Selected Research Accomplishments

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This note summarizes some of my research results and puts them in context with related developments. A non-technical *Research Description* and a *List of Publications* with brief abstracts and connections between the papers are the available at the top of my homepage.

(1) *A local excess intersection approach*

As indicated by the nearly two thousand citations of W. Fulton's treatise on this topic, Excess Intersection Theory plays a very important role in modern algebraic geometry. It is often used to determine the contributions to the Euler class of a vector bundle from degenerate components of the vanishing locus of a specific, typically geometrically meaningful, section. This approach is rigid and global in nature and involves blowing up the degenerate loci to smooth them out. My PhD thesis introduces an approach to this problem which is soft and local in nature. It applies in finite-dimensional singular and infinite-dimensional Fredholm settings. In contrast to the standard approach, it determines the contributions from (not necessarily compact) strata of the vanishing locus and involves no blowups (which may not be even defined in some of the applicable settings). The local excess intersection approach fits ideally with the virtual fundamental class construction in symplectic topology, as the latter is fundamentally about incorporating degenerate contributions. The rudiments of this approach, in both finite-dimensional and infinite-dimensional settings, appear in [18]. It is developed further and applied to a wide range of problems in classical enumerative geometry in [19]. Along with (2) below, this approach in a Fredholm setting underpins the proof of the long-standing BCOV mirror symmetry conjecture described in (3).

(2) *A sharp version of Gromov's compactification for pseudoholomorphic maps*

Gromov's 1985 *Inventiones* paper introduced the notion of pseudoholomorphic map (from a Riemann surface to a symplectic manifold), established a compactness theorem for sequences of such maps, and demonstrated the usefulness of the new notion through applications in symplectic topology. The compact spaces arising from Gromov's work give rise to curve-counting invariants of symplectic manifolds, now known as Gromov-Witten (or GW-) invariants. A drawback of Gromov's compactification is that it is generally too big for maps from Riemann surfaces of positive genera, even under reasonably ideal circumstances. The main strata of these spaces consisting of maps from smooth domains generally have smaller dimensions than some of the boundary strata consisting of maps from nodal domains, in contrast to what one would have expected in classical algebraic geometry. Ruan-Tian's first paper from the early/mid-1990s, which extracted invariants from Gromov's work, speculated that there exist sharper versions of Gromov's compactification for maps from positive-genera Riemann surfaces which take into account information analogous to that of arithmetic genus in algebraic geometry and still give rise to curve-counting invariants in symplectic manifolds. These expectations are confirmed in [23] and [24]. The former also shows that the new compactification is in fact sharp in genus 1, subject to certain naturality conditions. While the standard GW-invariants are known to exhibit a number of important *algebraic* properties, the reduced genus 1 GW-invariants of [24] are more natural from the *geometric* point of view. Similarly to the genus 0 invariants, they give a signed integer count of curves in good cases (the standard genus 1 GW-invariants are usually rational numbers in these cases). By [13], they also satisfy a version of Quantum Lefschetz Hyperplane Principle relating GW-invariants of a complete intersection to the GW-invariants of the ambient projective space (the standard genus 1 GW-invariants do not satisfy this principle). Since the standard and reduced genus 1 GW-invariants differ by the genus 0 invariants, this relation made it possible to compute the genus 1 GW-invariants of complete intersections and eventually led to a proof of the BCOV Mirror Symmetry Conjecture for the genus 1 GW-invariants of Calabi-Yau complete intersection threefolds (the only proof so far).

(3) *Proof of the BCOV Mirror Symmetry Conjecture for genus 1 Gromov-Witten invariants*

Among the most striking mathematical predictions of string theory are the so-called mirror symmetry formulas for GW-invariants. They generally concern Calabi-Yau threefolds, which are of particular importance in physics, and originate with the 1991 prediction of Candelas-de la Ossa-Green-Parkes for the genus 0 GW-invariants of the quintic threefold (a degree 5 hypersurface in $\mathbb{CP}^4$). This prediction was first verified in the mid-1990s by Givental and Lian-Liu-Yau. This development was of such importance that it shortly led to at least three other proofs of the 1991 prediction, two books with expository accounts, and a public dispute described in the 2006 New Yorker's *Manifold Destiny* article. The next case in terms of complexity, the 1993 prediction of Bershadsky-Cecotti-Ooguri-Vafa for the genus 1 GW-invariants of the quintic threefold remained essentially unapproachable for about 10 years. While the only issue in the genus 0 case was an equivariant localization computation, in the genus 1 case a number of other general geometric issues had to be dealt with before such a computation could even be set up. Approaches to verifying this prediction were suggested in the early/mid-2000s by Gathmann and Maulik-Pandharipande, but neither of the two approaches has succeeded yet. In 2007, I completed a 4-year project that finally established the BCOV prediction. In contrast to the earlier attempts, this project involved a thorough analysis of fundamental properties of moduli spaces of genus 1 pseudoholomorphic maps and of related properties of GW-invariants. It resulted in 12 publications [12,13,15–17,20–26], most of which had no apparent connection with the quintic threefold or mirror symmetry more generally. For example, [15,16] describe a smooth compactification of the Hilbert scheme of smooth genus 1 curves in complex projective spaces.

(4) *Towards a theory of singular symplectic varieties*

Algebraic (sub)varieties are the central objects of study in algebraic geometry. Curves and divisors, i.e. subvarieties of dimension and codimension 1 over the ground field, have long been of particular importance; they can be viewed as dual to each other. Gromov's 1985 *Inventiones* paper introducing pseudoholomorphic curves techniques into symplectic topology has led to its numerous connections with algebraic geometry and to the appearance of symplectic divisors in different contexts, including Gompf's symplectic sum construction, degeneration and decomposition formulas for GW-invariants (by G. Tian, Caporaso-Harris, Li-Ruan, J. Li, and B. Parker), McLean's work on affine symplectic geometry, and Sheridan's work on homological mirror symmetry. However, all of the known applications in symplectic topology had involved either *smooth* symplectic divisors or *almost Kahler* normal crossings divisors, as there had been no *symplectic* notion of *singular* divisor. Gromov's book asked about the feasibility of introducing such notions into symplectic topology over 30 years ago. The Gross-Siebert program, running for about 15 years now, is perhaps the best known among the many developments so far indirectly suggesting such feasibility and more directly motivating the desirability for a symplectic topology theory of singularities. Symplectic topology notions of singular (sub)varieties should involve only some intrinsic symplectic data, but at the same time ensure the existence of auxiliary almost complex structures needed for making it useful. A simple, purely symplectic notion of normal crossings (or NC) divisor is introduced in [1] and shown to be weakly homotopy equivalent to an almost

Kahler notion. The latter is trivial to do in the smooth setting, but becomes a highly delicate and technical issue in the NC setting. Nevertheless, our weak homotopy equivalence result is straightforward to use. It ensures that any construction on the deformation equivalence classes of almost Kahler divisors automatically descends to the deformation equivalence classes of NC symplectic divisors. As an example, this implies that the curve-counting invariants defined by B. Parker in the almost Kahler category are in fact *symplectic* invariants (GW-invariants relative to an NC symplectic divisor). The principle established in [1] is applied in [2] to introduce a smoothing construction for NC symplectic singularities which includes a multifold version of Gompf's symplectic sum construction and to produce a plethora of potentially new symplectic manifolds. A construction producing NC degenerations of symplectic manifolds is described in [3]; it includes a multifold version of Lerman's now classical symplectic cut construction. The constructions of [2] and [3] are shown to be mutual inverses in the appropriate sense in [4].

(5) *Real Gromov-Witten theory and enumerative geometry in all genera*

The enumerative geometry of curves in *complex* algebraic varieties goes back to the nineteenth century and has since spurred the development of many other areas of mathematics (such as Schubert Calculus and Excess Intersection Theory). In reasonable varieties (such as $\mathbb{CP}^2$), the number of complex curves passing through a generic collection of constraints depends only on the genus and the degree of the curves and on the (co)homology classes of the constraints. It was realized early on that the number of curves in *real* algebraic varieties generally depends on the choice of generic representatives for the cohomology classes. A central question in real enumerative geometry has thus been to find lower bounds for the number of such curves that depend only on the cohomology classes of the constraints. The first fundamental advance in this direction, made by Welschinger in 2003, provided *signed* counts of *rational* (genus 0) curves in real algebraic (and more generally symplectic) varieties of dimensions 2 and 3 dependent only on the cohomology classes of the constraints. Welschinger's signed counts have since been translated into the languages of tropical geometry and open/real GW-theory. These interpretations have led to many computations of Welschinger's invariants, including a mirror formula for the genus 0 real GW-invariants of the quintic threefold. However, an analogue of Welschinger's bounds for curves of positive genera (even genus 1) had remained elusive until [8]. It is the culmination of a 3-year investigation [5–7, 11] of the notorious orientability and codimension-one obstructions to defining invariant counts of real and open curves in symplectic manifolds. Like Welschinger's construction of genus 0 real GW-invariants, the construction of positive-genus real GW-invariants in [8] comes from an unexpected angle which makes the solution much simpler than had been expected. This work is accompanied by comparisons and computations in [9, 10, 14] and in a separate appendix, which provide corroborative evidence for the construction itself.

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