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}V\big),\end{aligned}\tag{2.7}$$

where $\bar{\partial}$ is the holomorphic $\bar{\partial}$-operator for some $\mathfrak{j}\in\mathcal{J}_\Sigma$ and a holomorphic structure in V and

$$A \in \Gamma\big(\Sigma; \mathrm{Hom}_{\mathbb{R}}(V, (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} V)\big)$$

is a zeroth-order deformation term. By [15, Proposition 3.6], a real Cauchy-Riemann operator on a real bundle pair is Fredholm in the appropriate completions if $\mathfrak{j}\in\mathcal{J}_c$; by [15, Remark 3.7], it need not be Fredholm if $\mathfrak{j}\notin\mathcal{J}_c$. A continuous family of such Fredholm operators D_t over a topological space $\mathcal{H}$ determines a line bundle over $\mathcal{H}$, called the determinant line bundle of $\{D_t\}$ and denoted $\det D$; see [23, Section A.2] and [38] for a construction.

Families of real Cauchy-Riemann operators often arise by pulling back data from a target manifold by smooth maps as follows. Suppose (X, J) is an almost complex manifold with an anti-complex involution $\phi\colon X \longrightarrow X$ and $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ is a real bundle pair. Let ∇ be a connection on V and

$$A \in \Gamma\big(X; \mathrm{Hom}_{\mathbb{R}}(V, (T^*X, J)^{0,1} \otimes_{\mathbb{C}} V)\big).$$

For any map $u : \Sigma \longrightarrow X$ and $\mathfrak{j}\in\mathcal{J}_\Sigma$, let ∇^u denote the induced connection in u^*V and

$$A_{\mathfrak{j};u} = A \circ \partial_{\mathfrak{j}} u \in \Gamma(\Sigma; \mathrm{Hom}_{\mathbb{R}}(u^*V, (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} u^*V)\big).$$

If c is a boundary involution on Σ and $u\circ c=\phi\circ u$ on $\partial\Sigma$, the homomorphisms

$$\bar{\partial}_u^\nabla = \frac{1}{2}(\nabla^u + \mathfrak{i} \circ \nabla^u \circ \mathfrak{j}), \;\; D_u \equiv \bar{\partial}_u^\nabla + A_{\mathfrak{j};u} \colon \Gamma(\Sigma; u^*V)^{u^*\tilde{\phi}} \longrightarrow \Gamma_{\mathfrak{j}}^{0,1}(\Sigma; u^*V)$$

are real Cauchy-Riemann operators on $(u^*V, u^*\tilde{\phi}) \longrightarrow (\Sigma, c)$ that form families of real Cauchy-Riemann operators over families of maps. We denote the determinant line bundle of such a family by $\det(D_{V,\tilde{\phi}})$.

2.2 The signs of automorphisms of the Deligne-Mumford spaces

Let Σ be an oriented genus g surface with $m>0$ boundary components so that $2g+m\geq 3$. For any diffeomorphism $h\colon \Sigma \longrightarrow \Sigma$, let

$$|h| = \begin{cases} 0, & \text{if } h \text{ is orientation-preserving;} \\ 1, & \text{if } h \text{ is orientation-reversing;} \end{cases} \tag{2.8}$$

and denote by $\mathrm{sgn}_h \in \{0, 1\}$ the sign of the permutation induced by h on the set of boundary components of Σ. Such a diffeomorphism induces an automorphism

$$\mathrm{DM}_h\colon \mathcal{M}_\Sigma \equiv \mathcal{M}_{\Sigma,\mathbf{0}} \longrightarrow \mathcal{M}_\Sigma, \qquad [\mathfrak{j}] \longrightarrow \big[(-1)^{|h|} h^*\mathfrak{j}\big], \tag{2.9}$$

with the notation as in (2.5). The Deligne-Mumford moduli space $\mathcal{M}_\Sigma$ of bordered Riemann surfaces is orientable; see Lemma 2.1 below. Proposition 2.5 determines the sign of the automorphism DM_h; the sign of its analogue on $\mathcal{M}_{\Sigma,\mathbf{k}}$ can then be easily obtained by proceeding as in the proof of [13, Corollary 1.8].

Lemma 2.1. *Let Σ be an oriented genus g surface with m boundary components so that $2g+m\geq 3$. An ordering of the boundary components of Σ canonically determines an orientation of the Deligne-Mumford moduli space $\mathcal{M}_\Sigma$.*

Proof. For each complex structure $\mathfrak{j}$ on Σ, there is a unique metric $g_{\mathfrak{j}}$ on Σ of constant scalar curvature -1 such that each boundary component $(\partial\Sigma)_i$ of Σ is a $g_{\mathfrak{j}}$-geodesic. We denote by $L_i(\mathfrak{j})$ the length of $(\partial\Sigma)_i$ with respect to $g_{\mathfrak{j}}$. The boundary length map

$$\mathcal{M}_\Sigma \longrightarrow (\mathbb{R}^+)^m, \qquad [\mathfrak{j}] \longrightarrow \big(L_1(\mathfrak{j}),\ldots,L_m(\mathfrak{j})\big), \tag{2.10}$$

is a fibration; the fiber over a point $(L_1,\ldots,L_m)$ is the moduli space $\mathcal{M}_\Sigma(L_1,\ldots,L_m)$ of bordered Riemann surfaces with fixed lengths of the boundary components equal to $(L_1,\ldots,L_m)$. The latter moduli space carries a Weil-Petersson volume form; see [25, Section 2]. Thus, it is canonically oriented. By the homotopy exact sequence for this fibration, the fundamental groups of the fiber and the total space are isomorphic by the inclusion homomorphism and so every loop homotopes to a fiber. Since each fiber and its normal bundle are canonically oriented, the total space is also canonically oriented. □

Lemma 2.2. *Let Σ be an oriented genus g surface with m boundary components so that $2g+m\ge 3$ and $\overline{\Sigma}$ be the same surface with the opposite orientation. The sign of the diffeomorphism*

$$\mathcal{C}_\Sigma\colon \mathcal{M}_\Sigma \longrightarrow \mathcal{M}_{\bar{\Sigma}}, \qquad [\mathfrak{j}] \longrightarrow [-\mathfrak{j}],$$

with respect to the orientations of Lemma 2.1 is $(-1)^{3g-3+m}$.

Proof. The diffeomorphism $\mathcal{C}_\Sigma$ induces a diffeomorphism $\widetilde{\mathcal{C}}_{\Sigma;\mathbf{1}}$ between the Teichmüller spaces $\mathcal{T}_\Sigma(\mathbf{1})$ and $\mathcal{T}_{\overline{\Sigma}}(\mathbf{1})$ of bordered surfaces with unit length boundary components. It is sufficient to determine the sign of $\widetilde{\mathcal{C}}_{\Sigma;\mathbf{1}}$ in the Fenchel-Nielsen coordinates

$$(\ell_1,\vartheta_1,\ldots,\ell_{3g-3+m},\vartheta_{3g-3+m}) \qquad \text{and} \qquad (\bar{\ell}_1,\bar{\vartheta}_1,\ldots,\bar{\ell}_{3g-3+m},\bar{\vartheta}_{3g-3+m})$$

on these spaces obtained from the same pair-of-pants decomposition of $\Sigma=\bar{\Sigma}$; see [1, (II.3.2)], for example. Since $g_{-\mathfrak{j}}=g_{\mathfrak{j}}$, but the orientations of the cutting circles are reversed,

$$\widetilde{\mathcal{C}}_{\Sigma;\mathbf{1}}(\ell_1,\vartheta_1,\ldots,\ell_{3g-3+m},\vartheta_{3g-3+m}) = (\ell_1,-\vartheta_1,\ldots,\ell_{3g-3+m},-\vartheta_{3g-3+m})\,.$$

The claim now follows from [25, Theorem 2.1]. □

Corollary 2.3. *Let Σ be an oriented genus g surface with m boundary components so that $2g+m\ge 3$. If $h\colon \Sigma\longrightarrow\Sigma$ is an orientation-reversing diffeomorphism preserving the boundary components of Σ, the sign of DM_h is $(-1)^{g+m-1}$.*

Proof. Since h preserves the boundary components and $g_{-h^*\mathfrak{j}} = h^*g_{\mathfrak{j}}$, DM_h acts on the fibers of (2.10). By Lemma 2.1, it is thus sufficient to show that the diffeomorphism

$$\mathcal{T}_\Sigma(\mathbf{1}) \longrightarrow \mathcal{T}_{\overline{\Sigma}}(\mathbf{1}), \qquad [\mathfrak{j}] \longrightarrow [h^*\mathfrak{j}],$$

is orientation-preserving with respect to the Weil-Petersson orientations. This diffeomorphism takes the Fenchel-Nielsen coordinates with respect to the pair of pants decomposition $\mathcal{P}$ to the Fenchel-Nielsen coordinates with respect to the pair of pants decomposition $h^{-1}(\mathcal{P})$. The claim now follows from [25, Theorem 2.1]. □

Lemma 2.4. *Let Σ be an oriented genus g surface with m boundary components so that $2g+m\ge 3$. For every pair of distinct boundary components $(\partial\Sigma)_i$ and $(\partial\Sigma)_j$ of Σ, there exists an orientation-preserving diffeomorphism h_{ij} of Σ interchanging $(\partial\Sigma)_i$ and $(\partial\Sigma)_j$ and preserving the remaining boundary components of Σ such that $\mathrm{DM}_{h_{ij}}$ is an orientation-reversing diffeomorphism.*

Proof. Choose a pair of pants decomposition of Σ such that $(\partial\Sigma)_i$ and $(\partial\Sigma)_j$ are contained in the same pair of paints; we denote the latter by Σ_{ij}. Let $h_{ij}:\Sigma\longrightarrow\Sigma$ be an orientation-preserving diffeomorphism which restricts to the identity on an open neighborhood of $\overline{\Sigma-\Sigma_{ij}}$ in Σ and interchanges $(\partial\Sigma)_i$ and $(\partial\Sigma)_j$. We show below that the induced diffeomorphism $\mathrm{DM}_{h_{ij}}$ is orientation-reversing.

The diffeomorphism $\mathrm{DM}_{h_{ij}}$ naturally lifts to a diffeomorphism $\widetilde{\mathrm{DM}}_{h_{ij}}$ on the Teichmüller space $\mathcal{T}_\Sigma$. It is sufficient to show that the action of $\widetilde{\mathrm{DM}}_{h_{ij}}$ on the Fenchel-Nielsen coordinates

$$(\ell_1,\vartheta_1,\ldots,\ell_{3g-3+m},\vartheta_{3g-3+m},L_1,\ldots,L_m) \tag{2.11}$$

on $\mathcal{T}_\Sigma$ is orientation-reversing. Since $g_{h^*\mathfrak{j}}=h^*g_{\mathfrak{j}}$ in the notation of the proof of Lemma 2.1, $\widetilde{\mathrm{DM}}_{h_{ij}}$ preserves the lengths ℓ_k of the cutting circles for the pair of pants decomposition and the lengths L_k of the boundary circles with $k\neq i,j$; the lengths L_i and L_j get interchanged by $\widetilde{\mathrm{DM}}_{h_{ij}}$. This establishes the claim in the case $\Sigma=\Sigma_{ij}$, i.e. $g=0$ and $m=3$. In the other cases, $\widetilde{\mathrm{DM}}_{h_{ij}}$ also preserves the twisting parameters ϑ_k associated with the cutting circles other than $\partial\Sigma_{ij}-\partial\Sigma$. In the description of [1, (II.3.2)], $\widetilde{\mathrm{DM}}_{h_{ij}}$ interchanges the origins of the unique shortest geodesics in Σ_{ij} from this boundary component to $(\partial\Sigma)_i$ and $(\partial\Sigma)_j$ and thus changes the associated twisting parameter ϑ_k by π. Thus, the action of $\widetilde{\mathrm{DM}}_{h_{ij}}$ on the coordinates (2.11) is orientation-reversing. □

Proposition 2.5. *Let Σ be an oriented genus g surface with m boundary components so that $2g+m\geq3$.*

(1) If $h:\Sigma\longrightarrow\Sigma$ is an orientation-preserving diffeomorphism, the sign of the automorphism DM_h on $\mathcal{M}_\Sigma$ is $(-1)^{\mathrm{sgn}_h}$.

(2) If $h:\Sigma\longrightarrow\Sigma$ is an orientation-reversing diffeomorphism, the sign of the automorphism DM_h on $\mathcal{M}_\Sigma$ is $(-1)^{g+m-1+\mathrm{sgn}_h}$.

Proof. By Lemma 2.4, there exists an orientation-preserving diffeomorphism $\hat{h}:\Sigma\longrightarrow\Sigma$ such that the diffeomorphism $\hat{h}^{-1}\circ h$ preserves the boundary components of Σ and

$$\mathrm{sgn}\,\mathrm{DM}_{\hat{h}}=(-1)^{\mathrm{sgn}_h}\,.$$

If h is orientation-preserving, $\hat{h}^{-1}\circ h$ is then an element of $\mathcal{D}_\Sigma$ and so

$$\mathrm{DM}_h\circ\mathrm{DM}_{\hat{h}}^{-1}=\mathrm{DM}_h\circ\mathrm{DM}_{\hat{h}^{-1}}=\mathrm{DM}_{\hat{h}^{-1}\circ h}=\mathrm{id}_{\mathcal{M}_\Sigma}\,.$$

This establishes the first claim.

By Corollary 2.3 and Lemma 2.4, there exists an orientation-reversing diffeomorphism $\hat{h}:\Sigma\longrightarrow\Sigma$ such that the diffeomorphism $\hat{h}^{-1}\circ h$ preserves the boundary components of Σ and

$$\mathrm{sgn}\,\mathrm{DM}_{\hat{h}}=(-1)^{g+m-1+\mathrm{sgn}_h}\,.$$

If h is orientation-reversing, $\hat{h}^{-1}\circ h$ is then an element of $\mathcal{D}_\Sigma$ and so $\mathrm{sgn}\,\mathrm{DM}_h=\mathrm{sgn}\,\mathrm{DM}_{\hat{h}}$ as in the previous case. This establishes the second claim. □

By [15, Lemma 6.1], the conclusion of Proposition 2.5 also applies to the moduli space of domains with crosscaps, $\mathcal{M}^c_{\Sigma,\mathbf{0}}$.

2.3 Topological observations

In light of [11, Theorem 8.1.1], [21, Theorem 6.36], and [33, Theorem 1.1], the vanishing of $w_2(L)$ or $w_2(L)+w_1(L)^2$ modulo the image of $H^2(X;\mathbb{Z}_2)$ in $H^2(L;\mathbb{Z}_2)$ plays an important role in the orientability question for moduli spaces of J-holomorphic maps from bordered Riemann surfaces to a symplectic manifold X with boundary mapping to a Lagrangian submanifold L; see Section 5.2 for more details. However, as can be seen immediately from [13, Theorem 1.1], it is in fact the vanishing of $w_2(L)$ modulo the image of $H^2(X;\mathbb{Z}_2)$ in $H^2(L;\mathbb{Z}_2)$ *and* the elements of $H^2(L;\mathbb{Z}_2)$ vanishing on all tori which is relevant to the orientability question. By [15, Theorem 1.1], the situation for moduli spaces of J-holomorphic maps from Riemann sh-surfaces is similar. In this section, we study some topological aspects of classes in $H^2(M;\mathbb{Z}_2)$, for a topological space M, vanishing on all maps from tori to M and classes vanishing on all maps from closed oriented surfaces to M. By Lemma 2.7, the latter are often squares of classes from $H^1(M;\mathbb{Z}_2)$.

Throughout this paper, we take $\mathbb{I}=[0,1]$. We recall that every compact connected unorientable surface Σ is the connected sum of m copies of $\mathbb{RP}^2$ and

$$H_1(\Sigma;\mathbb{Z}) \approx \mathbb{Z}^{m-1}\oplus\mathbb{Z}_2$$

for some $m\in\mathbb{Z}$; see [27, Theorem 77.5]. We begin with two observations made in [15].

Lemma 2.6 ([15, Lemma 2.2]). *Let Σ be a compact connected unorientable surface and $b_\Sigma\in H_1(\Sigma;\mathbb{Z})$ be the nontrivial torsion class. If $\kappa\in H^1(\Sigma;\mathbb{Z}_2)$,*

$$\big\langle \kappa^2,[\Sigma]_{\mathbb{Z}_2}\big\rangle = \langle \kappa, b_\Sigma\rangle\,, \tag{2.12}$$

where $[\Sigma]_{\mathbb{Z}_2}\in H_2(\Sigma;\mathbb{Z}_2)$ is the fundamental class with $\mathbb{Z}_2$-coefficients.

Lemma 2.7 ([15, Corollary 2.3]). *For any topological space M,*

$$\big\{w\in H^2(M;\mathbb{Z}_2)\colon\ w(B)=0\ \forall\, B\in H_2(M;\mathbb{Z})\big\}\supset\big\{\kappa^2\colon\ \kappa\in H^1(M;\mathbb{Z}_2)\big\}.$$

If $H_1(M;\mathbb{Z})$ is finitely generated, the reverse inclusion holds if and only if $H_1(M;\mathbb{Z})$ has no 4-torsion.

Definition 2.8. Let M be a topological space.

(1) A free homotopy class $b\in\pi_1(M)$ is a Klein boundary if there exists a continuous map $F\colon\mathbb{I}\times S^1\longrightarrow M$ such that

$$\big[F|_{0\times S^1}\big]=b\in\pi_1(M) \qquad\text{and}\qquad F|_{1\times S^1}=F|_{0\times S^1}\circ\mathfrak{c}_{S^1}\,,$$

where $\mathfrak{c}_{S^1}\colon S^1\longrightarrow S^1$ is the restriction of the standard conjugation on $\mathbb{C}$.

(2) A class $w\in H^2(M;\mathbb{Z}_2)$ is atorical (resp. spin) if $f^*w=0$ for every continuous map $f\colon\mathbb{T}\longrightarrow M$ (resp. for every closed oriented surface Σ and continuous map $f\colon\Sigma\longrightarrow M$).

If b is a Klein boundary, $2[b]=0\in H_1(M;\mathbb{Z})$. Conversely, if $[b]\in H_1(M;\mathbb{Z})$ and $2[b]=0$, there exist

- a compact oriented surface Σ with two boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$,

- orientation-preserving diffeomorphisms $\varphi_1\colon S^1 \longrightarrow (\partial\Sigma)_1$ and $\varphi_2\colon S^1 \longrightarrow (\partial\Sigma)_2$, and
- a continuous map $F\colon \Sigma \longrightarrow M$ such that $\big[F\circ\varphi_1\big]=[b]\in H_1(M;\mathbb Z)$ and $F\circ\varphi_1 = F\circ\varphi_2$.

Such a map F descends to a continuous map $\hat F$ from the unorientable surface

$$\hat\Sigma \equiv \Sigma/\!\sim, \qquad z\sim\varphi_1(\varphi_2^{-1}(z)) \quad \forall\, z\in(\partial\Sigma)_2\,.$$

The image of each boundary component $(\partial\Sigma)_i$ in $\hat\Sigma$ represents the nonzero two-torsion element $b_{\hat\Sigma}$ of $H_1(\hat\Sigma;\mathbb Z)$ and $\hat F_* b_{\hat\Sigma}=[b]$.

If $b\in\pi_1(M)$ is a Klein boundary, a map F as in Definition 2.8(1) descends to a continuous map from the Klein bottle,

$$\hat F\colon (\mathbb I\times S^1)/\!\sim\longrightarrow M, \qquad \text{where} \quad (1,z)\sim(0,\bar z)\ \forall\, z\in S^1\subset\mathbb C, \quad \hat F\big([s,z]\big)=F(s,z),$$

such that the loop $z\longrightarrow F(0,z)$ represents $b\in\pi_1(M)$.

Lemma-Definition 2.9. *Suppose M is a topological space.*

(1) Let $b\in\pi_1(M)$ be a Klein boundary and $w\in H^2(M;\mathbb Z_2)$ be atorical. The number

$$\lfloor w,b\rfloor \equiv \big\langle w, \hat F_*[\mathbb K]_{\mathbb Z_2}\big\rangle \in \mathbb Z_2, \tag{2.13}$$

where $\hat F\colon \mathbb K\longrightarrow M$ is induced by a map F as in Definition 2.8(1), is independent of the choice of F.

(2) Let $b\in H_1(M;\mathbb Z)$ be a two-torsion class and $w\in H^2(M;\mathbb Z_2)$ be spin. The number

$$\lfloor w,b\rfloor \equiv \big\langle w, \hat F_*[\hat\Sigma]_{\mathbb Z_2}\big\rangle \in \mathbb Z_2, \tag{2.14}$$

where $\hat F\colon \hat\Sigma\longrightarrow M$ is a continuous map from an unorientable surface $\hat\Sigma$ such that $\hat F_* b_{\hat\Sigma}=b$ for the unique nonzero two-torsion element $b_{\hat\Sigma}$ of $H_1(\hat\Sigma;\mathbb Z)$, is independent of the choice of $\hat F$.

(3) In either case,

$$\lfloor \kappa^2,b\rfloor = \langle\kappa,b\rangle \qquad \forall\ \kappa\in H^1(M;\mathbb Z). \tag{2.15}$$

Proof. (1) Let $F,F'\colon \mathbb I\times S^1\longrightarrow M$ be continuous maps such that

$$\big[F|_{0\times S^1}\big]=\big[F'|_{0\times S^1}\big]=b\in\pi_1(M), \qquad F|_{1\times S^1}=F|_{0\times S^1}\circ\mathfrak c_{S^1}, \qquad F'|_{1\times S^1}=F'|_{0\times S^1}\circ\mathfrak c_{S^1}\,.$$

The class $\hat F_*[\mathbb K]_{\mathbb Z_2}+\hat F'_*[\mathbb K]_{\mathbb Z_2}$ is represented by a continuous map $f\colon\mathbb T\longrightarrow M$, obtained by connecting the images of $F|_{0\times S^1}$ and $F'|_{0\times S^1}$ by cylinders. Since w is atorical, it follows that

$$\big\langle w,\hat F_*[\mathbb K]_{\mathbb Z_2}\big\rangle+\big\langle w,\hat F'_*[\mathbb K]_{\mathbb Z_2}\big\rangle=\big\langle w, f_*[\mathbb T]_{\mathbb Z_2}\big\rangle=0,$$

and so $\lfloor w,b\rfloor$ is well-defined.

(2) The proof is similar.
(3) This follows from (2.12). □

If $f: \mathbb{RP}^2 \longrightarrow M$ is a continuous map and $\alpha: S^1 \longrightarrow \mathbb{RP}^2$ represents the nonzero element of $H_1(\mathbb{RP}^2;\mathbb{Z})$, then $f\circ\alpha$ represents a Klein boundary b in M. A map F as in Definition 2.8(1) can be obtained by precomposing f with a map $g\colon \mathbb{I}\times S^1 \longrightarrow \mathbb{RP}^2$ such that

$$g|_{1\times S^1} = g|_{0\times S^1}\circ \mathfrak{c}_{S^1} \qquad \text{and} \qquad g_*[0\times S^1] = [\alpha] \in H_1(\mathbb{RP}^2;\mathbb{Z}).$$

In this case,

$$\lfloor w, b\rfloor = \big\langle w, f_*[\mathbb{RP}^2]_{\mathbb{Z}_2}\big\rangle \in \mathbb{Z}_2$$

for any $w\in H^2(M;\mathbb{Z}_2)$ atorical. The map on the $\mathbb{Z}_2$-quotients induced by the map $f_{\mathbb{P}^1,\mathbb{K}}$ in the proof of Lemma 3.4 is an example of such a map g.

The two-torsion classes in $H_1(M;\mathbb{Z})$ that are not Klein boundaries play no special role in the orientability problem in real Gromov-Witten theory, but unfortunately there appears to be no simple algebraic characterization of Klein boundaries. Similarly, there appears to be no simple algebraic characterization of atorical classes in $H^2(M;\mathbb{Z}_2)$, which are the most relevant to the orientability problem in open and real Gromov-Witten theory, but Lemma 2.7 provides such a characterization for the smaller collection of spin classes in many cases. Since every element of $H_2(M;\mathbb{Z})$ can be represented by a continuous map $F\colon \Sigma\longrightarrow M$ for a closed oriented surface, a class $w\in H^2(M;\mathbb{Z}_2)$ is spin if and only if w vanishes on the image of $H_2(M;\mathbb{Z})$ in $H_2(M;\mathbb{Z}_2)$, under the homomorphism induced by the surjective homomorphism $\mathbb{Z}\longrightarrow\mathbb{Z}_2$. By the Universal Coefficient Theorem for Cohomology [26, Theorem 53.1], there is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}\big(H_1(M;\mathbb{Z}),\mathbb{Z}_2\big) \longrightarrow H^2(M;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\mathbb{Z}_2\big) \longrightarrow 0\,.$$

Thus, the spin classes $w\in H^2(M;\mathbb{Z})$ are the elements of the group

$$\mathrm{Ext}\big(H_1(M;\mathbb{Z}),\mathbb{Z}_2\big) \equiv \mathrm{Hom}\big(B_1(M),\mathbb{Z}_2\big)\big/\big\{\eta|_{B_1(M)}\colon\ \eta\in\mathrm{Hom}(Z_1(M),\mathbb{Z}_2)\big\}, \tag{2.16}$$

where $B_1(M)$ and $Z_1(M)$ are the group of boundaries of 2-chains and the group of 1-cycles, respectively. If a one-cycle b represents a two-torsion element of $H_1(M;\mathbb{Z})$, $2b\in B_1(M)$ and

$$\lfloor w,[b]\rfloor = \tilde{w}(2b)$$

for any $\tilde{w}\in\mathrm{Hom}(B_1(M),\mathbb{Z}_2)$ representing w.

3 Equivariant cohomology

In this section, we recall basic notions in equivariant cohomology, in the case the group is $\mathbb{Z}_2$, and apply them to real bundle pairs. We classify the real bundle pairs over the torus and the Klein bottle, with certain fixed-point-free involutions, and compute w_2^{ϕ} of all rank 1 real bundle pairs over symmetric surfaces with fixed-point-free involutions; see Lemma 3.3 and Corollary 3.5, respectively. We conclude with two examples illustrating the intriguing nature of w_2^{ϕ} of real bundle pairs.

3.1 Basic notions

The group $\mathbb{Z}_2$ acts freely on the contractible space $\mathbb{E}\mathbb{Z}_2 \equiv S^\infty$ with the quotient $\mathbb{B}\mathbb{Z}_2 \equiv \mathbb{RP}^\infty$. An involution $\phi\colon M \longrightarrow M$, where M is a topological space, corresponds to a $\mathbb{Z}_2$-action on M. We denote by

$$\mathbb{B}_\phi M = \mathbb{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} M$$

the corresponding Borel construction and by

$$H^*_\phi(M) \equiv H^*(\mathbb{B}_\phi M; \mathbb{Z}_2), \quad H^\phi_*(M) \equiv H_*(\mathbb{B}_\phi M; \mathbb{Z}_2), \quad H^\phi_*(M;\mathbb{Z}) \equiv H_*(\mathbb{B}_\phi M; \mathbb{Z})$$

the corresponding $\mathbb{Z}_2$-equivariant cohomology and homology of M. Let

$$M \longrightarrow \mathbb{B}_\phi M \longrightarrow \mathbb{B}\mathbb{Z}_2 = \mathbb{RP}^\infty \tag{3.1}$$

be the fibration induced by the projection $p_1\colon \mathbb{E}\mathbb{Z}_2 \times M \longrightarrow \mathbb{E}\mathbb{Z}_2$.

If $(V, \tilde\phi) \longrightarrow (M, \phi)$ is a real bundle pair,

$$\mathbb{B}_{\tilde\phi} V \equiv \mathbb{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} V \longrightarrow \mathbb{B}_\phi M$$

is a real vector bundle; this is the quotient of the vector bundle $p_2^* V \longrightarrow \mathbb{E}\mathbb{Z}_2 \times M$ by the natural lift of the free $\mathbb{Z}_2$-action on the base. Let

$$w_i^{\tilde\phi}(V) \equiv w_i(\mathbb{B}_{\tilde\phi} V) \in H^i_\phi(M)$$

be the $\mathbb{Z}_2$-equivariant Stiefel-Whitney classes of $V \longrightarrow M$. If M is a point and $V = \mathbb{C} = \mathbb{R} \oplus \mathfrak{i}\mathbb{R}$,

$$\mathbb{B}_{\tilde\phi} V = \mathbb{RP}^\infty \times \mathbb{R} \oplus \mathcal{O}_{\mathbb{RP}^\infty}(-1) \longrightarrow \mathbb{RP}^\infty,$$

where $\mathcal{O}_{\mathbb{RP}^\infty}(-1)$ is the tautological line bundle; thus, $w_1^{\tilde\phi}(V)$ is the generator of $H^1_\phi(\mathrm{pt})$ in this case. The non-equivariant Stiefel-Whitney classes of V are recovered from the equivariant Stiefel-Whitney classes of V by restricting to the fiber of the fibration (3.1). If $f\colon \Sigma \longrightarrow M$ is a continuous map commuting with involutions c on Σ and ϕ on M, the involution $\tilde\phi$ on V induces an involution $f^*\tilde\phi$ on f^*V lifting c and

$$w_i^{f^*\tilde\phi}(f^*V) = \{\mathbb{B}_{\phi,c} f\}^* w_i^{\tilde\phi}(V) \in H^i_c(\Sigma), \tag{3.2}$$

where

$$\mathbb{B}_{\phi,c} f\colon \mathbb{B}_c\Sigma \longrightarrow \mathbb{B}_\phi M, \qquad \{\mathbb{B}_{\phi,c} f\}\big([e,z]\big) = \big[e, f(z)\big], \tag{3.3}$$

is the map induced by f.

If an involution $c\colon \Sigma \longrightarrow \Sigma$ has no fixed points, the projection $p_2\colon \mathbb{E}\mathbb{Z}_2 \times \Sigma \longrightarrow \Sigma$ descends to a fibration

$$\mathbb{E}\mathbb{Z}_2 \longrightarrow \mathbb{B}_c\Sigma \stackrel{q}{\longrightarrow} \Sigma/\mathbb{Z}_2\,. \tag{3.4}$$

Since $\mathbb{E}\mathbb{Z}_2$ is contractible, this fibration is a homotopy equivalence, with a homotopy inverse provided by any section of q; in the case of the antipodal map (2.1), such a section is explicitly described in [15, Section 2.2]. In particular, q induces isomorphisms

$$q^*\colon H^*(\Sigma/\mathbb{Z}_2) \longrightarrow H^*_c(\Sigma), \qquad q_*\colon H^c_*(\Sigma;\mathbb{Z}) \longrightarrow H_*(\Sigma/\mathbb{Z}_2;\mathbb{Z}). \tag{3.5}$$

Any section of q embeds $\Sigma/\mathbb{Z}_2$ as a homotopy retract, and every two such sections are homotopic. Thus, if $f\colon\Sigma\longrightarrow M$ is a continuous map commuting with the involutions c on Σ and ϕ on M, we also denote by

$$\mathbb{B}_{\phi,c}f\colon \Sigma/\mathbb{Z}_2 \longrightarrow \mathbb{B}_\phi M$$

the composition of $\mathbb{B}_{\phi,c}f\colon \mathbb{B}_c\Sigma\longrightarrow\mathbb{B}_\phi M$ with any section of q; this is well-defined and unambiguous up to homotopy. If Σ is a compact manifold and $c\colon\Sigma\longrightarrow\Sigma$ is smooth, let

$$[\Sigma]^c_{\mathbb{Z}_2}=[\Sigma/\mathbb{Z}_2]_{\mathbb{Z}_2}\in H^c_*(\Sigma) \qquad\text{and}\qquad [f]^c_{\mathbb{Z}_2}=\{\mathbb{B}_{\phi,c}f\}_*[\Sigma]^c_{\mathbb{Z}_2}\in H^\phi_*(M)$$

denote the $\mathbb{Z}_2$-fundamental class of Σ and its equivariant push-forward, respectively; if $\Sigma/\mathbb{Z}_2$ is oriented, we similarly define

$$[\Sigma]^c_{\mathbb{Z}}=[\Sigma/\mathbb{Z}_2]_{\mathbb{Z}}\in H^c_*(\Sigma;\mathbb{Z}) \qquad\text{and}\qquad [f]^c_{\mathbb{Z}}=\{\mathbb{B}_{\phi,c}f\}_*[\Sigma]^c_{\mathbb{Z}}\in H^\phi_*(M;\mathbb{Z}).$$

If $p\colon(V,\tilde{c})\longrightarrow(\Sigma,c)$ is a real bundle pair, $V/\mathbb{Z}_2\longrightarrow\Sigma/\mathbb{Z}_2$ is a real vector bundle and

$$\mathbb{B}_{\tilde{\phi}}V \longrightarrow q^*(V/\mathbb{Z}_2)\equiv\big\{\big([e,x],[v]\big)\in\mathbb{B}_c\Sigma\times(V/\mathbb{Z}_2)\colon\ [x]=[p(v)]\big\},$$
$$[e,v]\longrightarrow\big([e,p(v)],[v]\big),$$

is a vector bundle isomorphism covering the identity on $\mathbb{B}_c\Sigma$. Thus,

$$w^{\tilde{c}}_i(V)=w_i\big(q^*(V/\mathbb{Z}_2)\big)=q^*w_i(V/\mathbb{Z}_2)\in H^i_c(\Sigma;\mathbb{Z}_2). \tag{3.6}$$

We next recall two statements from [15].

Proposition 3.1 ([15, Proposition 2.1])**.** *Let $(L_1,\tilde{\phi}_1),(L_2,\tilde{\phi}_2)\longrightarrow(M,\phi)$ be rank 1 real bundle pairs over a topological space with an involution. If M is paracompact,*

$$w_2^{\tilde{\phi}_1\otimes_{\mathbb{C}}\tilde{\phi}_2}(L_1\otimes_{\mathbb{C}}L_2)=w_2^{\tilde{\phi}_1}(L_1)+w_2^{\tilde{\phi}_2}(L_2). \tag{3.7}$$

Corollary 3.2 ([15, Corollary 2.4])**.** *Let (M,ϕ) be a topological space with an involution and $(L,\tilde{\phi})\longrightarrow(M,\phi)$ be a rank 1 real bundle pair.*

(1) If M is simply connected and $w_2(L)=0$, $w_2^{\tilde{\phi}}(L)$ is a square class.

(2) If M is paracompact and $(L,\tilde{\phi})$ admits a real square root, $w_2^{\tilde{\phi}}(L)=0$.

3.2 Real bundle pairs over surfaces

The involution $\mathfrak{a}_{\mathbb{K}}$ on the Klein bottle $\mathbb{K}$ given by

$$\mathfrak{a}_{\mathbb{K}}\colon \mathbb{K}=\mathbb{I}\times S^1/\!\sim\longrightarrow\mathbb{K},\quad [s,z]\longrightarrow[s,-z],\qquad\text{where}\quad (1,z)\sim(0,\bar{z})\ \ \forall\,z\in S^1\subset\mathbb{C},$$

has no fixed points. The next lemma classifies real bundle pairs over $(\mathbb{K},\mathfrak{a}_{\mathbb{K}})$ and $(\mathbb{T},\mathfrak{a}_{\mathbb{T}})$, where $\mathfrak{a}_{\mathbb{T}}=\mathrm{id}_{S^1}\times\mathfrak{a}$.

Lemma 3.3. *Let $(\Sigma,c)=(\mathbb{T},\mathfrak{a}_{\mathbb{T}}),(\mathbb{K},\mathfrak{a}_{\mathbb{K}})$ and $n\in\mathbb{Z}^+$. A rank n real bundle pair $(V,\tilde{c})\longrightarrow(\Sigma,c)$ is isomorphic to the trivial real bundle pair $(\Sigma\times\mathbb{C}^n,c\times\mathfrak{c}_{\mathbb{C}^n})$ if and only if*

$$\big\langle w_2^{\tilde{c}}(V),[\Sigma]^c\big\rangle=\begin{cases}0, & \textit{if } (\Sigma,c)=(\mathbb{T},\mathfrak{a}_{\mathbb{T}}),\\ \binom{n}{2}+2\mathbb{Z}, & \textit{if } (\Sigma,c)=(\mathbb{K},\mathfrak{a}_{\mathbb{K}}).\end{cases}$$

Proof. (1) Since the torus case is addressed by [15, Lemma 2.2] and the proof of [15, Lemma 2.3], it is enough to consider the case $(\Sigma,c)=(\mathbb{K},\mathfrak{a}_{\mathbb{K}})$. Let

$$V_\pm = \big(\mathbb{I}\times S^1\times\mathbb{C}\big)/\sim, \quad (0,z,v)\sim(1,\bar{z},\pm v) \quad \forall\, z\in S^1,\, v\in\mathbb{C},$$

where the involutions $\tilde{c}_\pm$ are induced by the standard conjugation on $\mathbb{C}$. By [7, Lemma 2.3], every rank n real bundle pair over $(S^1,\mathfrak{a})$ is trivial and admits precisely two homotopy classes of isomorphisms covering id_{S^1}. The non-trivial class contains the isomorphism given by the constant function on S^1 with the value equal to the diagonal matrix with one entry -1 and the remaining entries 1. By composing isomorphisms of the trivial rank n real bundle pair over $(S^1,\mathfrak{a})$ covering the conjugation $\mathfrak{c}_{S^1}$ on S^1 with $\mathfrak{c}_{S^1}\times\mathrm{id}_{\mathbb{C}^n}$, we see that there are also precisely two homotopy classes of such isomorphisms. Thus, $(V,\tilde{c})$ is isomorphic to one of the real bundle pairs

$$(V,\tilde{c}) = (nV_+,n\tilde{c}_+), (V_-\oplus(n-1)V_+, \tilde{c}_-\oplus(n-1)\tilde{c}_+);$$

note that $(nV_+,n\tilde{c}_+)$ is the trivial rank n real bundle pair. These real bundle pairs canonically decompose into two $\mathbb{Z}_2$-equivariant real vector bundles, induced by the real and imaginary axes in $\mathbb{C}$. By (3.6),

$$\big\langle w_2^{\tilde{c}}(V),[\mathbb{K}]^{\mathfrak{a}_{\mathbb{K}}}\big\rangle = \big\langle w_2(V/\mathbb{Z}_2),[\mathbb{K}/\mathbb{Z}_2]\big\rangle = \big\langle w_2(V_{\mathbb{R}}\oplus V_{\mathrm{i}\mathbb{R}}),[\mathbb{K}/\mathbb{Z}_2]\big\rangle,$$

where

$$V_{\mathbb{R}} = n(V_+)_{\mathbb{R}}, (V_-)_{\mathbb{R}}\oplus(n-1)(V_+)_{\mathbb{R}}, \qquad V_{\mathrm{i}\mathbb{R}} = n(V_+)_{\mathrm{i}\mathbb{R}}, (V_-)_{\mathrm{i}\mathbb{R}}\oplus(n-1)(V_+)_{\mathrm{i}\mathbb{R}}$$

are the $\mathbb{Z}_2$-quotients of the real and imaginary parts of V over the Klein bottle

$$\mathbb{K}/\mathbb{Z}_2 \equiv \mathbb{I}\times\mathbb{I}/\sim, \qquad (0,t)\sim(1,1-t), \quad (s,0)\sim(s,1) \quad \forall\, s,t\in\mathbb{I}. \tag{3.8}$$

We note that

$$\begin{aligned}(V_\pm)_{\mathbb{R}} &= \big(\mathbb{I}\times\mathbb{I}\times\mathbb{R}\big)/\sim, \qquad (0,t,v)\sim(1,1-t,\pm v), \quad (s,0,v)\sim(s,1,v) \qquad \forall\, s,t\in\mathbb{I},\, v\in\mathbb{R},\\ (V_\pm)_{\mathrm{i}\mathbb{R}} &= \big(\mathbb{I}\times\mathbb{I}\times\mathbb{R}\big)/\sim, \qquad (0,t,v)\sim(1,1-t,\pm v), \quad (s,0,v)\sim(s,1,-v) \quad \forall\, s,t\in\mathbb{I},\, v\in\mathbb{R}\end{aligned}$$

over the Klein bottle (3.8).

(2) The first $\mathbb{Z}_2$-homology of the Klein bottle (3.8) is generated by the loops

$$\alpha,\beta\colon \mathbb{I}\longrightarrow\mathbb{K}/\mathbb{Z}_2, \qquad \alpha(t)=[0,t], \quad \beta(s)=[s,1/2],$$

with the intersections $\alpha^2=0$, $\alpha\cdot\beta,\beta^2=1$; we denote the Poincaré duals of the homology classes represented by these loops by the same symbols. Since the restriction of $(V_-)_{\mathbb{R}}$ to α is trivial and to β is the Mobius band line bundle,

$$\big\langle w_1((V_-)_{\mathbb{R}}),\alpha\big\rangle=0, \quad \big\langle w_1((V_-)_{\mathbb{R}}),\beta\big\rangle=1 \quad\Longrightarrow\quad w_1((V_-)_{\mathbb{R}})=\alpha\in H^1(\mathbb{K}/\mathbb{Z}_2;\mathbb{Z}_2).$$

Since the restrictions of $(V_-)_{\mathrm{i}\mathbb{R}}$ to α and β are the Mobius band line bundles,

$$\big\langle w_1((V_-)_{\mathrm{i}\mathbb{R}}),\alpha\big\rangle=1, \quad \big\langle w_1((V_-)_{\mathrm{i}\mathbb{R}}),\beta\big\rangle=1 \quad\Longrightarrow\quad w_1((V_-)_{\mathrm{i}\mathbb{R}})=\beta\in H^1(\mathbb{K}/\mathbb{Z}_2;\mathbb{Z}_2).$$

We conclude that

$$w(V_-/\mathbb{Z}_2) = 1+(\alpha+\beta)+\alpha\beta = 1+(\alpha+\beta)+\beta^2\,.$$

On the other hand, the restriction of $(V_+)_{\mathrm{i}\mathbb{R}}$ to α is the Mobius band line bundle and to β is trivial. Thus,

$$\big\langle w_1((V_+)_{\mathrm{i}\mathbb{R}}),\alpha\big\rangle=1,\quad \big\langle w_1((V_+)_{\mathrm{i}\mathbb{R}}),\beta\big\rangle=0 \quad\Longrightarrow\quad w_1((V_+)_{\mathrm{i}\mathbb{R}})=\alpha+\beta\in H^1(\mathbb{K}/\mathbb{Z}_2;\mathbb{Z}_2).$$

Since $(V_+)_{\mathbb{R}}$ is the trivial line bundle, we conclude that

$$w(V_+/\mathbb{Z}_2)=1+(\alpha+\beta)\,.$$

Putting the two conclusions together, we find that

$$\begin{split} w_2\big(n(V_+/\mathbb{Z}_2)\big)&=\binom{n}{2}(\alpha+\beta)^2=\binom{n}{2}\beta^2\,,\\ w_2\big((V_-\oplus(n-1)V_+)/\mathbb{Z}_2\big)&=\binom{n}{2}(\alpha+\beta)^2+\beta^2=\binom{n}{2}\beta^2+\beta^2. \end{split}$$

Since $\beta^2\neq0$, this establishes the claim. □

Let $\eta\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1$ be as in (1.1) and $\tilde\eta$ be the lift of η to the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)\longrightarrow\mathbb{P}^1$ given by

$$\tilde\eta\big(\ell,(u,v)^{\otimes2}\big)=\big(\eta(\ell),(-\bar v,\bar u)^{\otimes2}\big)\qquad\forall\,\big(\ell,(u,v)\big)\in\mathcal{O}_{\mathbb{P}^1}(-1)\subset\mathbb{P}^1\times\mathbb{C}^2.$$

Lemma 3.4. *With notation as above,*

$$w_2^{\tilde\eta}(\mathcal{O}_{\mathbb{P}^1}(-2))\neq0\in H^2_\eta(\mathbb{P}^1).$$

Proof. The real map $f_{\mathbb{P}^1,\mathbb{K}}\colon(\mathbb{K},\mathfrak{a}_{\mathbb{K}})\longrightarrow(\mathbb{P}^1,\eta)$ given by

$$[s,z]\longrightarrow\Big[\cos\frac{\pi s}{2}+\mathrm{i}z\sin\frac{\pi s}{2},\mathrm{i}\sin\frac{\pi s}{2}+z\cos\frac{\pi s}{2}\Big]\qquad\forall\,s\in\mathbb{I},\ z\in S^1\subset\mathbb{C}$$

sends $0\times S^1$ to the circle $|u|=|v|$ in $\mathbb{P}^1$ and then spins S^1 around the points $[1,\pm1]\in\mathbb{P}^1$ so that $1\times S^1$ is again mapped to the circle $|u|=|v|$, but in the conjugate way. Let $\tilde f_{\mathbb{P}^1,\mathbb{K}}=f_{\mathbb{P}^1,\mathbb{K}}\circ q$, where $q\colon\mathbb{I}\times S^1\longrightarrow\mathbb{K}$ is the quotient map. We trivialize the bundle $\tilde f^*_{\mathbb{P}^1,\mathbb{K}}\mathcal{O}_{\mathbb{P}^1}(-2)$ by

$$\begin{split} \mathbb{I}\times S^1\times\mathbb{C}&\longrightarrow\tilde f^*_{\mathbb{P}^1,\mathbb{K}}\mathcal{O}_{\mathbb{P}^1}(-2),\\ (s,z,\lambda)&\longrightarrow\Big(s,z,\mathrm{i}\bar z\lambda\big(\cos\frac{\pi s}{2}+\mathrm{i}z\sin\frac{\pi s}{2},\mathrm{i}\sin\frac{\pi s}{2}+z\cos\frac{\pi s}{2}\big)^{\otimes2}\Big). \end{split}$$

The conjugation on $\mathbb{I}\times S^1\times\mathbb{C}$ induced by $\tilde\eta$ via this trivialization is the standard one:

$$\mathbb{I}\times S^1\times\mathbb{C}\longrightarrow\mathbb{I}\times S^1\times\mathbb{C},\qquad(s,z,\lambda)\longrightarrow(s,-z,\bar\lambda).$$

Since

$$f^*_{\mathbb{P}^1,\mathbb{K}}\mathcal{O}_{\mathbb{P}^1}(-2)=\tilde f^*_{\mathbb{P}^1,\mathbb{K}}\mathcal{O}_{\mathbb{P}^1}(-2)/\!\sim,\qquad(1,z,\lambda)\sim(0,\bar z,-\lambda)\ \ \forall\,(z,\lambda)\in S^1\times\mathbb{C}\,,$$

the real bundle pair $f^*_{\mathbb{P}^1,\mathbb{K}}\mathcal{O}_{\mathbb{P}^1}(-2)$ is $(V_-,\tilde c_-)\longrightarrow(\mathbb{K},\mathfrak{a}_{\mathbb{K}})$, in the notation of the proof of Lemma 3.3. Thus, by the proof of Lemma 3.3,

$$f^*_{\mathbb{P}^1,\mathbb{K}}\,w_2^{\tilde\eta}\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big)=w_2^{\tilde c_-}(V_-)\neq0.$$

This establishes the claim. □

Corollary 3.5. *Let $(\hat{\Sigma},\sigma)$ be a symmetric surface so that $\hat{\Sigma}^\sigma=\emptyset$. If $(L,\tilde{\phi})\longrightarrow(\hat{\Sigma},\sigma)$ is a rank 1 real bundle pair,*

$$\big\langle w_2^{\tilde{\phi}}(L),[\hat{\Sigma}]_{\mathbb{Z}_2}^{\sigma}\big\rangle=\frac{1}{2}\langle c_1(L),[\hat{\Sigma}]_{\mathbb{Z}}\rangle+2\mathbb{Z}.$$

Proof. By Lemma 3.4 and Proposition 3.1, the claim holds for the real bundle pairs

$$(\mathcal{O}_{\mathbb{P}^1}(-2a),\tilde{\eta}^a)\equiv(\mathcal{O}_{\mathbb{P}^1}(-2),\tilde{\eta})^{\otimes a}\longrightarrow(\mathbb{P}^1,\eta)$$

with $a\in\mathbb{Z}$. By [4, Propositions 4.2], these are all the rank 1 real bundle pairs over $(\hat{\mathbb{P}}^1,\eta)$. We thus assume that $\hat{\Sigma}\neq\mathbb{P}^1$ for the remainder of the proof.

By [28, Theorem 1.2], there exists a $\mathbb{Z}_2$-invariant circle $S\subset\hat{\Sigma}$. Let $U\subset\hat{\Sigma}$ be a $\mathbb{Z}_2$-invariant tubular neighborhood of S and $\hat{\Sigma}'$ be the two-nodal surface obtained from $\hat{\Sigma}$ by collapsing the boundary circles of U. The involution σ descends to an involution σ' on $\hat{\Sigma}'$, which has no fixed points, so that the quotient map $q:\hat{\Sigma}\longrightarrow\hat{\Sigma}'$ intertwines the two involutions. The image of $\bar{U}$ in $\hat{\Sigma}'$ is an irreducible component C of $\hat{\Sigma}'$ homeomorphic to $\mathbb{P}^1$ and preserved by σ'; let C' denote the remaining component of $\hat{\Sigma}'$. Given $a\in\mathbb{Z}$, let $(L',\tilde{\phi}')\longrightarrow(\hat{\Sigma}',\sigma')$ be the real bundle pair so that

$$\big\langle c_1(L'),[C]_{\mathbb{Z}}\big\rangle=-2a,\qquad (L',\phi')|_{C'}=(C'\times\mathbb{C},\sigma'|_{C'}\times\mathfrak{c}_{\mathbb{C}}).$$

By the previous paragraph,

$$\begin{aligned}\big\langle w_2^{\tilde{\phi}'}(L'),[\hat{\Sigma}']_{\mathbb{Z}_2}^{\sigma'}\big\rangle&=\big\langle w_2^{\tilde{\phi}'}(L'),[C]_{\mathbb{Z}_2}^{\sigma'}\big\rangle+\big\langle w_2^{\tilde{\phi}'}(L'),[C']_{\mathbb{Z}_2}^{\sigma'}\big\rangle\\&=\frac{1}{2}\big\langle c_1(L'),[C]_{\mathbb{Z}}\big\rangle+0+2\mathbb{Z}=\frac{1}{2}\big\langle c_1(L'),[\hat{\Sigma}']_{\mathbb{Z}}\big\rangle+2\mathbb{Z}\,.\end{aligned}$$

Since the degree of q is 1, it follows that

$$\begin{aligned}\big\langle w_2^{q^*\tilde{\phi}'}(q^*L'),[\hat{\Sigma}]_{\mathbb{Z}_2}^{\sigma}\big\rangle&=\big\langle q^*w_2^{\tilde{\phi}'}(L'),[\hat{\Sigma}]_{\mathbb{Z}_2}^{\sigma}\big\rangle=\big\langle w_2^{\tilde{\phi}'}(L'),q_*[\hat{\Sigma}]_{\mathbb{Z}_2}^{\sigma}\big\rangle=\big\langle w_2^{\tilde{\phi}'}(L'),[\hat{\Sigma}']_{\mathbb{Z}_2}^{\sigma'}\big\rangle\\&=\frac{1}{2}\big\langle c_1(L'),[\hat{\Sigma}']_{\mathbb{Z}}\big\rangle+2\mathbb{Z}=\frac{1}{2}\big\langle c_1(L'),q_*[\hat{\Sigma}]_{\mathbb{Z}}\big\rangle+2\mathbb{Z}=\frac{1}{2}\big\langle c_1(q^*L'),[\hat{\Sigma}]_{\mathbb{Z}}\big\rangle+2\mathbb{Z}.\end{aligned}$$

This establishes the claim for the real bundle pairs $(L,\tilde{\phi})=q^*(L',\tilde{\phi}')$, for each $a\in\mathbb{Z}$ as above. By [4, Propositions 4.2], these are all the rank 1 real bundle pairs over $(\hat{\Sigma},\sigma)$. □

Corollary 3.6. *Suppose (M,ϕ) is a topological space with an involution and $(L,\tilde{\phi})\longrightarrow(M,\phi)$ is a rank 1 real bundle pair so that $w_2^{\tilde{\phi}}(L)\in H^2_{\phi}(M)$ is a spin class. Let $\alpha\colon(S^1,\mathfrak{a})\longrightarrow(M,\phi)$ be a real map. If $f:\Sigma\longrightarrow M$ is a continuous map from an oriented bordered Riemann surface such that $\partial f=\alpha$, then*

$$\lfloor w_2^{\tilde{\phi}}(L),[\alpha]_{\mathbb{Z}_2}^{\phi}\rfloor=\frac{1}{2}\big\langle\hat{f}^*c_1(L),[\hat{\Sigma}]_{\mathbb{Z}}\big\rangle+2\mathbb{Z},$$

where $\hat{f}:(\hat{\Sigma},\sigma)\longrightarrow(M,\phi)$ is the double of f. If Σ is a disk, the conclusion holds even if $w_2^{\tilde{\phi}}(L)\in H^2_{\phi}(M)$ is just atorical.

Proof. The map $\mathbb{B}_{\phi,\sigma}\hat{f}:\hat{\Sigma}/\mathbb{Z}_2\longrightarrow\mathbb{B}_{\phi}M$ takes the image of $\partial\Sigma$ in $\hat{\Sigma}/\mathbb{Z}_2$, which represents the nonzero two-torsion class in $H_1(\hat{\Sigma}/\sigma;\mathbb{Z})$, to $\mathbb{B}_{\phi,\mathfrak{a}}\alpha$. Thus,

$$\begin{aligned}\lfloor w_2^{\tilde{\phi}}(L),[\alpha]_{\mathbb{Z}_2}^{\phi}\rfloor&=\lfloor w_2(\mathbb{B}_{\tilde{\phi}}L),[\mathbb{B}_{\phi,\mathfrak{a}}\alpha]_{\mathbb{Z}_2}\rfloor=\big\langle w_2(\mathbb{B}_{\tilde{\phi}}L),\{\mathbb{B}_{\phi,\sigma}\hat{f}\}_*[\hat{\Sigma}/\mathbb{Z}_2]_{\mathbb{Z}_2}\big\rangle\\&=\langle w_2^{\tilde{\phi}}(L),\{\mathbb{B}_{\phi,\sigma}\hat{f}\}_*[\hat{\Sigma}/\mathbb{Z}_2]_{\mathbb{Z}_2}\rangle=\langle\hat{f}^*w_2^{\tilde{\phi}}(L),[\hat{\Sigma}]_{\mathbb{Z}_2}^{\sigma}\rangle;\end{aligned}$$

see Section 2.3 for the second equality. The claim now follows from Corollary 3.5. □

3.3 Some examples

We now give two examples. The first describes rank 1 real bundle pairs $(V,\tilde{\phi})$ over a simply connected space (M,ϕ) such that $w_2^{\tilde{\phi}}(V)$ is a non-trivial square. The second example describes a rank 1 real bundle pair $(V,\tilde{\phi})$ so that $w_2^{\tilde{\phi}}(V)$ is atorical, but not spin. These examples imply that there is no simple description of the condition of $w_2^{\tilde{\phi}}(V)$ vanishing on the tori $\mathbb{B}_{\phi,\mathfrak{a}_{\mathbb{T}}}f$; see (3.3) and the beginning of Section 3.2 for the notation.

Example 3.7. For each $m\in\mathbb{Z}$, define

$$\begin{gathered}\eta_{2m-1}\colon \mathbb{P}^{2m-1}\longrightarrow\mathbb{P}^{2m-1}\qquad\text{by}\\ \big[W_1,W_2,\ldots,W_{2m-1},W_{2m}\big]\longrightarrow\big[-\overline{W}_2,\overline{W}_1,\ldots,-\overline{W}_{2m},\overline{W}_{2m-1}\big].\end{gathered}$$

In particular, $\eta_1=\eta$. Let $(\mathcal{O}_{\mathbb{P}^{2m-1}}(-2),\tilde{\eta}_{2m-1})\longrightarrow(\mathbb{P}^{2m-1},\eta_{2m-1})$ be the real bundle pair given by

$$\tilde{\eta}_{2m-1}\big(\ell,(W_1,\ldots,W_{2m})^{\otimes2}\big)=\big(\eta_{2m-1}(\ell),(-\overline{W}_2,\overline{W}_1,\ldots,-\overline{W}_{2m},\overline{W}_{2m-1})^{\otimes2}\big).$$

Fix $a\in\mathbb{Z}$ and take

$$(M,\phi)=(\mathbb{P}^{2m-1},\eta_{2m-1}),\quad (V,\tilde{\phi})=(\mathcal{O}_{\mathbb{P}^{2m-1}}(2a),\tilde{\eta}_{2m-1})\equiv(\mathcal{O}_{\mathbb{P}^{2m-1}}(-2),\tilde{\eta}_{2m-1})^{\otimes(-a)}.$$

By Corollary 3.2(2), $w_2^{\tilde{\phi}}(V)$ is a square class. Since the fixed locus of ϕ is empty, the natural projection

$$\mathbb{B}_\phi M\longrightarrow M/\mathbb{Z}_2$$

is a homotopy equivalence and $w_2^{\tilde{\phi}}(V)$ corresponds to $w_2(V/\mathbb{Z}_2)$. We show that the rank 2 vector bundle

$$V/\mathbb{Z}_2\equiv\mathcal{O}_{\mathbb{P}^{2m-1}}(2a)/\mathbb{Z}_2\longrightarrow M/\mathbb{Z}_2\equiv\mathbb{P}^{2m-1}/\mathbb{Z}_2 \tag{3.9}$$

is non-orientable, is non-split if $a\neq0$, and has a non-zero w_2 if a is odd. Since it is sufficient to verify these statements for the restriction of this bundle to $\mathbb{P}^1/\mathbb{Z}_2=\mathbb{RP}^2$, we can assume that $m=1$. An orientation on $\mathcal{O}_{\mathbb{P}^1}(2a)/\mathbb{Z}_2$ would lift to an orientation on $\mathcal{O}_{\mathbb{P}^1}(2a)$ preserved by $\tilde{\eta}_1$; since $\tilde{\eta}_1$ is orientation-reversing, $\mathcal{O}_{\mathbb{P}^1}(2a)/\mathbb{Z}_2$ is not orientable. A splitting of $\mathcal{O}_{\mathbb{P}^1}(2a)/\mathbb{Z}_2$ would induce a splitting of $\mathcal{O}_{\mathbb{P}^1}(2a)$ into two real line bundles, each of which must be trivial, since $\mathbb{P}^1$ is simply connected. Since the euler class of $\mathcal{O}_{\mathbb{P}^1}(2a)$ is $2a$, this is impossible if $a\neq0$, and so $\mathcal{O}_{\mathbb{P}^1}(2a)/\mathbb{Z}_2$ does not split as a sum of line bundles if $a\neq0$. By Corollary 3.5, $w_2^{\tilde{\eta}_1}(\mathcal{O}_{\mathbb{P}^1}(2a))\neq0$ if a is odd. We note that the bundles (3.9) are pairwise distinct, since an isomorphism between a pair of them would lift to an isomorphism of the bundles $\mathcal{O}_{\mathbb{P}^1}(2a)$ as real vector bundles.

Example 3.8. We now describe a rank 1 real bundle pair $(V,\tilde{\phi})\longrightarrow(M,\phi)$ so that $w_2^{\tilde{\phi}}(V)$ vanishes on every homology class represented by a torus (and in particular on the tori of the form $\mathbb{B}_{\phi,\mathfrak{a}_{\mathbb{T}}}f$), but not on the image of $H_2^\phi(M;\mathbb{Z})$ in $H_2^\phi(M)$ and thus is not a square class or even spin. Let $\pi:\Sigma_3\longrightarrow\Sigma_2$ be a double cover of a genus 2 closed oriented surface by a genus 3 surface and $\phi\colon\Sigma_3\longrightarrow\Sigma_3$ be the deck transformation (which is orientation-preserving). Since ϕ has no fixed points,

$$H^2_\phi(\Sigma_3)=H^2(\Sigma_2;\mathbb{Z}_2).$$

The trivial rank 1 real bundle pair $(\Sigma_3\times\mathbb{C},\tilde{\phi}_0)\longrightarrow(\Sigma_3,\phi)$ induces a rank 2 bundle over Σ_2 which splits into the line bundles

$$L_{\mathbb{R}}\equiv(\Sigma_3\times\mathbb{R})/\mathbb{Z}_2\approx\Sigma_2\times\mathbb{R},\qquad L_{\mathfrak{i}\mathbb{R}}\equiv(\Sigma_3\times\mathfrak{i}\mathbb{R})/\mathbb{Z}_2\longrightarrow\Sigma_2.$$

The line bundle L_{iR} is not trivial: it restricts to the Mobius band line bundle along any loop in Σ_2 not in the image of a loop from Σ_3. Thus, there exists a line bundle $L \longrightarrow \Sigma_2$ such that

$$w_1(L_{i\mathbb{R}})w_1(L) \neq 0 \qquad \Longrightarrow \qquad w_2\big((\Sigma_3 \times \mathbb{C})/\mathbb{Z}_2 \otimes_{\mathbb{R}} L\big) \neq 0.$$

The involution ϕ naturally lifts to an involution $\tilde{\phi}_L$ on $\pi^*L \longrightarrow \Sigma_3$. We take

$$(V, \tilde{\phi}) = (\Sigma_3 \times \mathbb{C} \otimes_{\mathbb{R}} \pi^*L, \tilde{\phi}_0 \otimes_{\mathbb{R}} \tilde{\phi}_L) \longrightarrow (\Sigma_3, \phi).$$

Since $w_2^{\tilde{\phi}}(V)$ corresponds to

$$w_2(V/\mathbb{Z}_2) = w_2\big((\Sigma_3 \times \mathbb{C})/\mathbb{Z}_2 \otimes_{\mathbb{R}} L\big) \in H^2(M; \Sigma_2),$$

$w_2^{\tilde{\phi}}(V)$ does not vanish on the image of $H_2^{\phi}(M;\mathbb{Z})$ in $H_2^{\phi}(M)$. However, it vanishes on the homology classes represented by tori, since every map $\mathbb{T} \longrightarrow \Sigma_2$ is of degree 0.

4 Relative signs

This section applies the relative sign principle introduced in [14, Section 6] in the basic case $(\Sigma, c) = (D^2, \mathrm{id}_{S^1})$ to arbitrary oriented sh-surfaces (Σ, c). For a real Cauchy-Riemann operator D on a real bundle pair $(V, \tilde{c})$ over an oriented sh-surface (Σ, c) and a point $x_i \in (\partial\Sigma)_i$ on each boundary component with $|c_i| = 0$, it is convenient to define

$$\widetilde{\det}(D) = \det(D) \otimes \bigotimes_{\substack{|c_i|=0 \\ \langle w_1(V^{\tilde{c}}), (\partial\Sigma)_i \rangle = 0}} \big(\Lambda_{\mathbb{R}}^{\mathrm{top}} V_{x_i}^{\tilde{c}}\big)^* ; \tag{4.1}$$

a similar, but not identical, twisted determinant line is introduced in [33, Section 5]. If $(V, \tilde{c})$ is induced from a real bundle pair over a smooth manifold (X, ϕ) with an orientable maximal totally real subbundle, the additional factors in $\widetilde{\det}(D)$ are systematically orientable and can be essentially ignored (dropping them would change (4.12) by the sign of the permutation induced by h on the set of boundary components with $|c_i| = 0$ and $\langle w_1(V^{\tilde{\phi}}), b_i \rangle = 0$). We study the changes in the orientation of $\widetilde{\det}(D)$ under various operations on D.

The (twisted) orientations of the determinant line of a real Cauchy-Riemann operator D and of the moduli space of J-holomorphic maps from an oriented sh-surface at a point u are determined by certain collections of trivializations, which we call orienting collections. The conjugation operation of Section 4.1 transforms such a collection to an orienting collection for the conjugate operator $\bar{D}$. There is a natural isomorphism between $\widetilde{\det}(D)$ and $\widetilde{\det}(\bar{D})$; we call the sign of this isomorphism with respect to an orienting collection and its conjugate the relative sign of the conjugation. We compute it in Proposition 4.1 below and show that in particular it is independent of the choice of the initial orienting collection. This computation is similar to the analogous computations in [12, Section 4] and [33, Section 2], but we do not assume the existence of any ambient structure. This leads to more general orientability results and fits well with studying local systems of orientations when the index bundles are not orientable. Even in the basic case of the disk, the approach of comparing relative orientations without imposing additional structure is more systematic, helps avoid the sort of mistake made in [10, Proposition 11.5], which continued on as [5, Proposition 2.1], and combines cases that appear to be unnecessarily separated in [12].

A diffeomorphism h of Σ, possibly interchanging the boundary components of Σ, likewise induces an isomorphism between $\widetilde{\det}(D)$ and $\widetilde{\det}(h^*D)$. If the operators D and h^*D are homotopic in a natural way, the orienting collection for D can be transferred along a path γ to an orienting collection for h^*D. We compute the sign of the natural isomorphism between $\widetilde{\det}(D)$ and $\widetilde{\det}(h^*D)$ with respect to the corresponding orientations in Proposition 4.2. It is again independent of the initial orienting collection, but does depend on the choice of the transferring path γ unless an appropriate determinant line bundle is orientable.

In Section 5.3, we determine analogous signs for the determinant line bundles of the moduli spaces of J-holomorphic maps from the disk and the annulus with the five possible boundary involutions. This allows us to establish more general versions of Theorems 1.1 and 1.2.

4.1 Conjugation

Let (Σ, c) be an oriented sh-surface, $\mathfrak{j}\in\mathcal{J}_c$, and $D=\bar\partial+A$ be a compatible real Cauchy-Riemann operator on a real bundle pair $(V,\tilde c)$ over (Σ,c). We denote by $\bar V$ the complex vector bundle obtained from V by multiplying the complex structure by (-1); $(\bar V,\tilde c)$ is still a real bundle pair over (Σ,c). Let

$$\bar D=\bar\partial+A\colon \Gamma(\Sigma;\bar V)^{\tilde c}\longrightarrow \Gamma^{0,1}_{-\mathfrak{j}}(\Sigma;\bar V).$$

Thus, D and $\bar D$ are the same operators between the same real vector spaces with different complex structures.

By [15, Proposition 5.4], an orientation of $\widetilde{\det}\, D$ is induced by

(OC1) for each boundary component $(\partial\Sigma)_i$ with $|c_i|=0$, a choice of trivialization of the canonically oriented vector bundle $V^{\tilde c}\oplus 3\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\tilde c}$ over $(\partial\Sigma)_i$,

(OC2) for each boundary component $(\partial\Sigma)_i$ with $|c_i|=1$, a choice of trivialization of the real bundle pair $(V,\tilde c)$ over $(\partial\Sigma)_i$.

We will call such a collection of trivializations an orienting collection for D. It induces an orienting collection of trivializations for $\bar D$:

($\mathrm{OC1}_c$) for each boundary component $(\partial\Sigma)_i$ with $|c_i|=0$, take the same choices of trivializations as for D;

($\mathrm{OC2}_c$) for each boundary component $(\partial\Sigma)_i$ with $|c_i|=1$, compose the corresponding choice for D with the standard conjugation on $\mathbb{C}^{\mathrm{rk}_{\mathbb{C}}V}$.

We will call the latter collection of trivializations the conjugate orienting collection.

Since D and $\bar D$ have the same kernel and cokernel, the identity maps

$$\ker\, D\longrightarrow \ker\, \bar D \qquad\text{and}\qquad \mathrm{cok}\, D\longrightarrow \mathrm{cok}\,\bar D$$

induce an isomorphism

$$\widetilde{\det}(D)\longrightarrow\widetilde{\det}(\bar D). \tag{4.2}$$

We call the sign of this isomorphism with respect to the orientations induced by an orienting collection for D and its conjugate for $\bar{D}$ the relative sign of the conjugation on D. By the next proposition, which generalizes the conclusion of the main step in the proof of [14, Lemma 5.1], it is independent of the orienting collection for D and of D itself.

For a real vector bundle $W \longrightarrow (\partial\Sigma)_i$, we define

$$\tilde{w}_1\big(W,(\partial\Sigma)_i\big) = \begin{cases} 1, & \text{if } \langle w_1(W), [(\partial\Sigma)_i]_{\mathbb{Z}_2}\rangle \neq 0; \\ 0, & \text{if } \langle w_1(W), [(\partial\Sigma)_i]_{\mathbb{Z}_2}\rangle = 0. \end{cases} \tag{4.3}$$

Proposition 4.1. *Let (Σ, c) be a genus g oriented sh-surface, $\mathfrak{j}\in\mathcal{J}_c$, and D be a real Cauchy-Riemann operator on a real bundle pair $(V,\tilde{c})$ over (Σ,c) compatible with $\mathfrak{j}$. The relative sign of the conjugation on D is $(-1)^{\epsilon_D}$, where*

$$\epsilon_D = \frac{1}{2}\bigg(\mu(V,\tilde{c}) + \sum_{|c_i|=0} \tilde{w}_1\big(V^{\tilde{c}},(\partial\Sigma)_i\big)\bigg) + \big(1-g+|c|_0+|c|_1\big)\mathrm{rk}_{\mathbb{C}}V + 2\mathbb{Z} \in \mathbb{Z}_2\,, \tag{4.4}$$

for every orienting collection for D.

Proof. Let $n=\mathrm{rk}\,V$. By the proof of [15, Theorem 1.1], $\det(D)$ is oriented by first pinching a circle near each boundary component $(\partial\Sigma)_i$ with $|c_i|=1$ to obtain

- a real Cauchy-Riemann operator D' on a real bundle pair $(V',\tilde{c}')$ over an oriented sh-surface $(\Sigma',\mathfrak{j}')$ with the boundary components $(\partial\Sigma)_i$ with $|c_i|=0$ and an interior marked point $p_i\in\Sigma'$ for each boundary component $(\partial\Sigma)_i$ with $|c_i|=1$ such that

$$g(\Sigma')=g(\Sigma), \qquad \big(V',\tilde{c}'\big)\big|_{(\partial\Sigma)_i} = \big(V,\tilde{c}\big)\big|_{(\partial\Sigma)_i} \text{ if } |c_i|=0, \quad \text{and} \quad \mu(V',\tilde{c}')=\mu(V,\tilde{c}),$$

- for each boundary component $(\partial\Sigma)_i$ with $|c_i|=1$, a real Cauchy-Riemann operator D_i on a real bundle pair $(V_i,\tilde{c}_i)$ over the disk with boundary $(\partial\Sigma)_i$ and boundary involution c_i such that

$$(V_i,\tilde{c}_i) = (V,\tilde{c})\big|_{(\partial\Sigma)_i} \qquad \text{and} \qquad \mu(V_i,\tilde{c}_i)=0.$$

The lines $\det(D)$ and $\det(\bar{D})$ are then oriented via isomorphisms

$$\begin{split} \det(D) &\approx \det(D') \otimes \bigotimes_{|c_i|=1}\big(\det(D_i)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(V'_{p_i})^*\big), \\ \det(\bar{D}) &\approx \det(\bar{D}') \otimes \bigotimes_{|c_i|=1}\big(\det(\bar{D}_i)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(\bar{V}'_{p_i})^*\big). \end{split} \tag{4.5}$$

For each i with $|c_i|=1$, $\det(D_i)$ is oriented by homotoping D_i to the standard real $\bar\partial$-operator on $D^2\times\mathbb{C}^n$ with the antipodal boundary involution via the chosen trivialization of $V|_{(\partial\Sigma)_i}$. The latter operator is surjective and its kernel consists of constant functions with values in $\mathbb{R}^n\subset\mathbb{C}^n$; this determines an orientation on $\det(D_i)$. In particular, the identity map from $\det(D_i)$ to $\det(\bar{D}_i)$ is orientation-preserving for each i with $|c_i|=1$. On the other hand, $\Lambda^{\mathrm{top}}_{\mathbb{R}}V'_{p_i}$ and $\Lambda^{\mathrm{top}}_{\mathbb{R}}\bar{V}'_{p_i}$ are oriented by the complex orientations of V' and $\bar{V}'$ and so the sign of the identity isomorphism between the last factors in (4.5) is $(-1)^n$ for each i with $|c_i|=1$. This accounts for $|c|_1 n$ in (4.4).

By the proof of [13, Theorem 1.1], $\det(D')$ and $\det(\bar{D}')$ in (4.5) are oriented via isomorphisms

$$\begin{aligned}\det(D')\otimes\det(D_1)^{\otimes 4}&\approx\det(D_{+3})\otimes\det(D_1),\\ \det(\bar{D}')\otimes\det(\bar{D}_1)^{\otimes 4}&\approx\det(\bar{D}_{+3})\otimes\det(\bar{D}_1),\end{aligned}\tag{4.6}$$

where D_1 and D_{+3} are real Cauchy-Riemann operators on the bundles

$$V_1\equiv\Lambda^{\mathrm{top}}_{\mathbb{C}}V',\ V_{+3}\equiv V'\oplus 3\Lambda^{\mathrm{top}}_{\mathbb{C}}V'\longrightarrow\Sigma'$$

with the involutions $\tilde{c}_1$ and $\tilde{c}_{+3}$ induced by $\tilde{c}'$ and $\bar{D}_1$ and $\bar{D}_{+3}$ are the conjugates of D_1 and D_{+3}, respectively. The identity map between the second factors on the left-hand sides of the two expressions in (4.6) is clearly orientation-preserving.

By the proof of [13, Proposition 3.1], $\det(D_{+3})$ is oriented by first pinching a circle near each boundary component $(\partial\Sigma')_i$ to obtain

- a real Cauchy-Riemann operator D^{cl}_{+3} on a bundle V^{cl}_{+3} over a closed Riemann surface Σ^{cl} with a marked point p_i for each boundary component $(\partial\Sigma')_i$ such that

$$g(\Sigma^{\mathrm{cl}})=g(\Sigma')=g(\Sigma)\qquad\text{and}\qquad 2\deg\big(V^{\mathrm{cl}}_{+3}\big)=\mu\big(V_{+3},\tilde{c}_{+3}\big)=4\mu\big(V,\tilde{c}\big)\,,$$

- for each boundary component $(\partial\Sigma')_i$, the standard Cauchy-Riemann operator $D_{+3;i}$ on the trivial bundle $D^2\times\mathbb{C}^{n+3}$ obtained by extending the chosen trivialization of

$$V'^{\tilde{c}'}\oplus 3\Lambda^{\mathrm{top}}_{\mathbb{R}}V'^{\tilde{c}'}=V^{\tilde{c}}\oplus 3\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\tilde{c}}$$

  over $(\partial\Sigma)_i$ to a tubular neighborhood.

The lines $\det(D_{+3})$ and $\det(\bar{D}_{+3})$ are then oriented via isomorphisms

$$\begin{aligned}\det(D_{+3})&\approx\det(D^{\mathrm{cl}}_{+3})\otimes\bigotimes_{|c_i|=0}\big(\det(D_{+3;i})\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(V^{\mathrm{cl}}_{+3}|_{p_i})^*\big),\\ \det(\bar{D}_{+3})&\approx\det(\bar{D}^{\mathrm{cl}}_{+3})\otimes\bigotimes_{|c_i|=0}\big(\det(\bar{D}_{+3;i})\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(\bar{V}^{\mathrm{cl}}_{+3}|_{p_i})^*\big).\end{aligned}\tag{4.7}$$

The line $\det(D^{\mathrm{cl}}_{+3})$ is oriented by deforming D^{cl}_{+3} to a complex Cauchy-Riemann operator on V^{cl}. Thus, the sign of the identity isomorphism between the first factors on the right-hand sides in (4.7) contributes

$$\mathrm{ind}_{\mathbb{C}}D^{\mathrm{cl}}_{+3}=\deg V^{\mathrm{cl}}_{+3}+\big(1-g(\Sigma^{\mathrm{cl}})\big)(n+3)=2\mu\big(V,\tilde{c}\big)+\big(1-g(\Sigma)\big)(n+3)$$

to (4.4). For each i with $|c_i|=0$, $D_{+3;i}$ is a surjective operator and its kernel consists of constant functions with values in $\mathbb{R}^{n+3}\subset\mathbb{C}^{n+3}$; this determines an orientation on $\det(D_{+3;i})$. In particular, the identity map from $\det(D_{+3;i})$ to $\det(\bar{D}_{+3;i})$ is orientation-preserving for each i with $|c_i|=0$. On the other hand, $\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\mathrm{cl}}_{+3}|_{p_i}$ and $\Lambda^{\mathrm{top}}_{\mathbb{R}}\bar{V}^{\mathrm{cl}}_{+3}|_{p_i}$ are oriented by the complex orientations of V^{cl}_{+3} and $\bar{V}^{\mathrm{cl}}_{+3}$ and so the sign of the identity isomorphism between the last factors in (4.7) is $(-1)^{n+3}$ for each i with $|c_i|=0$. This contributes $|c|_0(n+3)$ to (4.4).

By the proof of [13, Proposition 3.2], $\det(D_1)$ is also oriented by first pinching a circle near each boundary component $(\partial\Sigma')_i$ to obtain

- a real Cauchy-Riemann operator $D_1^{\rm cl}$ on a line bundle $V_1^{\rm cl}$ over $\Sigma^{\rm cl}$ such that

$$2\deg\,V_1^{\rm cl}=\mu(V',\tilde c')-\sum_{|c_i|=0}\tilde w_1\big(V'^{\tilde c},(\partial\Sigma')_i\big)=\mu(V,\tilde c)-\sum_{|c_i|=0}\tilde w_1\big(V^{\tilde c},(\partial\Sigma)_i\big)\,,$$

- for each boundary component $(\partial\Sigma')_i$, a real Cauchy-Riemann operator $D_{1;i}$ on a real bundle pair $(V_{1;i},\tilde c_{1;i})$ over the disk with boundary $(\partial\Sigma)_i$ and trivial boundary involution such that

$$(V_{1;i},\tilde c_{1;i})=(V_1,\tilde c_1)\big|_{(\partial\Sigma)_i}\qquad\text{and}\qquad \mu(V_{1;i},\tilde c_{1;i})\in\{0,1\}.$$

The lines $\det(D_1)$ and $\det(\bar D_1)$ are then oriented via isomorphisms

$$\begin{split}\det(D_1)&\approx\det(D_1^{\rm cl})\otimes\bigotimes_{|c_i|=0}\big(\det(D_{1;i})\otimes(V_1^{\rm cl}|_{p_i})^*\big),\\ \det(\bar D_1)&\approx\det(\bar D_1^{\rm cl})\otimes\bigotimes_{|c_i|=0}\big(\det(\bar D_{1;i})\otimes(\bar V_1^{\rm cl}|_{p_i})^*\big).\end{split}\tag{4.8}$$

The line $\det(D_1^{\rm cl})$ is oriented by deforming $D_1^{\rm cl}$ to a complex Cauchy-Riemann operator on the line bundle $V_1^{\rm cl}$. Thus, the sign of the identity isomorphism between the first factors on the right-hand sides in (4.8) contributes

$$\mathrm{ind}_{\mathbb C}D_1^{\rm cl}=\deg V_1^{\rm cl}+1-g(\Sigma^{\rm cl})=\frac12\bigg(\mu(V,\tilde c)-\sum_{|c_i|=0}\tilde w_1\big(V^{\tilde c},(\partial\Sigma)_i\big)\bigg)+1-g(\Sigma)$$

to (4.4). For each i with $|c_i|=0$, the identity isomorphism between the last factors in (4.8) is orientation-reversing because the lines $V_1^{\rm cl}|_{p_i}$ and $\bar V_1^{\rm cl}|_{p_i}$ are oriented by their complex orientations. This contributes $|c|_0$ to (4.4).

For each i with $|c_i|=0$ and $\mu(V_{1;i},\tilde c_{1;i})=0$, i.e. $\tilde w_1(V^{\tilde c},(\partial\Sigma)_i)=0$, the operator $D_{1;i}$ is surjective and the evaluation homomorphism

$$\ker D_{1;i}\longrightarrow V_{1;i}^{\tilde c_{1;i}}\big|_{x_i}=\Lambda^{\rm top}_{\mathbb R}V^{\tilde c}\big|_{x_i}$$

is an isomorphism and induces the same orientations on $\det(D_{1;i})$ and $\det(\bar D_{1;i})$ from any orientation of the right-hand side. For each i with $|c_i|=0$ and $\mu(V_{1;i},\tilde c_{1;i})=1$, i.e. $\tilde w_1(V^{\tilde c},(\partial\Sigma)_i)=1$, the operator $D_{1;i}$ is surjective and the evaluation homomorphism

$$\ker D_{1;i}\longrightarrow V_{1;i}^{\tilde c_{1;i}}\big|_{x_i}\oplus V_{1;i}^{\tilde c_{1;i}}\big|_{x_i'}=\Lambda^{\rm top}_{\mathbb R}V^{\tilde c}\big|_{x_i}\oplus\Lambda^{\rm top}_{\mathbb R}V^{\tilde c}\big|_{x_i'}\tag{4.9}$$

is an isomorphism for any $x_i'\in(\partial\Sigma)_i-x_i$. This isomorphism again induces orientations on $\det(D_{1;i})$ and $\det(\bar D_{1;i})$ from orientations of the right-hand side. However, in this case, the two orientations of the second component on the right-hand side of (4.9) are opposite, since they are obtained by transporting the same orientation of the first component in the positive direction along $(\partial\Sigma)_i$ with respect to the orientations induced by $\mathfrak j$ for $\det(D_{1;i})$ and $-\mathfrak j$ for $\det(\bar D_{1;i})$. Thus, in either case, the isomorphism between the first components in the i-th factors in (4.8) contributes $\tilde w_1(V^{\tilde c},(\partial\Sigma)_i)$ to (4.4). This completes the proof. □

In the case $|c|_1=0$, (4.4) agrees with the pin$^+$ formula of [33, Proposition 2.12]. In turn, the latter specializes to [12, Theorem 1.3] whenever Σ is the disk and $V^{\tilde c}$ is orientable.

4.2 Interchange of boundary components

Let (Σ,c) be an oriented sh-surface, $h\colon\Sigma\longrightarrow\Sigma$ be a diffeomorphism such that $h\circ c=c\circ h$ on $\partial\Sigma$, and $|h|,\mathrm{sgn}_h\in\{0,1\}$ be as at the beginning of Section 2.2. We define $\mathrm{sgn}_h^0\in\{0,1\}$ to be the sign of the permutation induced by h on the set of boundary components $(\partial\Sigma)_i$ with $|c_i|=0$. For $k=0,1$, let

$$\mathcal{T}_{h;k}=\mathbb{I}\times\bigsqcup_{|c_i|=k}(\partial\Sigma)_i/\!\sim,\qquad (1,z)\sim\big(0,c^{|h|}(h(z))\big)\ \ \forall\, z\in\bigsqcup_{|c_i|=k}(\partial\Sigma)_i\,,;$$

where $c^0\equiv\mathrm{id}$; this is a union of tori if $|h|=0$ and of tori and Klein bottles if $|h|=1$. The boundary involutions c_i induce an involution $c_{h;k}$ on $\mathcal{T}_{h;k}$, which is trivial if $k=0$ and has no fixed points if $k=1$.

If (Σ,c) and h are as above, (X,ϕ) is a manifold with an involution, $\mathbf{k}\equiv(k_1,\ldots,k_{|c|_0+|c|_1})$ is a tuple of non-negative integers, and $\mathbf{b}$ is as in (2.4), we will call

$$(u_0,\mathbf{x}_0,\mathfrak{j}_0),(u_1,\mathbf{x}_1,\mathfrak{j}_1)\in\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}\times\mathcal{J}_c$$

h-related if

$$u_1=\phi^{|h|}\circ u_0\circ h,\qquad h(\mathbf{x}_1)=\mathbf{x}_0,\quad\text{and}\quad \mathfrak{j}_1=(-1)^{|h|}h^*\mathfrak{j}_0.$$

If $\gamma\equiv(u_t,\mathbf{x}_t,\mathfrak{j}_t)$ is any path in $\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}\times\mathcal{J}_c$ such that $(u_0,\mathbf{x}_0,\mathfrak{j}_0)$ and $(u_1,\mathbf{x}_1,\mathfrak{j}_1)$ are h-related and $k=0,1$, let $[\gamma]_{h;k}\in H_2(X;\mathbb{Z}_2)$ denote the push-forward of the fundamental homology class of $\mathcal{T}_{h;k}$ with $\mathbb{Z}_2$-coefficients by the continuous map

$$\gamma_{h;k}\colon\ \mathcal{T}_{h;k}\longrightarrow X,\qquad [t,z]\longrightarrow u_t(z)\ \ \forall\,[t,z]\in\mathcal{T}_{h;k}\,;$$

this map is $\mathbb{Z}_2$-equivariant. Let $[\gamma]_{h;1}^c\in H_2^\phi(X)$ denote the equivariant push-forward of the $\mathbb{Z}_2$-equivariant fundamental class $[\mathcal{T}_{h;1}]_{\mathbb{Z}_2}^{c_{h;1}}\in H_2^{c_{h;1}}(\mathcal{T}_{h;1})$ by $\gamma_{h;1}$.

Let (Σ,c), h, (X,ϕ), and γ be as above, $(V,\tilde\phi)\longrightarrow(X,\phi)$ be a real bundle pair, D_0 and D_1 be the real Cauchy-Riemann operators on the real bundles pairs $u_0^*(V,\tilde\phi)$ and $u_1^*(V,\tilde\phi)$ induced from a connection ∇ and a deformation A on V as in the last paragraph of Section 2.1, and

$$(V^h,D_0^h)=\begin{cases}(V,D_0), & \text{if } h \text{ is orientation-preserving;}\\ (\bar V,\bar D_0), & \text{if } h \text{ is orientation-reversing.}\end{cases}$$

If $(u_0,\mathfrak{j}_0)$ and $(u_1,\mathfrak{j}_1)$ are h-related,

$$h^{-1}\colon(\Sigma,(-1)^{|h|}\mathfrak{j}_0)\longrightarrow(\Sigma,\mathfrak{j}_1)\qquad\text{and}\qquad u_0^*\tilde\phi^{|h|}\colon u_0^*(V^h,\tilde\phi)\longrightarrow u_1^*(V,\tilde\phi)\tag{4.10}$$

are isomorphisms and thus induce an isomorphism

$$\widetilde{\det}(D_0^h)\equiv\det(D_0^h)\otimes\bigotimes_{\substack{|c_i|=0\\ \langle u_0^*w_1(V^{\tilde\phi}),(\partial\Sigma)_i\rangle=0}}\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}V^{\tilde\phi}_{u_0(x_i)}\big)^*\longrightarrow\widetilde{\det}(D_1)\equiv\det(D_1)\otimes\bigotimes_{\substack{|c_i|=0\\ \langle u_1^*w_1(V^{\tilde\phi}),(\partial\Sigma)_i\rangle=0}}\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}V^{\tilde\phi}_{u_1(x_i)}\big)^*,\tag{4.11}$$

where x_i is the first marked point on the boundary component $(\partial\Sigma)_i$. The path γ can be used to transfer an orienting collection for D_0 to an orienting collection for D_1, which we will call *the γ-transferred orienting collection*. The next proposition describes the sign of the above isomorphism.

Proposition 4.2. *Suppose (X,ϕ) is a manifold with an involution, $(V,\tilde\phi)\longrightarrow(X,\phi)$ is a real bundle pair, (Σ,c) is an oriented sh-surface, $\mathbf{k}\equiv(k_1,\ldots,k_{|c|_0+|c|_1})$ is a tuple of non-negative integers, $\mathbf{b}$ is as in (2.4), $\gamma\equiv(u_t,\mathbf{x}_t,\mathfrak{j}_t)$ is a path in $\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}\times\mathcal{J}_c$, and $h:\Sigma\longrightarrow\Sigma$ is a diffeomorphism such that $(u_0,\mathbf{x}_0,\mathfrak{j}_0)$ and $(u_1,\mathbf{x}_1,\mathfrak{j}_1)$ are h-related. If $V^{\tilde\phi}\longrightarrow X^\phi$ is not orientable, assume that $k_i>0$ for every boundary component $(\partial\Sigma)_i$ such that $|c_i|=0$ and $\langle w_1(V^{\tilde\phi}),b_i\rangle=0$. Denote by D_0 and D_1 the real Cauchy-Riemann operators on the bundle pairs $u_0^*(V,\tilde\phi)$ and $u_1^*(V,\tilde\phi)$ induced as in Section 2.1 and choose an orienting collection for D_0. The sign of the isomorphism (4.11) is $(-1)^{\epsilon^{\tilde\phi}_{\gamma,h}}$, where*

$$\epsilon^{\tilde\phi}_{\gamma,h}=\big\langle w_2(V^{\tilde\phi}),[\gamma]_{h;0}\big\rangle+\big\langle w_2^{\Lambda^{\mathrm{top}}_{\mathbb{C}}\tilde\phi}(\Lambda^{\mathrm{top}}_{\mathbb{C}}V),[\gamma]^c_{h;1}\big\rangle+(\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h+\mathrm{sgn}^0_h+2\mathbb{Z}\in\mathbb{Z}_2\,, \tag{4.12}$$

if $\widetilde{\det}(\bar D_0)$ and $\widetilde{\det}(D_1)$ are oriented using the conjugate and γ-transferred orienting collections, respectively.

Proof. Let $n=\mathrm{rk}_{\mathbb{C}}V$. The action of h on the set of boundary components of Σ preserves the subsets

$$\big\{(\partial\Sigma)_i\colon\,|c_i|=0\big\}\qquad\text{and}\qquad\big\{(\partial\Sigma)_i\colon\,|c_i|=1\big\} \tag{4.13}$$

and thus breaks these subsets into cycles corresponding to topological components of $\mathcal{T}_{h;0}$ and $\mathcal{T}_{h;1}$. Since the maps h^{-1} and $u_0^*\tilde\phi^{|h|}$ in (4.10) are holomorphic and $\mathbb{C}$-linear, respectively, it is sufficient to compare the orientation of the right-hand side of (4.11) with the orientation induced by the push-forward of the orienting collection for D_0^h by $(h^{-1},u_0^*\tilde\phi^{|h|})$.

For a subset $(\partial\Sigma)_{i_1},\ldots,(\partial\Sigma)_{i_k}$ of boundary components of Σ such that $|c_{i_1}|=0$ and

$$h\big((\partial\Sigma)_{i_j}\big)=(\partial\Sigma)_{i_{j+1}}\qquad\forall\ j=1,\ldots,k\,, \tag{4.14}$$

where $i_{k+1}\equiv i_1$, let

$$\mathcal{T}=\mathbb{I}\times\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}/\!\sim,\qquad(1,z)\sim\big(0,c^{|h|}(h(z))\big)\ \ \forall\,z\in\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}; \tag{4.15}$$

this is a connected component of $\mathcal{T}_{h;0}$. The maps h^{-1} and $u_0^*\tilde\phi^{|h|}$ in (4.10) induce trivializations of

$$\big(u_1^*V^{\tilde\phi}\oplus3\Lambda^{\mathrm{top}}_{\mathbb{R}}u_1^*V^{\tilde\phi}\big)\big|_{(\partial\Sigma)_{i_j}}=h^*\big(u_0^*V^{\tilde\phi}\oplus3\Lambda^{\mathrm{top}}_{\mathbb{R}}u_0^*V^{\tilde\phi}\big)\big|_{(\partial\Sigma)_{i_j}} \tag{4.16}$$

from the original orienting collection for D_0; if $|h|=1$, this is the same trivialization as the one induced from the conjugate trivialization for $D_0^h=\bar D_0$. Since $\pi_1(\mathrm{SO}(n+3))=\mathbb{Z}_2$, there are two homotopy classes of such trivializations. If the γ-transferred trivialization of the left-hand side in (4.16) agrees with the original trivialization of the right-hand side up to homotopy, let $\epsilon_j=0$; otherwise, let $\epsilon_j=1$. The oriented vector bundle $\gamma_{h;0}^*(V^{\tilde\phi}\oplus3\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\tilde\phi})|_{\mathcal{T}}$ is then isomorphic to

$$\mathbb{I}\times\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}\times\mathbb{R}^{n+3}/\!\sim,\qquad(1,z,v)\sim\big(0,h(z),g(z)v\big)\ \ \forall\,z\in\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}\,,\ v\in\mathbb{R}^{n+3}\,,$$

for some $g\colon \bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j} \longrightarrow \mathrm{SO}(n+3)$ such that $g|_{(\partial\Sigma)_{i_j}}$ is homotopic to the constant map $\mathbb{I}_{n+3}$ if and only if $\epsilon_j=0$. Since the oriented vector bundles over $\mathcal{T}$ (of rank at least 3) are classified by the value of their w_2 in $H^2(\mathcal{T};\mathbb{Z}_2)\approx\mathbb{Z}_2$,

$$\epsilon_1+\ldots+\epsilon_k = \big\langle \gamma_{h;0}^* w_2(V^{\tilde{\phi}}\oplus 3\Lambda_{\mathbb{R}}^{\mathrm{top}}V^{\tilde{\phi}}), [\mathcal{T}]_{\mathbb{Z}_2}\big\rangle = \big\langle \gamma_{h;0}^* w_2(V^{\tilde{\phi}}), [\mathcal{T}]_{\mathbb{Z}_2}\big\rangle. \tag{4.17}$$

By [15, Proposition 5.4], changing the choice (OC1) for a given boundary component $(\partial\Sigma)_i$ with $|c_i|=0$ changes the orientation of the determinant line bundle. Since the two choices (OC1) for D_1 obtained from the trivializing collection for u_0 via h and γ-transfer agree if and only if $\epsilon_j=0$, the possible differences in these choices for $(\partial\Sigma)_{i_1},\ldots,(\partial\Sigma)_{i_k}$ contribute (4.17) to (4.12). Summing over all cycles of the action of h on the set $\{(\partial\Sigma)_i\colon |c_i|=0\}$ gives the first term on the right-hand side of (4.12). However, h interchanges the order of the $(n+3)$-dimensional target spaces of the evaluation isomorphisms orienting $\det(D_{+3;i})$ as below (4.7). This contributes $(n+3)$ times the sign of the permutation on $\{(\partial\Sigma)_i\colon |c_i|=0\}$ induced by h.

For a subset $(\partial\Sigma)_{i_1},\ldots,(\partial\Sigma)_{i_k}$ of boundary components of Σ such that $|c_{i_1}|=1$ and (4.14) holds, let $\mathcal{T}$ be as in (4.15). The maps h^{-1} and $u_0^*\tilde{\phi}^{|h|}$ in (4.10) induce trivializations of $u_1^*(V,\tilde{\phi})\big|_{(\partial\Sigma)_i}$ from the orienting collection for D_0^h; this is the same trivialization as the trivialization of

$$h^*c^{|h|*}u_0^*(V,\tilde{\phi})\big|_{(\partial\Sigma)_i} = h^*u_0^*\phi^{|h|*}(V,\tilde{\phi})\big|_{(\partial\Sigma)_i} = u_1^*(V,\tilde{\phi})\big|_{(\partial\Sigma)_i} \tag{4.18}$$

induced by the pull-back from the orienting collection for D_0. If this trivialization agrees with the γ-transferred trivialization up to homotopy, let $\epsilon_j=0$; otherwise, let $\epsilon_j=1$. The real bundle $\gamma_{h;1}^*(V,\tilde{\phi})|_{\mathcal{T}_{h;1}}$ is then isomorphic to

$$\mathbb{I}\times\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}\times\mathbb{C}^n/\!\sim, \qquad (1,z,v)\sim\big(0,c^{|h|}(h(z)),g(z)v\big) \quad \forall\, z\in\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j},\ v\in\mathbb{C}^n,$$

for some $g\colon \bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j} \longrightarrow \mathrm{GL}_n\mathbb{C}$ with $g(c(z))=\overline{g(z)}$ such that $g|_{(\partial\Sigma)_{i_j}}$ is homotopic, in the space of such maps, to the constant map $\mathbb{I}_n$ if and only if $\epsilon_j=0$. By [7, Lemma 2.3], $\Lambda_{\mathbb{C}}^{\mathrm{top}}$ induces a bijection between sets of such homotopy classes for rank n and rank 1 real bundle pairs. By Lemma 3.3, rank 1 real bundle pairs over $\mathcal{T}$ are classified by the evaluation of their $\mathbb{Z}_2$-equivariant w_2-class on $[\mathcal{T}]^{c_{h;1}}$. Thus,

$$\epsilon_1+\ldots+\epsilon_k = \big\langle \gamma_{h;1}^* w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V), [\mathcal{T}]^{c_{h;1}}\big\rangle\,. \tag{4.19}$$

By [15, Proposition 5.4], changing the choice (OC2) for a given boundary component $(\partial\Sigma)_i$ with $|c_i|=1$ changes the orientation of the determinant line bundle. Since the two choices (OC2) for D_1 obtained from the trivializing collection for u_0 via h and γ-transfer agree if and only if $\epsilon_j=0$, the possible differences in these choices for $(\partial\Sigma)_{i_1},\ldots,(\partial\Sigma)_{i_k}$ contribute (4.19) to (4.12). Summing over all cycles of the action of h on the set $\{(\partial\Sigma)_i\colon |c_i|=1\}$ gives the second term on the right-hand side of (4.12). However, h interchanges the order of the n-dimensional target spaces for the evaluation isomorphisms orienting $\det(D_i)$ as below (4.5). This contributes n times the sign of the permutation on $\{(\partial\Sigma)_i\colon |c_i|=1\}$ induced by h. $\square$

4.3 Some consequences

We now combine the conclusions of Propositions 4.1 and 4.2 with some topological assumptions on (X,ϕ) related to the orientability of moduli spaces of real maps to (X,ϕ). Corollaries 4.4 and 4.5 below encode the orientability of $\widetilde{\det}(D)$ over a loop in the moduli space of J-holomorphic maps. They are used in Section 5.3 to establish more general versions of Theorems 1.1, 1.2, and 1.4.

Remark 4.3. Corollaries 4.4 and 4.5 also determine the signs of the corresponding isomorphisms on $\det(D)$ whenever $V^{\tilde\phi}\longrightarrow X^{\phi}$ is orientable: the difference with $\widetilde{\det}(D)$ is given by sgn_h^0 in the case of Corollary 4.4 and by sgn_h in the case of Corollary 4.5.

Corollary 4.4. *Let (X,ϕ) be a manifold with an involution, $(V,\tilde\phi)\longrightarrow(X,\phi)$ be a real bundle pair, and (Σ,c) be a genus g oriented sh-surface so that*

$$w_2(V^{\tilde\phi})\in H^2(X^{\phi};\mathbb{Z}_2)\qquad \textit{and}\qquad w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V)\in H^2_{\phi}(X)$$

are spin classes in the sense of Definition 2.8 if $|c|_0\neq0$ and $|c|_1\neq0$, respectively. Suppose $\mathbf{k}\equiv(k_1,\ldots,k_{|c|_0+|c|_1})$ is a tuple of non-negative integers, $\mathbf{b}$ is as in (2.4), $\gamma\equiv(u_t,\mathbf{x}_t,\mathfrak{j}_t)$ is a path in $\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,c}\times\mathcal{J}_c$, and $h\colon\Sigma\longrightarrow\Sigma$ is a diffeomorphism such that $(u_0,\mathbf{x}_0,\mathfrak{j}_0)$ and $(u_1,\mathbf{x}_1,\mathfrak{j}_1)$ are h-related. If $V^{\tilde\phi}\longrightarrow X^{\phi}$ is not orientable, assume that $k_i>0$ for every boundary component $(\partial\Sigma)_i$ such that $|c_i|=0$ and $\langle w_1(V^{\tilde\phi}),b_i\rangle=0$. Denote by D_0 and D_1 the real Cauchy-Riemann operators on the bundle pairs $u_0^*(V,\tilde\phi)$ and $u_1^*(V,\tilde\phi)$ induced as in Section 2.1, choose an orienting collection for D_0, and take the γ-transferred orienting collection for D_1.

(1) If h is orientation-preserving, the sign of the isomorphism (4.11) is $(-1)^{(\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h+\mathrm{sgn}_h^0}$.

(2) If h is orientation-reversing, the sign of the isomorphism (4.11) composed with the isomorphism (4.2) with $D=D_0$ is $(-1)^{\tilde\epsilon_{\gamma,h}^{\tilde\phi}}$, where

$$\begin{split}\tilde\epsilon_{\gamma,h}^{\tilde\phi}=\frac12\bigg(\langle c_1(V),B\rangle+\sum_{i=1}^{|c|_0}\tilde w_1\big(V^{\tilde\phi},b_i\big)\bigg)+\sum_{i=1}^{|c|_0}\lfloor w_2(V^{\tilde\phi}),b_i\rfloor+\sum_{i=|c|_0+1}^{|c|_0+|c|_1}\lfloor w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V),b_i\rfloor\\+\big(1-g+|c|_0+|c|_1+\mathrm{sgn}_h\big)\mathrm{rk}_{\mathbb{C}}V+\mathrm{sgn}_h^0+2\mathbb{Z}\in\mathbb{Z}_2,\end{split}\tag{4.20}$$

where $\lfloor w,b_i\rfloor$ is as in (2.14) if b_i is a two-torsion class and 0 otherwise.

Proof. (1) Suppose h is orientation-preserving. Since $[\gamma]_{h;0}$ and $[\gamma]_{h;1}^c$ are sums of classes represented by tori, while $w_2(V^{\tilde\phi})$ and $w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V)$ are spin classes

$$\langle w_2(V^{\tilde\phi}),[\gamma]_{h;0}\rangle=0\qquad\text{and}\qquad\big\langle w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V),[\gamma]_{h;1}^c\big\rangle=0.$$

The first claim now follows from (4.22).

(2) Suppose h is orientation-reversing. The class $[\gamma]_{h;0}$ is the sum of the classes represented by

$$\gamma_{h;0}\colon\mathcal{T}\equiv\mathbb{I}\times\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j}/\!\sim\longrightarrow X,\qquad(1,z)\sim\big(0,c(h(z))\big)\quad\forall\,z\in\bigsqcup_{j=1}^{k}(\partial\Sigma)_{i_j},$$

with $\mathcal{T}$ as in (4.15). Since h is orientation-reversing, $b_{i_{j+1}}=-b_{i_j}$ for every $j=1,\ldots,k$; in particular, $b_{i_1}=(-1)^k b_{i_1}$. If k is even, then $\mathcal{T}$ is a torus and so

$$\langle w_2(V^{\tilde{\phi}}),\gamma_{h;0*}[\mathcal{T}]_{\mathbb{Z}_2}\rangle = 0 = \sum_{j=1}^{k}\lfloor w_2(V^{\tilde{\phi}}),b_{i_j}\rfloor\,.$$

If k is odd, then $b_{i_j}=b_{i_1}$ is a two-torsion class for all $j=1,\ldots,k$ and $\mathcal{T}$ is a Klein bottle. Thus,

$$\langle w_2(V^{\tilde{\phi}}),\gamma_{h;0*}[\mathcal{T}]_{\mathbb{Z}_2}\rangle = \lfloor w_2(V^{\tilde{\phi}}),b_{i_1}\rfloor = \sum_{j=1}^{k}\lfloor w_2(V^{\tilde{\phi}}),b_{i_j}\rfloor\,;$$

the first equality holds by (2.14). Putting these two cases together, we find that

$$\langle w_2(V^{\tilde{\phi}}),[\gamma]_{h;0}\rangle = \sum_{i=1}^{|c|_0}\lfloor w_2(V^{\tilde{\phi}}),b_i\rfloor\,.$$

By the same argument,

$$\big\langle w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V),[\gamma]_{h;1}^c\big\rangle = \sum_{i=|c|_0+1}^{|c|_0+|c|_1}\lfloor w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V),b_i\rfloor\,.$$

Combining the last two equations with (4.22), we find that

$$\tilde{\epsilon}^{\tilde{\phi}}_{\gamma,h} = \epsilon^{\tilde{\phi}}_{\gamma,h}+\epsilon_{D_0} = \sum_{i=1}^{|c|_0}\lfloor w_2(V^{\tilde{\phi}}),b_i\rfloor + \sum_{i=|c|_0+1}^{|c|_0+|c|_1}\lfloor w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}}(\Lambda_{\mathbb{C}}^{\mathrm{top}}V),b_i\rfloor + (\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h + \mathrm{sgn}_h^0 + \epsilon_{D_0}\,,$$

where ϵ_{D_0} is the relative sign of the conjugation on D_0 computed by (4.4). Along with (2.6), this establishes the second claim. □

Corollary 4.5. *Let (X,ϕ) be a manifold with an involution and $(V,\tilde{\phi})\longrightarrow(X,\phi)$ be a real bundle pair such that*

$$w_2(V^{\tilde{\phi}}) = w + \varpi|_{X^\phi}$$

for a spin class $w\in H^2(X^\phi;\mathbb{Z}_2)$ and some $\varpi\in H^2(X;\mathbb{Z}_2)$. Suppose Σ is a genus g oriented surface with m ordered boundary components, $\mathbf{k}\equiv(k_1,\ldots,k_m)$ is a tuple of non-negative integers, $\mathbf{b}$ is as in (1.8), $\gamma\equiv(u_t,\mathbf{x}_t,\mathfrak{j}_t)$ is a path in $\mathfrak{B}_{\mathbf{k}}(X,\mathbf{b})^{\phi,\mathrm{id}_{\partial\Sigma}}\times\mathcal{J}_\Sigma$, and $h\colon\Sigma\longrightarrow\Sigma$ is a diffeomorphism such that $(u_0,\mathbf{x}_0,\mathfrak{j}_0)$ and $(u_1,\mathbf{x}_1,\mathfrak{j}_1)$ are h-related. If $V^{\tilde{\phi}}\longrightarrow X^\phi$ is not orientable, assume that $k_i>0$ for every boundary component $(\partial\Sigma)_i$ such that $\langle w_1(V^{\tilde{\phi}}),b_i\rangle=0$. Denote by D_0 and D_1 the real Cauchy-Riemann operators on the bundle pairs $u_0^*(V,\tilde{\phi})$ and $u_1^*(V,\tilde{\phi})$ induced as in Section 2.1, choose an orienting collection for D_0, and take the γ-transferred orienting collection for D_1.

(1) If h is orientation-preserving, the sign of the isomorphism (4.11) is $(-1)^{(\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h+\mathrm{sgn}_h}$.

(2) If h is orientation-reversing, the sign of the isomorphism (4.11) composed with the isomorphism (4.2) with $D=D_0$ is $(-1)^{\tilde{\epsilon}^{\tilde{\phi}}_{\gamma,h}}$, where

$$\tilde{\epsilon}^{\tilde{\phi}}_{\gamma,h} = \frac{1}{2}\bigg(\langle c_1(V),\mathfrak{d}(\beta)\rangle + \sum_{i=1}^{m}\tilde{w}_1\big(V^{\tilde{\phi}},b_i\big)\bigg) + \big\langle\varpi,\mathfrak{d}(\beta)\big\rangle + \sum_{i=1}^{m}\lfloor w,b_i\rfloor + \big(1-g+m+\mathrm{sgn}_h\big)\mathrm{rk}_{\mathbb{C}}V + \mathrm{sgn}_h + 2\mathbb{Z}\in\mathbb{Z}_2, \tag{4.21}$$

where $\lfloor w, b_i \rfloor$ is as in (2.14) if b_i is a two-torsion class and 0 otherwise.

Proof. By our assumptions on Σ, $|c|_0 = m$, $|c|_1 = 0$, $\mathrm{sgn}_h^0 = \mathrm{sgn}_h$, and $[\gamma]_{h;1}^c = 0$ in (4.4) and (4.12). Let

$$Z_h = \mathbb{I}\times\Sigma/\sim, \qquad (1,z) \sim \big(0,h(z)\big) \ \ \forall\, z\in\partial\Sigma,$$
$$F_\gamma\colon Z_h \longrightarrow X, \qquad F_\gamma\big([s,z]\big) = u_s(z) \ \ \forall\,(s,z)\in\mathbb{I}\times\Sigma\,.$$

Thus, F_γ determines a three-chain in X with coefficients in $\mathbb{Z}_2$ and boundary

$$\partial F_\gamma = u_0 \sqcup_h u_1 + \gamma_{h;0}.$$

In particular, for any $\varpi\in H^2(X;\mathbb{Z}_2)$,

$$\big\langle \varpi, [\gamma]_{h;0}\big\rangle = \big\langle \varpi, [u_0\sqcup_h u_1]_{\mathbb{Z}_2}\big\rangle\,.$$

Since $w_2(V^{\tilde\phi}) = w + \varpi|_{X^\phi}$, (4.12) becomes

$$\epsilon_{\gamma,h}^{\tilde\phi} = \langle w, [\gamma]_{h;0}\rangle + \big\langle \varpi, [u_0\sqcup_h u_1]_{\mathbb{Z}_2}\big\rangle + (\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h + \mathrm{sgn}_h\,. \tag{4.22}$$

Below we analyze the first two terms on the right-hand side of (4.22) in the two cases separately.

(1) Suppose h is orientation-preserving. Since $[\gamma]_{h;0}$ is a sum of classes represented by tori and w is a spin class, $\langle w, [\gamma]_{h;0}\rangle = 0$. Since $u_1 = u_0\circ h$ in this case, $[u_0\sqcup_h u_1]_{\mathbb{Z}_2} = 0$. Thus, the first claim follows from (4.22).

(2) Suppose h is orientation-reversing. By the same argument as in (2) of the proof of Corollary 4.4,

$$\langle w, [\gamma]_{h;0}\rangle = \sum_{i=1}^m \lfloor w, b_i\rfloor\,.$$

Since $[u_0\sqcup_h u_1]_{\mathbb{Z}} = \mathfrak{d}(\beta)$ in this case,

$$\big\langle \varpi, [u_0\sqcup_h u_1]_{\mathbb{Z}_2}\big\rangle = \big\langle \varpi, \mathfrak{d}(\beta)\big\rangle.$$

Combining the last two equations with (4.22), we find that

$$\tilde\epsilon_{\gamma,h}^{\tilde\phi} = \epsilon_{\gamma,h}^{\tilde\phi} + \epsilon_{D_0} = \sum_{i=1}^m \lfloor w, b_i\rfloor + \big\langle \varpi, \mathfrak{d}(\beta)\big\rangle + (\mathrm{rk}_{\mathbb{C}}V)\mathrm{sgn}_h + \mathrm{sgn}_h + \epsilon_{D_0}\,,$$

where ϵ_{D_0} is the relative sign of the conjugation on D_0 computed by (4.4). Along with (2.6), this establishes the second claim. □

Remark 4.6. If h is orientation-reversing and $w = \kappa^2$ for some $\kappa\in H^1(X^\phi;\mathbb{Z}_2)$, then (4.21) becomes

$$\begin{aligned}\tilde\epsilon_{\gamma,h}^{\tilde\phi} = \frac{1}{2}\bigg(\langle c_1(V), \mathfrak{d}(\beta)\rangle + \sum_{i=1}^m \tilde w_1\big(V^{\tilde c}, b_i\big)\bigg) + \big\langle \varpi, \mathfrak{d}(\beta)\big\rangle + \sum_{i=1}^m \lfloor \kappa, b_i\rfloor \\ +\big(1-g+m+\mathrm{sgn}_h V\big)\mathrm{rk}_{\mathbb{C}}V + \mathrm{sgn}_h + 2\mathbb{Z} \in \mathbb{Z}_2;\end{aligned} \tag{4.23}$$

see (2.15). The two sign formulas of [33, Proposition 5.1] are special cases of this formula: $(V,\tilde\phi) = (TX, \mathrm{d}\phi)$, $\mathfrak{j}_1 = \mathfrak{j}_0$, and either $\kappa = 0$ or $\kappa = w_1(TX^\phi)$, i.e. the Lagrangian X^ϕ is either relatively pin$^+$ or relatively pin$^-$, respectively. In turn, [33, Proposition 5.1] includes the orientability results of [12, 21].

5 Applications

Let (X,ω) be a symplectic manifold, $L\subset X$ be a Lagrangian submanifold, $\beta\in H_2(X,L;\mathbb{Z})$, and J be an ω-compatible almost complex structure. As shown in [11, Section 8.1.1], the moduli space

$$\mathfrak{M}_{D^2}(X,L,\beta;J) \equiv \big\{u\in C^\infty(D^2,X)\colon \bar\partial_{J,\mathfrak{j}_0}u=0,\, u(S^1)\subset L,\, u_*[D^2,S^1]=\beta\big\}/\sim$$

of holomorphic maps from the disk D^2 with the standard complex structure $\mathfrak{j}_0$ is orientable if (L is orientable and) the pair (X,L) admits a relative spin structure; furthermore, every such structure canonically determines an orientation on $\mathfrak{M}_{D^2}(X,L,\beta;J)$. This fundamental observation is extended in [33, Theorem 1.1] to arbitrary bordered Riemann surfaces $(\Sigma,\mathfrak{j}_0)$, with a fixed complex structure on the domain, and a relative pin$^\pm$ structure on the pair (X,L), if there is one. In this case, the moduli space

$$\mathfrak{M}'_\Sigma(X,L,\beta;J) \equiv \big\{u\in C^\infty(\Sigma,X)\colon \bar\partial_{J,\mathfrak{j}_0}u=0,\, u(\partial\Sigma)\subset L,\, u_*[\Sigma,\partial\Sigma]=\beta\big\}$$

need not be orientable, but its determinant line bundle is isomorphic to a tensor product of the pull-backs of $\Lambda^{\mathrm{top}}_{\mathbb{R}}(TL)$ by evaluation maps at boundary marked points. The pair (X,L) admits a relative pin$^\pm$ structure if and only if

$$w_2(L)=\varpi|_L \qquad \text{or} \qquad w_2(TL)=w_1(L)^2+\varpi|_L \tag{5.1}$$

for some $\varpi\in H^2(X;\mathbb{Z}_2)$. As observed in [13, Remark 1.9], there is nothing special about $w_1(L)$ in (5.1) for the purposes of the above isomorphism statement.

In Sections 5.1 and 5.2, we introduce generalizations of the notions of (relative) spin/pin structures which better capture the orientability in open Gromov-Witten theory. For a symplectic manifold (X,ω) with an anti-symplectic involution ϕ, we introduce a notion of compatibility of such structures with the involution that better captures the relevant features than a similar notion introduced in [12]; see Sections 5.3 and 5.4 for details.

5.1 Spin sub-structures

We recall that a spin structure on a real oriented vector bundle $W\longrightarrow M$ of $\mathrm{rk}_{\mathbb{R}}W\ge 3$ is a collection of homotopy classes of trivializations of $\alpha^*W\longrightarrow S^1$, one class for each loop $\alpha\colon S^1\longrightarrow M$, such that for every bordered surface Σ and continuous map $f\colon\Sigma\longrightarrow M$ the vector bundle $f^*W\longrightarrow\Sigma$ admits a trivialization restricting to the chosen homotopy classes of trivializations on the boundary components of Σ under their identification with S^1. A spin structure on a real oriented vector bundle $W\longrightarrow M$ of $\mathrm{rk}_{\mathbb{R}}W=2$ is a spin structure on $W\oplus M\times\mathbb{R}\longrightarrow M$. An oriented vector bundle $W\longrightarrow M$ admits a spin structure if and only if the oriented vector bundle

$$f^*W\oplus\Sigma\times\mathbb{R}\longrightarrow\Sigma \tag{5.2}$$

is trivial for every closed surface Σ and continuous map $f\colon\Sigma\longrightarrow M$, i.e. $w_2(W)=0$. However, [13, Theorem 1.1] implies that the triviality of (5.2) matters for the orientability problem only when $\Sigma=\mathbb{T}$ is the two-torus, and so the notion of spin structure is too restrictive from this point of view whenever M is not simply connected. As it is generally difficult to describe the atorical classes, we focus on the larger collection of spin classes in $H^2(M;\mathbb{Z}_2)$; see Definition 2.8. The next lemma describes a geometric realization of such classes.

Lemma 5.1. *Let M be a paracompact topological space. For every spin class $w \in H^2(M;\mathbb{Z}_2)$, there exists a complex line bundle $W \longrightarrow M$ such that $w_2(W) = w$.*

Proof. By the proof of the Universal Coefficient Theorem for Cohomology [26, Theorem 53.1], there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}\big(H_1(M;\mathbb{Z}),\mathbb{Z}\big) & \longrightarrow & H^2(M;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\mathbb{Z}\big) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}\big(H_1(M;\mathbb{Z}),\mathbb{Z}_2\big) & \longrightarrow & H^2(M;\mathbb{Z}_2) & \longrightarrow & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\mathbb{Z}_2\big) & \longrightarrow & 0\,, \end{array}$$

with the vertical arrows induced by the nonzero homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_2$. By [26, Exercise 52.4], the first vertical arrow in the above diagram is surjective. It follows that every element of $\mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$ comes from $H^2(M;\mathbb{Z})$ and thus equals $w_2(W)$ for some complex line bundle $W \longrightarrow M$. □

For example, if $w = \kappa^2$ for some $\kappa \in H^1(M;\mathbb{Z}_2)$, then $w = w_2(W' \oplus W')$ for some real line bundle $W' \longrightarrow M$, since these line bundles correspond to elements of $H^1(M;\mathbb{Z}_2)$. By Lemma 2.7, quite often every spin class is a square, in which case every spin class is of the form $w_2(W' \oplus W') = w_1(W')^2$ for some real line bundle $W' \longrightarrow M$.

Every oriented real vector bundle $W \longrightarrow S^1$ admits a trivialization. If $\mathrm{rk}\, W \geq 3$, there are two homotopy classes of trivializations, since $\pi_1(\mathrm{SO}(k)) = \mathbb{Z}_2$ for $k \geq 3$. If $W \longrightarrow S^1$ is the restriction of a vector bundle over D^2, there is a unique homotopy class of trivializations of $W \longrightarrow S^1$ that extend to a trivialization over D^2; this is used in the proof of Lemma 5.3 below.

Definition 5.2. Let $W \longrightarrow M$ be an oriented real vector bundle.

(1) If $\mathrm{rk}_{\mathbb{R}} W \geq 3$, a spin sub-structure on $W \longrightarrow M$ is a collection of homotopy classes of trivializations of $\alpha^* W \longrightarrow S^1$, one class for each loop $\alpha\colon S^1 \longrightarrow M$, such that for every

- connected oriented surface Σ with boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$,
- orientation-reversing diffeomorphism $\varphi_1\colon S^1 \longrightarrow (\partial\Sigma)_1$,
- orientation-preserving diffeomorphism $\varphi_2\colon S^1 \longrightarrow (\partial\Sigma)_2$, and
- continuous map $F\colon \Sigma \longrightarrow M$,

the vector bundle $F^* W \longrightarrow \Sigma$ admits a trivialization such that its pull-backs by φ_1 and φ_2 agree with the chosen homotopy classes of trivializations for the loops $F \circ \varphi_1, F \circ \varphi_2\colon S^1 \longrightarrow M$.

(2) If $\mathrm{rk}_{\mathbb{R}} W = 2$, a spin sub-structure on $W \longrightarrow M$ is a spin sub-structure on $W \oplus M \times \mathbb{R} \longrightarrow M$.

Lemma 5.3. *An oriented real vector bundle $W \longrightarrow M$ admits a spin sub-structure if and only if $w_2(W)$ is a spin class.*

Proof. We can assume that $\mathrm{rk}\, W \geq 3$ and M is connected.

(1) Suppose we have chosen a spin sub-structure $W \longrightarrow M$. Let $F_0\colon D^2 \longrightarrow M$ be a fixed continuous map (e.g. a constant map). By reversing the chosen homotopy classes of trivializations for all null-homologous loops if necessary (reversing for either all or for none of them), we can assume that

the restriction of a trivialization of $F_0^*W\longrightarrow D^2$ to S^1 agrees with the chosen homotopy class of trivializations for the loop $F_0|_{S^1}: S^1\longrightarrow M$. If $F_1: D^2\longrightarrow M$ is another continuous map, there exists a continuous map $F\colon \mathbb{I}\times S^1\longrightarrow M$ such that the map

$$f=F_0\sqcup F\sqcup F_1\colon S^2\approx\big(0\times D^2\sqcup\mathbb{I}\times S^1\sqcup 1\times D^2\big)/\!\sim\longrightarrow M,$$
$$0\times D^2\ni(0,z)\sim(0,z)\in\mathbb{I}\times S^1,\ 0\times D^2\ni(1,z)\sim(1,z)\in\mathbb{I}\times S^1\quad\forall\,z\!\in\! S^1\subset D^2,$$

is continuous and homotopically trivial.[3] Thus, $f^*W\longrightarrow S^2$ admits a trivialization that restricts to the chosen homotopy class of trivializations for the loop $F_0|_{S^1}$. So the chosen homotopy class of trivializations over a contractible loop consists of trivializations that extend over a bounding disk.

If $f\colon\Sigma\longrightarrow M$ is any continuous map from a closed oriented surface, the restriction of f^*W to the complement of two disjoint disks D_1^2 and D_2^2 admits a trivialization that restricts to the chosen homotopy classes of trivializations on ∂D_1^2 and ∂D_2^2. Since both of the latter extend over the corresponding disks, it follows that $f^*W\longrightarrow\Sigma$ is a trivial vector bundle and so

$$\big\langle w_2(W),f_*[\Sigma]_{\mathbb{Z}_2}\big\rangle=\big\langle w_2(f^*W),[\Sigma]_{\mathbb{Z}_2}\big\rangle=0.$$

Thus, $w_2(W)$ is a spin class.

(2) A spin sub-structure on $W\longrightarrow M$ can be obtained as follows. Pick a representative $\alpha\colon S^1\longrightarrow M$ for each element $[\alpha]$ of $H_1(M;\mathbb{Z})$ and a homotopy class of trivializations of α^*W. Given a loop $\beta\colon S^1\longrightarrow M$ with $[\beta]=[\alpha]$ in $H_1(M;\mathbb{Z})$, choose

- a connected oriented surface Σ with boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$,
- an orientation-reversing diffeomorphism $\varphi_1\colon S^1\longrightarrow(\partial\Sigma)_1$,
- an orientation-preserving diffeomorphism $\varphi_2\colon S^1\longrightarrow(\partial\Sigma)_2$, and
- a continuous map $F\colon\Sigma\longrightarrow M$,

so that $F\circ\varphi_1=\alpha$ and $F\circ\varphi_2=\beta$. Since an oriented vector bundle over a connected surface with at least one boundary component is trivial, there exists a trivialization of $f^*W\longrightarrow\Sigma$ extending the chosen trivialization over $(\partial\Sigma)_1$. We take the homotopy class of the trivializations for $\beta^*W\longrightarrow S^1$ to be the homotopy class of the restriction of this trivialization to $(\partial\Sigma)_2$.

Given another collection $(\Sigma',\varphi_1',\varphi_2',F')$ as above, let

$$\breve\Sigma=(\Sigma\sqcup\bar\Sigma')/\!\sim,\quad z\sim\varphi_1'(\varphi_1^{-1}(z))\ \forall\,z\!\in\!(\partial\Sigma)_1,\qquad \hat\Sigma=\breve\Sigma/\!\sim,\quad z\sim\varphi_2'(\varphi_2^{-1}(z))\ \forall\,z\!\in\!(\partial\Sigma)_2\,,$$

where $\bar\Sigma'$ denotes Σ' with the opposite orientation. The maps F and F' induce continuous maps $\breve F\colon\breve\Sigma\longrightarrow M$ and $\hat F\colon\hat\Sigma\longrightarrow M$. A trivialization of α^*W in the chosen homotopy class extends to a trivialization of $\breve F^*W\longrightarrow\breve\Sigma$ and

$$\hat F^*W\approx\breve\Sigma\times\mathbb{R}^k/\!\sim,\qquad(z,v)\sim\big(\varphi_2'(\varphi_2^{-1}(z)),g(z)v\big)\ \forall\,(z,v)\in(\partial\Sigma)_2\times\mathbb{R}^k\,,$$

[3] Project $[1/4,3/4]\times S^1$ onto a path $[1/4,3/4]\longrightarrow M$ between $F_0(0)$ and $F_1(0)$; then take F on $[0,1/4]\times S^1$ and $[3/4,1]\times S^1$ to be homotopies from $F_0|_{S^1}$ to the constant loop at $F_0(0)$ and from the constant loop at $F_1(0)$ to $F_1|_{S^1}$, respectively.

for some $g\colon (\partial\Sigma)_2 \longrightarrow \mathrm{SO}(k)$. Since oriented vector bundles over $\hat{\Sigma}$ with rank at least 3 are classified by their w_2 and $\pi_1(\mathrm{SO}(k)) \approx \mathbb{Z}_2$ for $k \ge 3$, g is homotopically trivial if and only if $w_2(\hat{F}^*W)=0$. By our assumption on $w_2(W)$,

$$\big\langle w_2(\hat{F}^*W), [\hat{\Sigma}]_{\mathbb{Z}_2}\big\rangle = \big\langle w_2(W), \hat{F}_*[\hat{\Sigma}]_{\mathbb{Z}_2}\big\rangle = 0.$$

Thus, g is homotopically trivial and so the trivializations of β^*W induced via F and F' are the same.

Suppose $(\Sigma,\varphi_1,\varphi_2,F)$ is as in Definition 5.2(1), α is the chosen representative for the homology class

$$[F\circ\varphi_1]_{\mathbb{Z}} = [F\circ\varphi_2]_{\mathbb{Z}} \in H_1(M;\mathbb{Z}),$$

and $(\Sigma',\varphi_1',\varphi_2',F')$ is as in the construction of the induced trivialization for $\beta \equiv F\circ\varphi_1$, replacing $(\Sigma,\varphi_1,\varphi_2,F)$. Let

$$\breve{F} = F' \sqcup F\colon\ \breve{\Sigma} = (\Sigma' \sqcup \Sigma)/\!\sim \longrightarrow M, \qquad z \sim \varphi_2'(\varphi_1^{-1}(z))\ \forall\, z \in (\partial\Sigma)_1.$$

The chosen trivialization of α^*W extends to a trivialization of $\breve{F}^*W$. By construction, the restrictions of the latter to $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$ lie in the chosen homotopy classes of trivializations for the loops $F\circ\varphi_1$ and $F\circ\varphi_2$. Thus, trivializations of $\{F\circ\varphi_1\}^*W$ and $\{F\circ\varphi_2\}^*W$ in the chosen homotopy classes extend to a trivialization of F^*W, as required. $\square$

A spin sub-structure can thus be viewed as a weak spin structure for oriented vector bundles with w_2 lying in the subgroup $\mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$ of $H^2(M;\mathbb{Z}_2)$, instead of being 0; this extension group is non-trivial if $H_1(M;\mathbb{Z})$ has even torsion. The group of *maps*

$$\vartheta\colon H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2$$

acts freely and transitively on the set of spin sub-structures on $W \longrightarrow M$, if this set is non-empty; ϑ changes the chosen homotopy class of trivializations along a loop $\alpha: S^1 \longrightarrow M$ if and only if $\vartheta(\alpha) \neq 0$. In the case of spin structures, the same role is played by

$$H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}_2),$$

the group of *homomorphisms* from $H_1(M;\mathbb{Z})$ to $\mathbb{Z}_2$.

Let $\mathfrak{c}_{S^1}: S^1 \longrightarrow S^1$ denote the restriction of the standard conjugation on $S^1 \subset \mathbb{C}$. If $\alpha\colon S^1 \longrightarrow M$, let

$$\alpha_{\mathfrak{c}} = \alpha\circ\mathfrak{c}_{S^1}\colon S^1 \longrightarrow M\,.$$

For any vector bundle $W \longrightarrow M$, the diffeomorphism $\mathfrak{c}_{S^1}$ induces the commutative diagram

$$\begin{array}{ccccccc} \alpha^*W & \xrightarrow{\tilde{\mathfrak{c}}_{S^1}} & \alpha_{\mathfrak{c}}^*W & \qquad & (z,w) & \longrightarrow & (\mathfrak{c}_{S^1}(z),w) \qquad \forall\, w \in W_{\alpha(z)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1 & \xrightarrow{\mathfrak{c}_{S^1}} & S^1 & & z & \longrightarrow & \mathfrak{c}_{S^1}(z) \qquad \forall\, z \in S^1\,. \end{array} \tag{5.3}$$

If $[\alpha] \in H_1(M;\mathbb{Z})$ is a two-torsion class, $[\alpha_c] = [\alpha]$. The next lemma describes the action of the above commutative diagram on homotopy classes of trivializations.

Lemma 5.4. *Let $W \longrightarrow M$ be an oriented real vector bundle with a chosen spin sub-structure and $\alpha: S^1 \longrightarrow M$ be a representative of a two-torsion element of $H_1(M;\mathbb{Z})$. The commutative diagram (5.3) takes the chosen homotopy class of trivializations of α^*W to the chosen homotopy class of trivializations of $\alpha_{\mathfrak{c}}^*W$ if and only if $\lfloor w_2(W),[\alpha]_{\mathbb{Z}_2}\rfloor = 0$, with $\lfloor w_2(W),[\alpha]_{\mathbb{Z}_2}\rfloor$ defined as in (2.14). In particular, if $w_2(W)=\kappa^2$ for some $\kappa\in H^1(M;\mathbb{Z}_2)$, the commutative diagram (5.3) respects a spin sub-structure on W if and only if $\langle\kappa,[\alpha]_{\mathbb{Z}_2}\rangle=0$.*

Proof. Since α represents a two-torsion element of $H_1(M;\mathbb{Z})$, there exist

- a connected oriented surface Σ with boundary components $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$,
- an orientation-reversing diffeomorphism $\varphi_1: S^1 \longrightarrow (\partial\Sigma)_1$,
- an orientation-preserving diffeomorphism $\varphi_2: S^1 \longrightarrow (\partial\Sigma)_2$, and
- a continuous map $F: \Sigma \longrightarrow M$,

such that $F\circ\varphi_1=\alpha$ and $F\circ\varphi_2=\alpha_{\mathfrak{c}}$. The map F descends to a continuous map

$$\hat{F}: \hat{\Sigma} = \Sigma/\sim \longrightarrow M, \qquad z \sim \varphi_2\big(\mathfrak{c}_{S^1}(\varphi_1^{-1}(z))\big)\ \forall\, z\in(\partial\Sigma)_1 .$$

Since the diffeomorphism $\varphi_2\circ\mathfrak{c}_{S^1}\circ\varphi_1^{-1}: (\partial\Sigma)_1 \longrightarrow (\partial\Sigma)_2$ is orientation-preserving, the closed compact surface $\hat{\Sigma}$ is not orientable. We note that

$$\hat{F}^*W = F^*W/\sim, \qquad v \sim \tilde{\mathfrak{c}}_{S^1}(v)\ \forall\, v\in\alpha^*W.$$

Since $\pi_1(\mathrm{SO}(k))=\mathbb{Z}_2$ for $k\geq 3$ and there is an oriented vector bundle over $\hat{\Sigma}$ with a nonzero w_2, the oriented vector bundles over $\hat{\Sigma}$ of rank at least 3 are classified by their w_2. Since the commutative diagram (5.3) respects the homotopy classes of the restrictions of a trivialization of $F^*W \longrightarrow \Sigma$ to $(\partial\Sigma)_1$ and $(\partial\Sigma)_2$ if and only if $\hat{F}^*W \longrightarrow \hat{\Sigma}$ is a trivial vector bundle, it follows that this is the case if and only if

$$0 = \big\langle w_2(\hat{F}^*W),[\Sigma]_{\mathbb{Z}_2}\big\rangle = \big\langle w_2(W),\hat{F}_*[\Sigma]_{\mathbb{Z}_2}\big\rangle \equiv \lfloor w_2(W),[\alpha]_{\mathbb{Z}_2}\rfloor.$$

The last claim of the lemma now follows from (2.15). □

5.2 Moduli spaces of open maps

We now define a relative version of the spin sub-structure of Definition 5.2 and show that it canonically induces orientations of the determinant lines of real Cauchy-Riemann operators in open Gromov-Witten theory.

Definition 5.5. Let L be a submanifold of a manifold X.

(1) A relative spin sub-structure on a real oriented vector bundle $F \longrightarrow L$ consists of

  (1a) a spin sub-structure on a real oriented bundle $W \longrightarrow L$,

  (1b) a real oriented vector bundle $E \longrightarrow X_{[3]}$, where $X_{[3]}$ is the 3-skeleton of X in a triangulation extending a triangulation of L, and

  (1c) a spin structure on $F\oplus W\oplus E \longrightarrow L\cap X_{[3]}$.

(2) A relative pin sub-structure on a real vector bundle $F \longrightarrow L$ is a relative spin sub-structure on the oriented vector bundle $F \oplus 3\Lambda_{\mathbb{R}}^{\mathrm{top}} F$.

(3) A relative spin/pin sub-structure on (X, L) is a relative spin/pin sub-structure on the real vector bundle $TL \longrightarrow L$.

(4) The pair (X, L) is relatively subspin/subpin if (X, L) admits a relative spin/pin structure.

A relative spin structure on $F \longrightarrow L$ in the sense of [11, Definition 3.1.1] is a relative spin sub-structure on $F \longrightarrow L$ with $W = L \times \{0\}$. The next corollary gives a purely cohomological criterion for the existence of a spin/pin sub-structures.

Corollary 5.6. *Let L be a submanifold of a manifold X and $F \longrightarrow L$ be a real vector bundle.*

(1) If F admits a relative spin or pin sub-structure, then $w_2(F) = w + \varpi|_L$ for some spin class $w \in H^2(L; \mathbb{Z}_2)$ and for some $\varpi \in H^2(X; \mathbb{Z}_2)$.

(2) If $w_2(F) = w + \varpi|_L$ for some spin class $w \in H^2(L; \mathbb{Z}_2)$ and for some $\varpi \in H^2(X; \mathbb{Z}_2)$, then F admits a pin sub-structure. If in addition F is orientable, then F admits a relative spin sub-structure.

Proof. (1) If F admits a relative spin sub-structure as in Definition 5.5, the vector bundle $F \oplus W \oplus E$ over $L \cap X_{[3]}$ admits a spin structure and so

$$0 = w_2\big((F \oplus W \oplus E)|_{L\cap X_{[3]}}\big) = w_2(F)|_{L\cap X_{[3]}} + w_2(W)|_{L\cap X_{[3]}} + w_2(E)|_{L\cap X_{[3]}}$$
$$\Longrightarrow \qquad w_2(F) = w_2(W) + \varpi|_L$$

for some $\varpi \in H^2(X; \mathbb{Z}_2)$. Since W admits a spin sub-structure, $w_2(W)$ is a spin class by Lemma 5.3. The same reasoning, with F replaced by $F \oplus 3\Lambda_{\mathbb{R}}^{\mathrm{top}} F$, applies in the pin case.

(2) It is sufficient to consider the orientable case. By Lemma 5.1, $w = w_2(W)$ for some oriented vector bundle $W \longrightarrow L$. By Lemma 5.3, W admits a spin sub-structure. By the usual obstruction theory reasoning, there exists an oriented rank 3 vector bundle $E \longrightarrow X_{[3]}$ such that $w_2(E) = \varpi|_{X_{[3]}}$.[4] Since

$$w_2\big((F \oplus W \oplus E)|_{L\cap X_{[3]}}\big) = w_2(F)|_{L\cap X_{[3]}} + w_2(W)|_{L\cap X_{[3]}} + w_2(E)|_{L\cap X_{[3]}}$$
$$= w_2(F)|_{L\cap X_{[3]}} + w|_{L\cap X_{[3]}} + \varpi|_{L\cap X_{[3]}} = 0,$$

the vector bundle $F \oplus W \oplus E$ over $L \cap X_{[3]}$ admits a spin structure. □

Let X be a smooth manifold, $L \subset X$ be a smooth submanifold, and Σ be a compact bordered surface, with ordered boundary components $(\partial\Sigma)_1, \ldots, (\partial\Sigma)_m$. Given

$$\mathbf{b} = (\beta, b_1, \ldots, b_m) \in H_2(X, L; \mathbb{Z}) \oplus H_1(L; \mathbb{Z})^m, \tag{5.4}$$

[4] There is a continuous map $f : X \longrightarrow K(\mathbb{Z}_2, 2)$ such that $\varpi = f^*\Omega$, where $K(\mathbb{Z}_2, 2)$ is the Eilenberg-MacLane space with $\pi_2 = \mathbb{Z}_2$ and Ω is the generator of $H^2(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$. It can be assumed that f takes $X_{[3]}$ to the 3-skeleton $K(\mathbb{Z}_2, 2)_{[3]}$ of $K(\mathbb{Z}_2, 2)$. Since $K(\mathbb{Z}_2, 2)$ and $\mathbb{B}\mathrm{SO}(3)$ are simply-connected with $\pi_2 = \mathbb{Z}_2$, there is a continuous map $F : K(\mathbb{Z}_2, 2)_{[3]} \longrightarrow \mathbb{B}\mathrm{SO}(3)$ inducing an isomorphism on π_2 and thus on the second $\mathbb{Z}$-homology. This implies that $\Omega = F^* w_2(\gamma_3)$, where $\gamma_3 \longrightarrow \mathbb{B}\mathrm{SO}(3)$ is the tautological oriented rank 3 vector bundle and so $\varpi|_{X_{[3]}} = w_2(f^*F^*\gamma_3)$.

we define

$$\mathfrak{B}_\Sigma(X,L,\mathbf{b}) = \big\{u\in C^\infty(\Sigma,X)\colon\ u(\partial\Sigma)\subset L,\ u_*[\Sigma,\partial\Sigma]=\beta,\ u_*[(\partial\Sigma)_i]=b_i\ \forall\, i=1,\ldots,m\big\}.$$

If in addition $\mathbf{k}=(k_1,\ldots,k_m)$ is a tuple of nonnegative integers, let

$$\mathfrak{B}_{\Sigma;\mathbf{k}}(X,L,\mathbf{b}) = \mathfrak{B}_\Sigma(X,L,\mathbf{b}) \times \prod_{i=1}^m \big((\partial\Sigma)_i^{k_i} - \Delta_{i,k_i}\big),$$

where

$$\Delta_{i,k_i} = \big\{(x_{i,1},\ldots,x_{i,k_i})\in(\partial\Sigma)_i^{k_i}\colon\ x_{i,j'}=x_{i,j} \text{ for some } j,j'=1\ldots,k_i,\ j\neq j'\big\}$$

is the big diagonal. Let

$$\begin{aligned}\mathcal{H}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b}) &= \big(\mathfrak{B}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b})\times\mathcal{J}_\Sigma\big)/\mathcal{D}_\Sigma,\\ \mathfrak{M}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b};J) &= \big\{[u,\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathfrak{j}]\in\mathcal{H}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b})\colon\ \bar\partial_{J,\mathfrak{j}}u=0\big\},\end{aligned}$$

if J is an almost complex structure on X. For each $i=1,\ldots,m$ and $j=1,\ldots,k_i$, we define

$$\mathrm{ev}_{i,j}\colon\ \mathcal{H}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b})\longrightarrow L$$

to be the evaluation map at the j-th marked point of the i-th boundary component.

Proposition 5.7. *Suppose X is a smooth manifold, $L\subset X$ is a smooth submanifold, $V\longrightarrow X$ is a complex vector bundle, $V_\mathbb{R}\subset V|_L$ is a totally real subbundle, Σ is a compact oriented surface with ordered boundary components $(\partial\Sigma)_1,\ldots,(\partial\Sigma)_m$, and $x_i\in(\partial\Sigma)_i$ for each $i=1,\ldots,m$.*

(1) For every complex structure $\mathfrak{j}$ on Σ and every map $u:(\Sigma,\partial\Sigma)\longrightarrow(X,L)$, a relatively pin sub-structure on $V_\mathbb{R}\longrightarrow L$ canonically induces an orientation on the twisted determinant line

$$\widetilde{\det}(D_u) = \det(D_u)\otimes \bigotimes_{\langle u^*w_1(V_\mathbb{R}),(\partial\Sigma)_i\rangle=0} \big(\Lambda^{\mathrm{top}}_\mathbb{R} V_\mathbb{R}|_{u(x_i)}\big)^*$$

of a real Cauchy-Riemann operator D_u on the bundle pair $(u^*V, u|_{\partial\Sigma}^*V_\mathbb{R})\longrightarrow(\Sigma,\partial\Sigma)$.

(2) Let $\mathbf{b}$ be as in (5.4) and $\mathbf{k}\equiv(k_1,\ldots,k_m)$ be a tuple of non-negative integers such that $k_i>0$ for each i with $\langle w_1(V_\mathbb{R}),b_i\rangle=0$. If $V_\mathbb{R}\longrightarrow L$ is relatively subpin, the twisted determinant line bundle

$$\widetilde{\det}(D_{V,V_\mathbb{R}})\longrightarrow\mathcal{H}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b})$$

induced by $(V,V_\mathbb{R})$ as in [13, Remark 1.3], with $x_i=x_{i;1}$, is orientable.

(3) Let $\mathbf{b}$ be as in (5.4) and $\mathbf{k}\equiv(k_1,\ldots,k_m)$ be a tuple of non-negative integers. If $V_\mathbb{R}\longrightarrow L$ is relatively subspin, the determinant line bundle

$$\det(D_{V,V_\mathbb{R}})\longrightarrow\mathcal{H}_{\Sigma,\mathbf{k}}(X,L,\mathbf{b})$$

is orientable.

Proof. The proof of [13, Theorem 1.1] reduces the non-orientable case to the orientable case, with the difference accounted for by twisting the determinant. Thus, we can assume that $V_{\mathbb{R}} \longrightarrow L$ is orientable. By the proof of [11, Proposition 8.1.4], an orientation on $\det(D_u)$ is canonically determined by a choice of homotopy classes of trivializations of $u_0|^*_{(\partial\Sigma)_i}V_{\mathbb{R}}$ for each boundary component $(\partial\Sigma)_i$ of Σ for a map

$$u_0\colon (\Sigma, \partial\Sigma) \longrightarrow (X_{[3]}, X_{[3]} \cap L).$$

Given a relatively spin sub-structure on $F = V_{\mathbb{R}}$ as in Definition 5.5, a homotopy class of trivializations of $u_0|^*_{(\partial\Sigma)_i}V_{\mathbb{R}}$ is canonically determined by

- the homotopy class of trivializations of $u_0|^*_{(\partial\Sigma)_i}W$ provided by the spin sub-structure on W and
- a homotopy class of trivializations of $u_0|^*_{(\partial\Sigma)_i}E$.

We choose a collection of homotopy classes of trivializations of $u_0|^*_{(\partial\Sigma)_i}E$, one for each boundary component, so that it is the restriction of a trivialization of $u^*E \longrightarrow \Sigma$. In the case $\Sigma = D^2$ considered in [11, Proposition 8.1.4], there is only one such collection. In general, any two such collections differ by changing the chosen homotopy class of trivializations on an even number of boundary components. Since changing the homotopy class of trivializations along a single boundary component changes the orientation of $\det(D_u)$, any two collections of trivializations of $u_0|^*_{(\partial\Sigma)_i}E$ that come from a trivialization of $u_0^*E \longrightarrow \Sigma$ induce the same orientation on $\det(D_u)$; see the proof of [33, Proposition 3.1] for a different presentation of this point. This establishes (1), which immediately implies (2) and (3). □

Corollary 5.8. *Suppose (X, ω) is a symplectic manifold, $L \subset X$ is a Lagrangian submanifold, $J \in \mathcal{J}_\omega$, Σ is a compact oriented surface with ordered boundary components $(\partial\Sigma)_1, \ldots, (\partial\Sigma)_m$, $\mathbf{b}$ is as in (5.4), and $\mathbf{k} \equiv (k_1, \ldots, k_m)$ is a tuple of non-negative integers.*

(1) If L is orientable and admits a relative spin sub-structure, the moduli space $\mathfrak{M}_{\Sigma,\mathbf{k}}(X, \mathbf{b}; J)$ is orientable. Furthermore, a choice of such a structure canonically determines an orientation on $\mathfrak{M}_{\Sigma,\mathbf{k}}(X, \mathbf{b}; J)$.

(2) If $k_i > 0$ for each boundary component $(\partial\Sigma)_i$ with $\langle u^*w_1(TL), (\partial\Sigma)_i\rangle = 0$ and L admits a relative pin sub-structure, the orientation line bundle of $\mathfrak{M}_{\Sigma,\mathbf{k}}(X, \mathbf{b}; J)$ is isomorphic to

$$\bigotimes_{\langle u^*w_1(TL), (\partial\Sigma)_i\rangle = 0} \mathrm{ev}^*_{i;1}(\Lambda^{\mathrm{top}}_{\mathbb{R}} TL).$$

Furthermore, a choice of such a structure determines such an isomorphism up to homotopy.

Proof. Both statements follow from Proposition 5.7; see the proof of [13, Corollary 1.8]. □

5.3 Moduli spaces of real maps

Let (X, ω, ϕ) be a symplectic manifold with an anti-symplectic involution and J be an ω-compatible almost complex structure on X such that $\phi^*J = -J$. Every real map from a symmetric surface $(\hat{\Sigma}, \sigma)$ to (X, ϕ) can be represented by a map from an oriented sh-surface (Σ, c), but not uniquely in general. This gives rise to coverings of moduli spaces of the former by moduli spaces of the latter.

These coverings are regular if the genus of $\hat{\Sigma}$ is 0 or 1 or if $\hat{\Sigma}-\hat{\Sigma}^\sigma$ is disconnected; there are 5 topological types of symmetric surfaces of genus 0 or 1. Orientability of moduli spaces of maps from oriented sh-surfaces is addressed in [15]. Propositions 4.1 and 4.2 determine when the deck transformations of these coverings are orientation-preserving. We apply them in this section to establish more general versions of Theorems 1.1, 1.2, and 1.4.

If (Σ,c) is an oriented sh-surface and $\mathbf{b}$ is as in (2.4), denote by $\mathcal{P}(\mathbf{b})$ the set of tuples obtained from the tuples

$$\mathbf{b}=\big(B,b_1,\ldots,b_{|c|_0},b_{|c|_0+1},\ldots,b_{|c|_0+|c|_1}\big) \quad\text{and}\quad \bar{\mathbf{b}}=\big(B,-b_1,\ldots,-b_{|c|_0},-b_{|c|_0+1},\ldots,-b_{|c|_0+|c|_1}\big)$$

by permuting the $b_1,\ldots,b_{|c|_0}$-entries and the $b_{|c|_0+1},\ldots,b_{|c|_0+|c|_1}$-entries (within each of the two sets). If $|c|_1=0$, i.e. (Σ,c) is a bordered surface without crosscaps, and $\mathbf{b}$ is as in (1.8), denote by $\mathcal{P}(\mathbf{b})$ the set of tuples obtained from the tuples

$$\mathbf{b}=\big(\beta,b_1,\ldots,b_{|c|_0}\big) \qquad\text{and}\qquad \bar{\mathbf{b}}=\big(-\phi_*\beta,-b_1,\ldots,-b_{|c|_0}\big)$$

by permuting the $b_1,\ldots,b_{|c|_0}$-entries. We define

$$\mathfrak{M}^\cup(X,\mathbf{b};J)^{\phi,c}=\bigcup_{\mathbf{b}'\in\mathcal{P}(\mathbf{b})}\mathfrak{M}(X,\mathbf{b}';J)^{\phi,c}, \qquad \mathfrak{M}^\cup_\Sigma(X,X^\phi,\mathbf{b};J)=\bigcup_{\mathbf{b}'\in\mathcal{P}(\mathbf{b})}\mathfrak{M}_\Sigma(X,X^\phi,\mathbf{b}';J)$$

in the two cases, respectively. If $h\colon\Sigma\longrightarrow\Sigma$ is a diffeomorphism commuting with c on $\partial\Sigma$, similarly to (2.9) we define

$$\begin{aligned}\mathfrak{M}_h\colon\mathfrak{M}^\cup(X,\mathbf{b};J)^{\phi,c}&\longrightarrow\mathfrak{M}^\cup(X,\mathbf{b};J)^{\phi,c}, & [u,\mathfrak{j}]&\longrightarrow\big[\phi^{|h|}\circ u\circ h,(-1)^{|h|}h^*\mathfrak{j}\big],\\ \mathfrak{M}_h\colon\mathfrak{M}^\cup_\Sigma(X,X^\phi,\mathbf{b};J)&\longrightarrow\mathfrak{M}^\cup_\Sigma(X,X^\phi,\mathbf{b};J), & [u,\mathfrak{j}]&\longrightarrow\big[\phi^{|h|}\circ u\circ h,(-1)^{|h|}h^*\mathfrak{j}\big],\end{aligned} \tag{5.5}$$

with the notation as in (2.5). We will call these automorphisms the natural automorphisms of $\mathfrak{M}^\cup(X,\mathbf{b};J)^{\phi,c}$ and $\mathfrak{M}^\cup_\Sigma(X,X^\phi,\mathbf{b};J)$, respectively.

Corollary 5.9. *Let (X,ω) be a symplectic $2n$-manifold with an anti-symplectic involution ϕ, $J\in\mathcal{J}_\phi$, Σ be a genus g oriented bordered surface with m boundary components, and $\mathbf{b}$ be as in (1.8). If either $2b_i=0$ for all i and $m-g\in 2\mathbb{Z}$ or $b_i=\pm b_j$ for some $i\neq j$, assume also that n is odd. If $X^\phi\subset X$ is orientable and there exist a spin class $w\in H^2(X^\phi;\mathbb{Z}_2)$ and a class $\varpi\in H^2(X;\mathbb{Z}_2)$ such that*

$$w_2(TX^\phi)=w+\varpi|_{X^\phi} \qquad\text{and}\qquad \frac{1}{2}\langle c_1(TX),\mathfrak{d}(\beta)\rangle+\langle\varpi,\mathfrak{d}(\beta)\rangle+\sum_{i=1}^m\lfloor w,b_i\rfloor\in 2\mathbb{Z}\,, \tag{5.6}$$

then the natural automorphisms of $\mathfrak{M}^\cup_\Sigma(X,X^\phi,\mathbf{b};J)$ are orientation-preserving with respect to the orientation induced by some relative spin sub-structure associated with w and ϖ as in Definition 5.5. If $w=0$, this is the case for the orientation induced by every relative spin structure associated with ϖ.

Proof. By Corollary 5.8, the moduli space $\mathfrak{M}_\Sigma(X,X^\phi,\mathbf{b};J)$ is orientable under our assumptions, and a relative spin sub-structure determines an orientation. We consider three cases separately.

(1) Suppose first that $\Sigma = D^2$. Let $\mathfrak{j}_0$ be the standard complex structure on the disk and

$$\begin{aligned}\widetilde{\mathfrak{M}}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J) = \big\{u\in\mathfrak{B}(X)^{\phi,\mathrm{id}_{S^1}}:\ \bar\partial_{J,\mathfrak{j}_0}u=0,\ u_*[D^2,S^1]=\beta,\ u_*[S^1]=b_1\big\}\\ \cup\big\{u\in\mathfrak{B}(X)^{\phi,\mathrm{id}_{S^1}}:\ \bar\partial_{J,\mathfrak{j}_0}u=0,\ u_*[D^2,S^1]=-\phi_*\beta,\ u_*[S^1]=-b_1\big\}.\end{aligned}$$

Since $\mathfrak{M}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)=\widetilde{\mathfrak{M}}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)/\mathrm{PGL}_2^0\mathbb{R}$, there is a canonical isomorphism

$$\det(D_{TX,\mathrm{d}\phi})=\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\widetilde{\mathfrak{M}}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)\big)\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_{\mathrm{id}}\mathrm{PGL}_2^0\mathbb{R})\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathfrak{M}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)\big)\,,$$

with $\det(D_{TX,\mathrm{d}\phi})$ as at the end of Section 2.1. In this case, (1.7) is the only automorphism of $\mathfrak{M}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)$ to consider, since all (orientation-preserving) automorphisms of D^2 are isotopic to the identity. By Corollary 4.5(2) and Remark 4.3, the action of this automorphism on $\det(D_{TX,\mathrm{d}\phi})$ at $u\in\widetilde{\mathfrak{M}}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)$ is orientation-preserving under our assumptions (5.6) if the J-holomorphic maps u and $\phi\circ u\circ\mathfrak{c}_{D^2}$ are homotopic (as continuous maps). Its action on $T\mathrm{PGL}_2^0\mathbb{R}$ is orientation-preserving as well, since there is a canonical isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_{\mathrm{id}}\mathrm{PGL}_2^0\mathbb{R})\approx T_1S^1\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_0D^2)$$

and the automorphism (1.7) reverses the orientations of both factors. This establishes both claims of Corollary 5.9 at the elements $[u]$ of $\mathfrak{M}^{\cup}_{D^2}(X,X^\phi,\mathbf{b};J)$ such that u and $\phi\circ u\circ\mathfrak{c}_{D^2}$ are homotopic.

Suppose the restrictions of u and $v\equiv\phi\circ u\circ\mathfrak{c}_{D^2}$ to $S^1\subset D^2$ are homologous, but the maps u and v are not necessarily homotopic. Suppose also that the lines $\det(D_{TX,\mathrm{d}\phi})$ at u and v are oriented by a spin sub-structure on TX^ϕ as in the proof of Proposition 5.7. In the terminology of the proof of Proposition 4.2, the trivializations of $TX^\phi\oplus W\oplus E$ at $v|_{S^1}$ transferred by a cobordism and pulled back from the trivialization of this bundle at $u|_{S^1}$ are the trivializations given by the spin structure on this bundle. By Lemma 5.4, the difference between the two trivializations of $v|_{S^1}^*W$ is given by

$$\lfloor w_2(W),[u|_{S^1}]_{\mathbb{Z}_2}\rfloor=\lfloor w,b_1\rfloor\,.$$

The difference between the trivialization of v^*E in the second bullet in the proof of Propositions 5.7 and the trivialization pulled back from the corresponding trivialization of u^*E is given by

$$\big\langle w_2(E),[u\sqcup_{\mathfrak{c}_{D^2}}v]_{\mathbb{Z}_2}\big\rangle=\big\langle\varpi,\mathfrak{d}(\beta)\big\rangle.$$

Along with Proposition 4.1, this implies that the sign of the action of the automorphism (1.7) on $\det(D_{TX,\mathrm{d}\phi})$ at u is still given by the left-hand side of the second expression in (5.6). Thus, the first claim of Corollary 5.9 holds in this case as well.

If the restrictions of u and $v\equiv\phi\circ u\circ\mathfrak{c}_{D^2}$ to $S^1\subset D^2$ are not homologous, we can simply choose a spin sub-structure on TX^ϕ, starting with a trivialization of $TX^\phi\oplus W\oplus E$ at $u|_{S^1}$, so that the action of the automorphism (1.7) on $\det(D_{TX,\mathrm{d}\phi})$ at u is orientation-preserving. So, the first claim of Corollary 5.9 holds again. If $w=0$, the $W=0$ case of the discussion in the previous paragraph (without transfers) applied to any spin structure on $TX^\phi\oplus E$ establishes the last claim of Corollary 5.9.

(2) Suppose next that Σ is a cylinder with ordered boundary components. Let

$$\begin{aligned}\widetilde{\mathfrak{M}}_{\Sigma}(X,X^{\phi},\mathbf{b};J) &= \big\{(u,r)\in\mathfrak{B}(X)^{\phi,\mathrm{id}_{\partial\Sigma}}\times\mathring{\mathbb{I}}:\ \bar\partial_{J,\mathfrak{j}_r}u=0,\ u_*[\Sigma,\partial\Sigma]=\beta,\\ &\qquad\qquad u_*[(\partial\Sigma)_1]=b_1,\ u_*[(\partial\Sigma)_2]=b_2\big\},\\ \widetilde{\mathfrak{M}}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J) &= \bigcup_{\mathbf{b}'\in\mathcal{P}(\mathbf{b})}\widetilde{\mathfrak{M}}_{\Sigma}(X,X^{\phi},\mathbf{b}';J),\end{aligned}$$

with $\mathfrak{j}_r\in\mathcal{J}_\Sigma$ as in Section 2.1. Since $\mathfrak{M}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)=\widetilde{\mathfrak{M}}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)/S^1$, there is a canonical isomorphism

$$\det(D_{TX,\mathrm{d}\phi})=\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\widetilde{\mathfrak{M}}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)\big)\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_1S^1)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathfrak{M}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)\big)\,.$$

Since every diffeomorphism of Σ preserving the orientation and the boundary components is isotopic to the identity, it is sufficient to consider the automorphisms induced by the diffeomorphisms h_Σ and $\mathfrak{c}_\Sigma$ defined in Section 2.1 and their composite. By Corollary 4.5 and Remark 4.3, the actions of the first two automorphisms on $\det(D_{TX,\mathrm{d}\phi})$ at $u\in\widetilde{\mathfrak{M}}^{\cup}_{D^2}(X,X^{\phi},\mathbf{b};J)$ are orientation-reversing under our assumptions (5.6) if u and its images under these automorphisms are homotopic. Their actions on S^1 are also orientation-reversing, since they are induced by the maps

$$S^1\longrightarrow S^1,\qquad z\longrightarrow 1/z=\bar z\,.$$

Thus, these automorphisms preserve an orientation on $\mathfrak{M}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)$. The same is the case of the composite automorphism. This establishes both claims of Corollary 5.9 at the elements $[u]$ of $\widetilde{\mathfrak{M}}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)$ homotopic to their images under each automorphism. The remaining cases are handled as in the disk case above.

(3) Suppose Σ is not a disk or a cylinder. The forgetful morphism

$$\mathfrak{f}\colon\mathfrak{M}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)\longrightarrow\mathcal{M}_\Sigma$$

canonically induces an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathfrak{M}^{\cup}_{\Sigma}(X,X^{\phi},\mathbf{b};J)\big)\approx\det(D_{TX,\mathrm{d}\phi})\otimes\mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathcal{M}_\Sigma\big).$$

The sign of the action on $\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathcal{M}_\Sigma)$ induced by a diffeomorphism $h:\Sigma\longrightarrow\Sigma$ is described by Proposition 2.5. The corresponding sign for $\det(D_{TX,\mathrm{d}\phi})$ is given by Corollary 4.5 without the extra sgn_h term in the homotopic cases; see Remark 4.3. Under our assumptions (5.6), the two signs are again the same. The non-homotopic cases are treated as in (1) above. □

By [7, Lemma 2.3], every rank n real bundle pair $(V,\tilde c)\longrightarrow(S^1,\mathfrak{a})$ is trivial, i.e. there is a vector bundle isomorphism $\Psi\colon V\longrightarrow S^1\times\mathbb{C}^n$ covering the identity on S^1 such that

$$\Psi\circ c=\{\mathrm{id}\times\mathfrak{c}_{\mathbb{C}^n}\}\circ\Psi,$$

where $\mathfrak{c}_{\mathbb{C}^n}$ is the standard conjugation on $\mathbb{C}^n$. Furthermore, there are two homotopy classes of such real trivializations and they correspond to the two homotopy classes of real trivializations of $\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\tilde c)$. In similarity with Definition 5.2, we define a spin sub-structure on a real bundle pair $(W,\tilde\phi)\longrightarrow(X,\phi)$ to be a collection of homotopy classes of trivializations of real bundle pairs $\alpha^*(W,\tilde\phi)\longrightarrow(S^1,\mathfrak a)$, one class for each real loop $\alpha\colon(S^1,\mathfrak a)\longrightarrow(X,\phi)$, such that for every real map

$$F\colon(\mathbb I\times S^1,\mathrm{id}_{\mathbb I}\times\mathfrak a)\longrightarrow(X,\phi),\qquad (s,z)\longrightarrow F_s(z),$$

a trivialization of the real bundle pair $F^*(W,\tilde\phi)\longrightarrow(\mathbb I\times S^1,\mathrm{id}_{\mathbb I}\times\mathfrak a)$ restricts to a trivialization of the real bundle pair $F_s^*(W,\tilde\phi)\longrightarrow(S^1,\mathfrak a)$ in the chosen homotopy class for each $s\in\mathbb I$.

Corollary 5.10. *Let (X,ω) be a symplectic $2n$-manifold with an anti-symplectic involution ϕ, (Σ,c) be a genus g oriented sh-surface, $J\in\mathcal J_\phi$, and $\mathbf b$ be as in (2.4). We also assume that*

- *n is odd if either $2b_i=0$ for all i and $|c|_0+|c|_1-g\in2\mathbb Z$ or $b_i=\pm b_j$ for some $i\neq j$;*
- *X^ϕ is orientable and $w_2(TX^\phi)\in H^2(X^\phi;\mathbb Z_2)$ is a spin class if $|c|_0\neq0$;*
- *$w_2^{\Lambda_{\mathbb C}^{\mathrm{top}}\mathrm d\phi}(\Lambda_{\mathbb C}^{\mathrm{top}}TX)\in H^2_\phi(X)$ is a spin class if $|c|_1\neq0$.*

If

$$\frac12\langle c_1(TX),B\rangle+\sum_{i=1}^{|c|_0}\lfloor w_2(TX),b_i\rfloor+\sum_{i=|c|_0+1}^{|c|_0+|c|_1}\lfloor w_2^{\Lambda_{\mathbb C}^{\mathrm{top}}\mathrm d\phi}(\Lambda_{\mathbb C}^{\mathrm{top}}TX),b_i\rfloor\in2\mathbb Z\,,\tag{5.7}$$

then the natural automorphisms of $\mathfrak M^\cup(X,\mathbf b;J)^{\phi,c}$ are orientation-preserving with respect to the orientation induced by some spin sub-structure on TX^ϕ and some spin sub-structure on $(TX,\mathrm d\phi)$; the former is needed only if $|c|_0\neq0$, while the latter is needed only if $|c|_1\neq0$. If $\Sigma=D^2$ and $c\neq\mathrm{id}_{S^1}$, the condition (5.7) can be dropped.

Proof. By the same argument as in the proof of Lemma 5.3, $(TX,\mathrm d\phi)$ admits a spin sub-structure if $w_2^{\Lambda_{\mathbb C}^{\mathrm{top}}\mathrm d\phi}(\Lambda_{\mathbb C}^{\mathrm{top}}TX)$ is a spin class. By [15, Corollary 6.2], $\mathfrak M^\cup(X,\mathbf b;J)^{\phi,c}$ is orientable under our assumptions. By the proofs of Propositions 5.7 and [15, Corollary 6.2], an orientation on $\mathfrak M^\cup(X,\mathbf b;J)^{\phi,c}$ is induced by a spin sub-structure on TX^ϕ and a spin sub-structure on $(TX,\mathrm d\phi)$; the former is needed only if $|c|_0\neq0$, while the latter is needed only if $|c|_1\neq0$. By Corollary 4.4, Remark 4.3, Proposition 2.5, [15, Lemma 6.1], (1) and (2) in the proof of Corollary 5.9, and the treatment of non-homotopic cases in (1) of the proof of Corollary 5.9, the sign of the action on $\det(D_{TX,\mathrm d\phi})$ induced by an orientation-reversing diffeomorphism $h\colon\Sigma\longrightarrow\Sigma$ is the left-hand side of (5.7) plus $n\,\mathrm{sgn}(\mathrm{DM}_h)$, where $\mathrm{sgn}(\mathrm{DM}_h)$ is the corresponding sign on $\mathcal D_c^*$ if Σ is a disk or a cylinder and on $\mathcal M_\Sigma$ otherwise; if h is orientation-preserving, the sign is just $n\,\mathrm{sgn}(\mathrm{DM}_h)$. Thus, the first claim of Corollary 5.10 is obtained by considering three cases as in the proof of Corollary 5.9. The last claim follows from Corollary 3.6. □

Remark 5.11. The conclusions of Corollaries 5.9 and 5.10 hold under more general circumstances. In particular, the cases with X^ϕ non-orientable can be handled, but with additional care.

We conclude with a short proof of an observation obtained by a rather delicate argument in [12].

Corollary 5.12 ([12, Proposition 3.14])**.** *If $m\in\mathbb Z^+$, the standard anti-holomorphic involution ϕ does not preserve any relative spin structure on the pair $(\mathbb P^{4m+1},\mathbb{RP}^{4m+1})$.*

Proof. Let $\varpi \in H^2(\mathbb{P}^{4m+1};\mathbb{Z}_2)$ and $\beta \in H^2(\mathbb{P}^{4m+1},\mathbb{RP}^{4m+1};\mathbb{Z})$ be the standard generators. Since $\mathbb{RP}^{4m+1}$ is orientable,

$$w_2(\mathbb{RP}^{4m+1}) = \varpi|_{\mathbb{RP}^{4m+1}} \qquad \text{and} \qquad \frac{1}{2}\langle c_1(\mathbb{P}^{4m+1}),\mathfrak{d}(\beta)\rangle + \langle \varpi,\mathfrak{d}(\beta)\rangle = 2m+2 \in 2\mathbb{Z},$$

the automorphism (1.7) on $\mathfrak{M}_{D^2}(\mathbb{P}^{4m+1},\mathbb{RP}^{4m+1},\beta;J)$ is orientation-preserving by Corollary 5.9. Since the minimal Maslov index of the pair $(\mathbb{P}^{4m+1},\mathbb{RP}^{4m+1})$ evaluated on β, i.e. $4n+2$, is not divisible by 4, [12, Theorem 1.1] implies that no relative spin structure on the pair $(\mathbb{P}^{4m+1},\mathbb{RP}^{4m+1})$ is preserved by ϕ. □

5.4 Floer theory

A number of striking implications of anti-symplectic involutions to Floer homology are described in [12]. In this section, we streamline some aspects of the approach in [12], modifying one of the key notions introduced in [12]. This allows us to extend some statements in [12] and significantly simplify some of the proofs, without altering the fundamental principles behind them.

If (X,ω) is a symplectic manifold with an anti-symplectic involution ϕ and $\beta \in H_2(X,X^\phi;\mathbb{Z})$, let $\mathfrak{d}(\beta) \in H_2(X;\mathbb{Z})$ denote the natural ϕ-double of β as before; see [14, Section 3].

Definition 5.13. Let (X,ω) be a symplectic manifold with an anti-symplectic involution ϕ such that X^ϕ is orientable. A ϕ-relative spin structure on (X,X^ϕ) consists of

(1) a real oriented vector bundle $E \longrightarrow X_{[3]}$, where $X_{[3]}$ is the 3-skeleton of X in a ϕ-invariant triangulation extending a triangulation of X^ϕ, such that

$$w_2(TX^\phi) = \varpi|_{X^\phi} \quad \text{and} \quad \frac{1}{2}\langle c_1(TX),\mathfrak{d}(\beta)\rangle + \langle \varpi,\mathfrak{d}(\beta)\rangle \in 2\mathbb{Z} \quad \forall\, \beta \in H_2(X,X^\phi;\mathbb{Z}), \tag{5.8}$$

with $\varpi \in H^2(X;\mathbb{Z}_2)$ defined by $\varpi|_{X_{[3]}} = w_2(E)$, and

(2) a spin structure on $TX^\phi \oplus E \longrightarrow X^\phi \cap X_{[3]}$.

By Footnote 4, (X,X^ϕ) admits a ϕ-relative spin structure if and only if there exists $\varpi \in H^2(X;\mathbb{Z}_2)$ satisfying (5.8). We note that our notion of ϕ-relatively spin structure is different from that of [12, Definition 3.11]; see more below. A natural equivalence on the set of relative spin structures is described by [12, Definition 3.4]; we will view two relative spin structures as identical if they are equivalent in this sense.

As explained in [12, Section 3.2], there are two relative spin structures on $(\mathbb{P}^{2n+1},\mathbb{RP}^{2n+1})$. They correspond to the same class ϖ, which is 0 if n is odd and the generator of $H^2(\mathbb{P}^{2n+1};\mathbb{Z}_2)$ if n is even, and are ϕ-relative spin in the sense of Definition 5.13. If (Y,ω_Y) is a symplectic manifold, the interchange of factors is an anti-symplectic involution on $(Y \times Y, \pi_1^*\omega_Y - \pi_2^*\omega_Y)$; the diagonal is the fixed locus. In this case, the class $\varpi = \pi_1^* c_1(TY)$ provides a ϕ-relative spin structure.

The significance of Definition 5.13 from the point of view of the Floer-theoretic applications in [12] is described by the next corollary, which is essentially a special case of Corollary 5.9.

Corollary 5.14. *Let (X,ω) be a symplectic manifold with an anti-symplectic involution ϕ, $J\in\mathcal{J}_\phi$, and $\beta\in H_2(X,X^\phi;\mathbb{Z})$. If $X^\phi\subset X$ is orientable and there exists $\varpi\in H^2(X;\mathbb{Z}_2)$ such that (5.8) holds, then the isomorphism*

$$\mathfrak{M}_{D^2}(X,X^\phi,\beta;J)\longrightarrow\mathfrak{M}_{D^2}(X,X^\phi,-\phi_*\beta;J)$$

given by (1.7) is orientation-preserving with respect to the orientations induced by any relative spin structure associated with ϖ.

In [12, Definition 3.11], a relative spin structure on $X^\phi\subset X$ is called ϕ-relative if it is preserved by the involution ϕ; no simple test, like (5.8), is provided for this property. The motivation for [12, Definition 3.11] appears to be the mistake in [10, Proposition 11.5], which misses the possibility that a relatively spin structure need not to be preserved when pulled back by the involution ϕ. The statement of [10, Proposition 11.5] is indeed valid for the ϕ-relative spin structures of [12, Definition 3.11], which follows immediately from [33, Proposition 5.1] and [12, Theorem 1.1]; the latter corrects the statement of [10, Proposition 11.5] by taking into account the change of the relative spin structure under the pull-back by ϕ. However, [12, Definition 3.11] does not seem ideally suited for the remarkable applications considered in [12]. The primary idea behind these applications is to study whether the involution (1.7) is orientation-preserving with respect to the orientation defined by a fixed relative spin structure on the two sides; the applications depend on this involution being orientation-preserving. This involution is orientation-preserving if the relative spin structure is ϕ-relative spin in the sense of [12, Definition 3.11] *and* the Maslov index of (TX,TX^ϕ) is divisible by 4, but not otherwise; see [12, Theorem 1.1]. For example, this is the case for $(\mathbb{P}^{2n+1},\mathbb{RP}^{2n+1})$ with n odd, but not even, and for $(Y\times Y,\pi_1^*\omega_Y-\pi_2^*\omega_Y)$ with $c_1(TY)$ even, but not odd. The nature of [12, Definition 3.11] forces the authors to split the consideration of $\mathbb{P}^{2n-1}$ and $(Y\times Y,\pi_1^*\omega_Y-\pi_2^*\omega_Y)$ in [12, Section 6.4] and in [12, Section 6.3.1], respectively, based on the parity of n and $c_1(TY)$, with a careful consideration of the cases which are not ϕ-relatively spin. There is no distinction between the two cases from the point of view of Definition 5.13 and Corollary 5.14. In the same spirit, we obtain the following extensions of results in [12].

Proposition 5.15. *Let (X,ω) be a compact symplectic manifold with an anti-symplectic involution ϕ.*

(1) The justification of the conclusion of [12, Theorem 1.5] applies also if X^ϕ is relative spin in X and ϵ_1 is increased by $\langle\varpi,\mathfrak{d}(\beta)\rangle$, with ϖ as in Definition 5.13.

(2) The justification of the conclusion of [12, Corollary 1.6] applies also if (X,X^ϕ) is ϕ-relative spin in the sense of Definition 5.13.

(3) The justification of the conclusion of [12, Corollary 1.8] applies also if $c_1(X)|_{\pi_2(X)}=0$, X^ϕ is orientable, and $w_2(X^\phi)=\varpi|_{X^\phi}$ for some $\varpi\in H^2(X;\mathbb{Z}_2)$ such that $\langle\varpi,\mathfrak{d}(\beta)\rangle=0$ for all $\beta\in\pi_2(X,X^\phi)$.

Proof. (1) The sign $(-1)^{\epsilon_1}$ in [12, Theorem 1.5] is determined by [12, Theorem 4.12] applied with a ϕ-relative spin structure in the sense of [12, Definition 3.11]; see the beginning of [12, Section 6.2]. The restriction on the relative spin structure in [12, Theorem 1.5] ensures that the pull-back relative spin structure used to orient the moduli space on the left-hand side of [12, (4.10)] is the same as the relative spin structure used to orient the moduli space on the right-hand side of [12, (4.10)]. For an arbitrary relative spin structure, the difference between the orientations induced by the two relative spin structures is given by [12, Proposition 3.10] and equals $(-1)^{\langle\varpi,\mathfrak{d}(\beta)\rangle}$, which needs to be combined with the sign of $(-1)^{\epsilon_1}$ in [12]. Alternatively and more directly, the sign of the action on the unmarked moduli space of disks in the proof of [12, Theorem 1.5] is $(-1)^{\epsilon}$, where

$$\epsilon = \frac{1}{2}\langle c_1(TX), \mathfrak{d}(\beta)\rangle + \langle\varpi, \mathfrak{d}(\beta)\rangle\,;$$

see the first part of the proof of Corollary 5.9, restricting to the $w=0$ case. The proof of [12, Theorem 4.12] then shows that the statement of [12, Theorem 4.12] applies to the map in [12, (4.10)] with the moduli space on the left-hand side oriented by the same relative spin structure as on the right-hand side, provided ϵ in [12, (4.10)] is increased by $\langle\varpi, \mathfrak{d}(\beta)\rangle$. The proof of [12, Theorem 1.5] in [12, Section 6.2] then applies without any changes under our weaker assumptions.

(2) By the first part of this proposition, the crucial identity

$$\mathfrak{m}_{0,\tau_*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$$

in the proof of [12, Corollary 1.6] in [12, Section 6.2] remains valid under our weaker assumptions. The rest of the proof in [12] applies without any changes.

(3) By the second part of this proposition, the proof of [12, Corollary 1.8] in [12, Section 6.2] applies without any changes under our weaker assumptions. □

For example, [12, Theorem 1.5] does not apply to complete intersections $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ of dimension at least 2 such that

$$n-|\mathbf{a}|\equiv 2 \mod 4 \qquad \text{and} \qquad a_1^2+\ldots+a_l^2\equiv|\mathbf{a}| \mod 4,$$

since these complete intersections do not admit a ϕ-relative spin structure in the sense of [12, Definition 3.11]. However, Proposition 5.15(1) applies to these symplectic manifolds, since they admit a ϕ-relative spin structure in the sense of Definition 5.13.

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