On Symplectic Sum Formulas in Gromov-Witten Theory: an Update

Aleksey Zinger 02/26/15

Abstract

This note describes the changes made in the second arXiv version of [FZ0] relative to the first. It also summarizes the contributions of and issues with [IP4], [IP5], and [LR]. The only differences with the 12/28/14 update are in (IP4p2) and (IP4p3).

Contents

1 The changes

The topological part of the first version of [FZ0] and the exposition on one of the three applications described in [IP5] are now contained in [FZ1, FZ2]. The purpose of these papers is to describe topological aspects of Gompf's symplectic sum construction in detail and to thoroughly explore Ionel-Parker's suggested rim tori refinements of the standard relative GW-invariants and of the usual symplectic sum formulas.

The only change in the content of the issue items listed in [FZ0, Section 2] concerns (IPt4) and (IPt5). Concerning (IPt4), the covers of the sort described in [IP4] can indeed be constructed even if the divisor V is disconnected. However, there is no topological description of these covers in [IP4], their group of deck transformations is specified incorrectly, the nature of the resulting "invariants" is mis-described, and the description of these invariants in two simple cases is completely wrong; see Section 2. Concerning (IPt5), there is no fundamental problem with defining a cohomology class on the product of rim tori covers by intersecting homology classes with closed oriented submanifolds. However, the required closed oriented submanifolds are not properly constructed and the consequences of the suggested refinement are misrepresented; see Section 3.

We have also corrected the review of Gompf's symplectic sum construction, which had originally followed closely that in [IP5]. The review of relative moduli spaces and GW-invariants has been moved before the more general statement of the symplectic sum formula and now includes an overview of the rim tori refinement suggested in [IP4]. The comparison of the formulations of the symplectic sum formula in [IP5], [LR], and [Lj2] has been split into two parts and now includes an overview of the rim tori refinement suggested in [IP5].

2 Roundup on [IP4]

The abstract and the 4-page summary of [IP4] suggest that this paper defines relative GW-invariants for arbitrary (X, ω, V) and more generally than the relative GW-invariants of [LR]. While the relative moduli spaces in [IP4] are defined for a wider class of almost complex structures on (X, ω, V) than in [LR], relative GW-invariants for (X, ω, V) are defined in [IP4] only in a narrow range of "semi-positive" cases. According to the last paragraph of [IP4, Section 1], the main construction of relative GW-invariants in [IP4] applies to arbitrary (X, ω, V) because of a VFC construction *in a separate paper [IP5]*, listed as *in preparation* (not *work in progress*) in the references. This citation first appeared in the 2001 arXiv version; it replaced Remark 1.8 in the 1999 arXiv version, which claimed that the semi-positive restriction can be removed because of the VFC construction of [LT]. However, applying this construction would have required gluing maps with rubber components, which is not done in [IP4]. The VFC construction advertised in [IP4] is claimed in [IP6] by building on [CM]. However, [CM] first appeared on arXiv almost 5.5 years after the 2001 version of [IP4]. Furthermore, for two of the most crucial analytic points, [IP6, Lemma 7.4] and [IP6, (11.4)], which require gluing maps with rubber components, the authors cite [IP4] and [IP5]; these two papers restrict to "semi-positive" cases precisely to avoid such gluing.

In my view, the contributions of [IP4] consist of:

(IP4c1) the conditions on the (J, ν) pairs of [RT1, RT2] in [IP4, Definition 3.2] that should lead to a geometric definition of relative GW-invariants in some "semi-positive" cases. These cases, which are not specified correctly in [IP4], form a smaller portion of the overall cases than in the absolute setting of [RT1, RT2]. The introduction of (J, ν) pairs in [RT1, RT2] was a fundamental innovation that immediately led to some applications (associativity of quantum product for semi-positive symplectic manifolds and enumeration of rational curves in $\mathbb{P}^n$). It later formed the basis for the virtual class constructions in symplectic topology. In [IP4], it is simply an adaptation of this innovation in combination with a rescaling similar to the earlier [LR].

(IP4c2) a very vague suggestion that the usual relative evaluation maps can be lifted to some natural covers of the symplectic divisor and its products and that this leads to refined relative GW-invariants in some sense.

The problems in [IP4] include the following.

(IP4p1) The notion of relative map described by [IP4, Definitions 7.1,7.2] allows the contact marked points to lie in any layer (instead of just the last one). If the contact marked points are not required to lie on the last layer, the relative moduli space cannot be Hausdorff. Since [IP4, Section 6] does force the contact marked points to lie on the last layer, this is just an omission, but in the definition of the most important notion of [IP4].

(IP4p2) The relative maps are defined in [IP4] in terms of elements of the kernel of the linearized $\bar{\partial}$-operator on the normal bundle. In the proof of Proposition 6.6, they are described as $(\widetilde{J}, \pi^*\nu)$-holomorphic maps for an almost complex structure $\widetilde{J}$ on the $\mathbb{P}^1$-bundle

$$\pi : \mathbb{P}_X V \equiv (\mathcal{N}_X V \oplus \mathcal{O}_V) \longrightarrow V$$

induced from V via an arbitrary connection on $\mathcal{N}_X V$. This is incorrect because $\widetilde{J}$ should be induced by a connection arising from a torsion-free connection on TX and the correct induced ν-term on $\mathbb{P}_X V$ does not (generally) vanish in the normal direction to V; see the beginning of Section 4.1 and the equation above Remark 4.8 in [FZ0].

(IP4p3) The rescaling argument in [IP4, Section 6] does not check that the bubbles in the different layers connect (which is done in [LR]). As [LR], [IP4] does not check that the resulting moduli space is Hausdorff. Overall, [IP4, Section 6] is an imperfect adaptation of the rescaling argument of [LR, Section 3.2] for more general (J, ν)-pairs.

(IP4p4) The failure to carry out the gluing in the simplest possible case in [IP5] raises doubts about the compatibility of the (J, ν)-pairs of [IP4] with gluing as necessary for any virtual class construction. In contrast, the compatibility with gluing in [RT1] is illustrated in the proof of the associativity of quantum multiplication.

(IP4p5) The topology on the desired rim tori covers of [IP4, Section 5] is not specified. The description of this cover is wrong about the group of its deck transformations and about the resulting GW-invariants in the simple cases of [IP5, Lemmas 14.5,14.8]; see [FZ0, Section 4.3] and [FZ2, Remarks 6.5,6.8].

(IP4p6) The lifts of the relative evaluation maps to the above covers are not unique and the refined relative GW-"invariants" generally depend on the choice of such a lift; see [FZ0, Section 4.4] and [FZ1, Sections 1.1,1.2]. This makes these "invariants" not computable outside of very rare cases. It is possible to use them for some qualitative applications though, as demonstrated in [FZ1, FZ2].

3 Roundup on [IP5]

According to the abstract, the long summary, and the main theorems in [IP5], i.e. *Symplectic Sum Theorem* and Theorems 10.6 and 12.3, the symplectic sum formulas in [IP5] are proved without any restrictions on X, Y, V, but most arguments are clearly restricted to "semi-positive" cases. According to the beginning of [IP5, Section 8], *the key analysis step* is *obtaining estimates on the linearization* of the $\bar{\partial}$-operator of an approximately J-holomorphic map. This part of the proof has 3 serious consecutive errors, i.e. with each sufficient to break it. The gluing argument has additional problems, leaving pretty much nothing correct in it (or the entire paper). I see no practical way of fixing it without using the SFT approach suggested by [LR] with the more regular almost complex structures of [LR].

The problems in [IP5] include the following.

(IP5p1) The operator in [IP5, (7.5)] is not the adjoint of the operator in [IP5, (7.4)] with respect to any inner-product, because the first component of its image does not satisfy the imposed average condition. This ruins the argument regarding the linearized operators being uniformly invertible at the start. The correction would have to be of L^2-type, which is not compatible with the required L^p_1-norms.

(IP5p2) Gauss's relation for curvatures, [IP5, (8.7)], is written in a peculiar way, resulting in a sign error. The sign error in [IP5, (8.7)] is crucial to establishing a uniform bound on the incorrect adjoint operator in [IP5, (7.5)].

(IP5p3) The argument at the bottom of page 984 in [IP5] implicitly presupposes that the limiting element η lies in the Sobolev space $L^{1,2}_{\mathbf{s}}$. This is the last step in establishing a uniform bound on the incorrect adjoint operator in [IP5, (7.5)].

(IP5p4) The justification for the uniform elliptic estimate in [IP5, Lemma 8.5] indicates why the degeneration of the domains does not cause a problem, but makes no comment about the degeneration of the target. It is unclear that it is in fact uniform with the chosen norms.

(IP5p5) The map Φ_λ in [IP5, Proposition 9.1] appears to be non-injective because the metrics on the target $\mathcal{Z}_\lambda$ collapse in the normal direction to the divisor V as $\lambda\longrightarrow 0$. The wording of the second-to-last paragraph on page 938 suggests that the norms are weighted to account for this collapse and the convergence estimate of [IP5, Lemma 5.4] could accommodate norms weighted heavier in the vertical direction, but the rather light weights in the norms of [IP5, Definition 6.5] appear far from sufficient.

(IP5p6) Neither the summary of [IP5] nor the proof of [IP5, Proposition 9.4] makes any mention of whether the quadratic error term in the expansion [IP5, (9.10)] of the $\bar{\partial}$-operator is uniformly bounded. The latter mentions only the need for the 0-th and 1-st order terms to be uniform (in (a) and (b) on page 939).

(IP5p7) As explained in the summary and in Section 12 in [IP5], the S-matrix appears in the main formulas (0.2) and (12.7) of [IP5] due to components of limiting maps sinking into V. Such components should correspond to maps into the rubber up to the $\mathbb{C}^*$-action on the target, just as happens in the relative maps setting of [IP4, Section 7]. This action, which is forgotten in the imprecise limiting argument of [IP5, Section 12], implies that such limits do not contribute to the GW-invariants of $X\#_V Y$ for dimensional reasons, and so the S-matrix should not appear in any symplectic sum formula of [IP5]. As shown in [FZ0, Section 6.5], the S-matrix does not matter anyway because it *acts as* the identity in all cases and not just in the cases considered in [IP5, Sections 14,15], when the S-matrix *is* the identity. I am not aware of anyone else who believes the S-matrix should have appeared in the first place.

In my view, the contributions of [IP5] consist of:

(IP5c1) a vague suggestion that refined relative GW-"invariants" give rise to a refined symplectic sum formula of some sort. This suggestion indeed leads to a refined relation between the GW-invariants of a smooth fiber $X\#_V Y$ and the singular fiber $X\cup_V Y$. Contrary to a claim in the abstract of [IP5], it does not usually lead to a refined relation between the GW-invariants of $X\#_V Y$ and any kind of relative GW-invariants of (X,V) and (Y,V) because the cohomology of products of the covers of [IP4, Section 5] usually does not admit a Kunneth decomposition; see [FZ0, Section 5.3] and [FZ2, Example 3.7]. Even when the GW-invariants of $X\#_V Y$ split into refined relative GW-"invariants" of (X,V) and (Y,V), this is rarely computationally useful because of dependence of the latter on the choice of the lift; see Section 2. I am not aware of a single application of this suggested refinement in the 14 years since the first arXiv version of [IP5]; some qualitative applications are obtained in [FZ2] though. Furthermore, the existence of the crucial refined degree gluing map [IP5, (3.10)] is never established; as indicated in [FZ2, Sections 3.1,4.1], this is a subtle issue that depends on consistent choices of certain coset representatives.

(IP5c2) fairly detailed, but not completely correct, alternative proofs of three formulas in enumerative geometry that had been previously by other methods. These are certainly nice illustrations of the power of the symplectic sum formula, especially once their exposition is properly cleaned up (the ideas behind the proofs are clear from [IP5]). However, these applications are not new results and the arguments still have gaps; none of the three main claims is even stated correctly. As noted in the middle of page 941 in [IP5], two of the applications are essentially symplectic sum re-formulations of the original proofs. The third application is fundamentally different from the original proof and also contains the most significant gap; see [FZ2, Remark 6.12]. This third application would have made for a nice IMRN level paper, but certainly not an Annals paper.

The general structure of the symplectic sum formula itself had been known well before the first arXiv version of [IP5] and even before [IP3]; the arXiv version of the latter predates the first version of [IP4] by 12.5 months and of [IP5] by 28.5 months. It describes relative GW-"invariants" that depend on the choice of (J,ν) because the last two conditions of [IP4, Definition 3.2] are not imposed in [IP3]; see [IP3, Theorem 2.5]. This dependence is accidentally mentioned even in [IP5, Definition 11.3]. The symplectic sum formula with cohomology insertions that do not come from the singular fiber in [IP5, Section 13] involves similar kinds of relative "invariants" that depend on auxiliary choices.

4 Roundup on [LR]

The introduction to [LR] suggests that the primary purpose of this paper was intended to be applications to the birational geometry of Calabi-Yau threefolds. The applications and related background parts of [LR], i.e. Sections 1,2, and 6, could have been written a lot more efficiently, but appear to be solid content-wise.

The remainder of [LR], just 43 lightly written journal pages, appears to have been more of a secondary consideration. It is organized in a haphazard way, in contrast to [IP4, IP5], and purports to establish the compactness and Hausdorffness of the relative moduli space, define relative GW-invariants via a new virtual cycle construction, and address all of the gluing issues needed to prove a symplectic sum formula for GW-invariants. In my view, its contributions consist of:

(LRc1) the notion of relative stable map. While [LR, Definition 3.14] takes up a whole page, I do not think it is possible to guess the desired meaning from it. However, knowing what it should be, one might believe that this is what the authors meant.

(LRc2) a suggestion to define relative stable maps and approach the symplectic sum formula via the SFT stretching of the target and the domain.

On the other hand, [LR] contains

(LRp1) no proof of anything significant (outside of applications),

(LRp2) no attempt at a proof of anything significant, (LRp3) no notion of morphism to the singular fiber suitable for a proof of a symplectic sum formula for GW-invariants; see [FZ0, Section 4.0].

These are the main reasons why it is possible to comment a lot more on [IP4, IP5] than on [LR].

According to A.-M. Li's response [Li], issues such as Hausdorffness of the relative moduli spaces and the injectivity and surjectivity of the gluing map are standard. These properties are clearly very specific to the setup. The exact same claims are made about the relative moduli spaces, either explicitly or implicitly, in the first three arXiv versions of [LR]. However, in these three versions, the equivalence relation on the relative maps does not involve the $\mathbb{C}^*$-action on the rubber components (only $\mathbb{R}$-action, corresponding to the log of the norm); see the middle of page 63 and Definition 4.16 in the third version, for example. A mention of any $\mathbb{C}^*$-action on maps appears only once, in the middle of page 72 in the third version, with a very cryptic wording.

Detailed comments on [Li] are available on my website.

5 Post Scriptum

Many papers cite [IP5] and [LR], but usually along with [Lj2] and without referring to a specific formula in any of the papers. There is no correct symplectic sum formula, even for primary GW-invariants, to cite in either in [IP5] or [LR] (three formulas in the latter combine into a nearly correct decomposition formula for primary GW-invariants). Even if the formulas in [IP5] and [LR] were correct, any citations of them for decompositions of descendant invariants or of virtual classes would have been erroneous anyway (most computations in GW-theory involve descendant invariants). The two formulas at the bottom of page 201 in [Lj2] are correct and include decompositions of descendant invariants and of virtual classes, but have no number associated with them.

In the 8 months since the first arXiv version of [FZ0], no one has admitted to being a referee for [IP4], [IP5], or [LR]. It appears that the referees for these papers prefer to exercise their right to hide, instead of either standing by their actions or taking responsibility for them.

Journal editors, especially those of the top journals, have enormous influence over the directions different fields of mathematics take. In my view, they have a moral responsibility to ensure that

(1) the evaluation of the suitability of a paper for the journal is handled fairly and promptly, especially when done by a potential competitor. Referees should be required to advise on the former with a reasonable general description of the paper within a month.

(2) the evaluation of the correctness of a paper is done thoroughly relative to the level of the journal. The editors handling [IP4, IP5] and [LR] clearly failed in this regard.

At least, journal editors cannot hide behind the anonymity accorded to the referees.

References

[CM] K. Cieliebak and K. Mohnke, *Symplectic hypersurfaces and transversality in Gromov-Witten theory*, J. Symplectic Geom. 5 (2007), 281–356.

[FZ0] M. Farajzadeh Tehrani and A. Zinger, *On Symplectic Sum Formulas in Gromov-Witten Theory*, math/1404.1898v2.

[FZ1] M. Farajzadeh Tehrani and A. Zinger, *On the rim tori refinement of relative Gromov-Witten invariants*, pre-print.

[FZ2] M. Farajzadeh Tehrani and A. Zinger, *On the refined symplectic sum formula for Gromov-Witten invariants*, pre-print.

[IP3] E. Ionel and T. Parker, *Gromov-Witten invariants of symplectic sums*, Math. Res. Lett. 5 (1998), 563–576.

[IP4] E. Ionel and T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. 157 (2003), no. 1, 45–96.

[IP5] E. Ionel and T. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. 159 (2004), no. 3, 935–1025.

[IP6] E. Ionel and T. Parker, *A natural Gromov-Witten virtual fundamental class*, 1302.3472.

[Li] A.-M. Li, *Remark on symplectic relative Gromov-Witten invariants and degeneration formula*, math/1405.3825.

[LR] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. 145 (2001), no. 1, 151–218.

[Lj2] J. Li, *A degeneration formula for GW-invariants*, J. Diff. Geom. 60 (2002), no. 1, 199–293.

[LT] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, in *Topics in Symplectic 4-Manifolds*, 47–83, Internat. Press 1998.

[RT1] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Diff. Geom. 42 (1995), no. 2, 259–367.

[RT2] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. 130 (1997), no. 3, 455–516.