Equivariant Localization and Mirror Symmetry

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April 4, 2018

Contents

Chapter 1

Equivariant Cohomology

We review the notion of equivariant cohomology of a topological space with a group action and study some of its properties in Sections 3 and 4; these sections contain everything needed to formulate the Localization Theorem of [2]. In Section 5, we explicitly determine the equivariant cohomology of the complex projective space with the standard $\mathbb{T}\equiv(S^1)^n$-action; this section is not used in the proof of the Localization Theorem, but is central to its applications in Chapter 4. The equivariant pushforward of a continuous equivariant map is constructed in Section 7; it is used in a simple setting in the proof of the Localization Theorem and in a more general setting in Chapter 4. The theorem itself is stated in Section 8 and proved in Section 9; the proof follows [2, Section 3].

1 Topology preliminaries

We begin with preliminaries concerning the topology of infinite-dimensional spaces that are central to the Localization Theorem.

Let

$$\mathbb{C}^\infty = \big\{(z_1,\ldots,z_n,0,0,\ldots)\colon\, n\in\mathbb{Z}^+,\, z_i\in\mathbb{C}\big\} = \bigcup_{n=1}^{\infty}\mathbb{C}^n = \lim_{n\longrightarrow\infty}\mathbb{C}^n\,.$$

The last equality specifies the topology of $\mathbb{C}^\infty$ as the direct limit of the topologies of $\mathbb{C}^n$ for the inclusions $\mathbb{C}^n\longrightarrow\mathbb{C}^{n+1}$. This means that a subset $U\subset\mathbb{C}^\infty$ is open if and only if $U\cap\mathbb{C}^n$ is open for all n. Let

$$S^\infty = \Big\{(z_1,z_2,\ldots)\in\mathbb{C}^\infty\colon\, \sum_{i=1}^{\infty}|z_i|^2=1\Big\}$$

with the subspace topology. The group $\mathbb{C}^*$ acts on $\mathbb{C}^\infty-0$ by

$$u\cdot(z_1,z_2,\ldots)=(uz_1,uz_2,\ldots) \qquad \forall\, u\in\mathbb{C}^*,\ (z_1,z_2,\ldots)\in\mathbb{C}^\infty, \tag{1.1}$$

and restricts to an action of S^1 on S^∞. Denote the quotients of both actions by $\mathbb{P}^\infty$.

Exercise 1.1. Show that

(a) the subspace topology of $S^\infty\subset\mathbb{C}^\infty$ is the same as the direct limit topology for the inclusions $S^{n-1}\longrightarrow S^n$;

(b) the action (1.1) of $\mathbb{C}^*$ on $\mathbb{C}^\infty - 0$ is continuous, as is the induced action of S^1 on S^∞;

(c) the inclusion maps $S^1 \longrightarrow \mathbb{C}^*$ and $S^\infty \longrightarrow \mathbb{C}^\infty - 0$ induce a homeomorphism

$$S^\infty/S^1 \longrightarrow (\mathbb{C}^\infty - 0)/\mathbb{C}^*;$$

(d) the quotient topology on $\mathbb{P}^\infty$ is the same as the direct limit topology for the inclusions $\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$, where $\mathbb{P}^n$ is the n-dimensional complex projective space.

Exercise 1.2. Show that the spaces S^∞ and $\mathbb{C}^\infty - 0$ are contractible.
`Hint:` Show that S^∞ retracts onto $S^\infty \cap (0 \times \mathbb{C}^\infty)$, i.e. the subspace of elements $v \in S^\infty$ whose first coordinate is zero.

For $c \in \mathbb{Z}$, define

$$f_c : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{by} \quad f_c(z) = \begin{cases} z^c, & \text{if } c \in \mathbb{Z}^{\geq 0}; \\ \bar{z}^{-c}, & \text{if } c \in \mathbb{Z}^{\leq 0}. \end{cases}$$

For an $n \times k$ matrix $M \equiv (m_{r,s})_{r \leq n, s \leq k}$ with integer coefficients and $r \leq n$, let

$$\langle M \rangle_r = \left\{ s \in \mathbb{Z}^+ : \ s \leq k, \ m_{r,s} \neq 0 \right\}.$$

If in addition each row of M is nonzero, i.e. $\langle M \rangle_r \neq \emptyset$ for every $r \leq n$, then we define

$$F_M : (\mathbb{C}^\infty)^k \longrightarrow (\mathbb{C}^\infty)^n \qquad \text{by}$$
$$\big(F_M\big((z_{s,j})_{s \leq k, j \in \mathbb{Z}^+}\big)\big)_{r,(i_s)_{s \in \langle M \rangle_r}} = \prod_{s \in \langle M \rangle_r} f_{m_{r,s}}(z_{s,i_s}) \quad \forall\, r \leq n, \ (i_s)_{s \in \langle M \rangle_r} \in (\mathbb{Z}^+)^{\langle M \rangle_r}; \tag{1.2}$$

this definition presupposes some identification of $(\mathbb{Z}^+)^{\langle M \rangle_r}$ with $\mathbb{Z}^+$. If the r-th row of M is zero, we define the r-th component of F_M to be the constant function with value $(1, 0, 0, \ldots) \in S^\infty$.

Exercise 1.3. Let M be an $n \times k$ integer matrix. Show that the map F_M

(a) is well-defined and continuous (in particular, each element in the image has finitely many nonzero components);

(b) restricts to continuous maps $(\mathbb{C}^\infty - 0)^k \longrightarrow (\mathbb{C}^\infty - 0)^n$ and $(S^\infty)^k \longrightarrow (S^\infty)^n$;

(c) induces a continuous map $\overline{F}_M : (\mathbb{P}^\infty)^k \longrightarrow (\mathbb{P}^\infty)^n$.

Exercise 1.4. Let $M \in \mathrm{GL}_n\mathbb{Z}$. Show that the map F_M in (1.2)

(a) is a homeomorphism;

(b) restricts to homeomorphisms on $(\mathbb{C}^\infty - 0)^n$ and $(S^\infty)^n$;

(c) induces a homeomorphism on $(\mathbb{P}^\infty)^n$.

An element of $\mathbb{P}^\infty$ is a complex line ℓ in $\mathbb{C}^\infty$ passing through the origin, i.e. ℓ is a one-dimensional linear subspace of $\mathbb{C}^n$ for some $n\in\mathbb{Z}^+$. Let

$$\gamma = \big\{(\ell, v)\in\mathbb{P}^\infty\times\mathbb{C}^\infty:\ v\in\ell\subset\mathbb{C}^\infty\big\}$$

be the tautological line bundle over $\mathbb{P}^\infty$.

Exercise 1.5. Show that

(a) the subspace topology of $\gamma\subset\mathbb{P}^\infty\times\mathbb{C}^\infty$ is the same as the direct limit topology for the inclusions $\gamma_{n-1}\longrightarrow\gamma_n$, where $\pi\colon\gamma_n\longrightarrow\mathbb{P}^n$ is the tautological line bundle over $\mathbb{P}^n$;

(b) the natural homomorphism
$$\mathbb{Z}\big[c_1(\gamma^*)\big]\longrightarrow H^*(\mathbb{P}^\infty;\mathbb{Z})$$
induced by sending $c_1(\gamma^*)^k$ to itself is a ring isomorphism.

By the Kunneth formula and Exercise 1.5(b), there is a natural isomorphism

$$H^*\big((\mathbb{P}^\infty)^n;\mathbb{Z}\big)\approx\mathbb{Z}[\alpha_1,\ldots,\alpha_n],\quad \deg\alpha_i=2, \tag{1.3}$$

where $\alpha_i=\pi_i^*c_1(\gamma^*)$ and $\pi_i:(\mathbb{P}^\infty)^n\longrightarrow\mathbb{P}^\infty$ is the projection onto the i-th component. By Exercise 1.3(c), an integer $n\times k$ matrix thus induces a commutative diagram

$$\begin{array}{ccc} H^*\big((\mathbb{P}^\infty)^n;\mathbb{Z}\big) & \xrightarrow{\ \overline{F}_M^*\ } & H^*\big((\mathbb{P}^\infty)^k;\mathbb{Z}\big) \\ \approx\downarrow & & \downarrow\approx \\ \mathbb{Z}[\alpha_1,\ldots,\alpha_n] & \longrightarrow & \mathbb{Z}[\alpha_1,\ldots,\alpha_k]\,. \end{array} \tag{1.4}$$

Exercise 1.6. Let $M\equiv(m_{r,s})_{r\le n,s\le k}$ be an integer matrix. Show that

$$\overline{F}_M^*\alpha_r=\sum_{s=1}^k m_{r,s}\alpha_s\qquad\forall\ r=1,\ldots,n,$$

i.e. the restriction of the bottom arrow in (1.4) to the degree 2 part is given by the transpose $M^\dagger$ of M.

Exercise 1.7. Let M be an $n\times k$ integer matrix and M' be a $k\times\ell$ integer matrix. Show that

$$\overline{F}_{MM'}^*=\overline{F}_{M'}^*\circ\overline{F}_M^*\colon H^*\big((\mathbb{P}^\infty)^n;\mathbb{Z}\big)\longrightarrow H^*\big((\mathbb{P}^\infty)^\ell;\mathbb{Z}\big)\,.$$

2 Lie groups preliminaries

A Lie group is a smooth manifold and a group at the same time such that the group multiplication

$$G\times G\longrightarrow G,\qquad (g_1,g_2)\longrightarrow g_1g_2,$$

is a smooth map. The Lie algebra of a Lie group G is the tangent space $T_{\mathrm{id}}G$ of G at the identity $\mathrm{id}\in G$. The only Lie groups relevant for our purposes are abelian: the tori $\mathbb{T}\equiv(S^1)^n$ or $(\mathbb{C}^*)^n$ and their products with the finite cyclic groups $\mathbb{Z}_k\equiv\mathbb{Z}/k\mathbb{Z}$.

Lemma 2.1. *For every Lie group homomorphism*

$$f\colon (S^1)^k\equiv\mathbb{R}^k/\mathbb{Z}^k \longrightarrow (S^1)^n\equiv\mathbb{R}^n/\mathbb{Z}^n\,,$$

there exists an $n\times k$ *integer matrix* M *such that*

$$f(v+\mathbb{Z}^k) = Mv+\mathbb{Z}^n \qquad \forall\ v\in\mathbb{R}^k\,.$$

Proof. Since $\mathbb{R}^k$ is contractible, there exists a continuous map $\widetilde{f}\colon\mathbb{R}^k\longrightarrow\mathbb{R}^n$ such that

$$f(v+\mathbb{Z}^k) = \widetilde{f}(v)+\mathbb{Z}^n \qquad \forall\, v\in\mathbb{R}^k;$$

see [27, Lemma 79.1]. Since f is a group homomorphism, $\widetilde{f}(0)\in\mathbb{Z}^n$. Thus, by shifting $\widetilde{f}$ by a constant in $\mathbb{Z}^n$, we can assume that $\widetilde{f}(0)=0$. Since f is a group homomorphism, the continuous map

$$\mathbb{R}^k\times\mathbb{R}^k \longrightarrow \mathbb{R}^n, \qquad (v_1,v_2)\longrightarrow \widetilde{f}(v_1+v_2)-\widetilde{f}(v_1)-\widetilde{f}(v_2),$$

takes values in $\mathbb{Z}^n$ and thus is constant. Since this map sends $(0,0)$ to 0,

$$\widetilde{f}(v_1+v_2) = \widetilde{f}(v_1)+\widetilde{f}(v_2) \qquad \forall\, v_1,v_2\in\mathbb{R}^k\,.$$

Since $\widetilde{f}$ is continuous, it follows that $\tilde{f}$ is a vector space homomorphism and thus there exists an $n\times k$ real matrix M such that $\widetilde{f}(v)=Mv$ for $v\in\mathbb{R}^k$. Since $\widetilde{f}(\mathbb{Z}^k)\subset\mathbb{Z}^n$, M has integer coefficients. □

Exercise 2.2. Suppose $k\le n$, $\mathbb{T}=(S^1)^n$, and $\mathbb{T}'=(S^1)^k$. Show that any injective Lie group homomorphism $f:\mathbb{T}'\longrightarrow\mathbb{T}$ extends to a Lie group isomorphism $g\colon \mathbb{T}'\times(S^1)^{n-k}\longrightarrow\mathbb{T}$.

Let G be a group acting on a space X, H be a group acting on a space Y, and $f\colon G\longrightarrow H$ be a group homomorphism. A map $F\colon X\longrightarrow Y$ is called f-equivariant if

$$F(g\cdot x) = f(g)\cdot F(x) \qquad \forall\ g\in G,\ x\in X.$$

Exercise 2.3. Let X and Y be topological spaces with continuous actions of groups G and H, respectively, and $f\colon G\longrightarrow H$ be a group homomorphism. Show that F descends to a continuous map $X/G\longrightarrow Y/H$ between the quotient spaces.

Exercise 2.4. Let M be an $n\times k$ integer matrix with every row nonzero. Show that the map F_M in (1.2) is equivariant with respect to the homomorphism $f_M:\mathbb{T}^k\longrightarrow\mathbb{T}^n$ determined by M as in Lemma 2.1.

Corollary 2.5. *Every Lie group homomorphism* $f\colon (S^1)^k\longrightarrow(S^1)^n$ *induces a ring homomorphism*

$$f^*\colon H^*\big((S^\infty)^n/(S^1)^n;\mathbb{C}\big) \longrightarrow H^*\big((S^\infty)^k/(S^1)^k;\mathbb{C}\big)$$

with the property that

$$\{f\circ g\}^* = g^*\circ f^*\colon H^*\big((S^\infty)^n/(S^1)^n;\mathbb{C}\big) \longrightarrow H^*\big((S^\infty)^\ell/(S^1)^\ell;\mathbb{C}\big) \tag{2.1}$$

for every Lie group homomorphism $f\colon (S^1)^\ell\longrightarrow(S^1)^k$.

Proof. By Lemma 2.1, every homomorphism $f:(S^1)^k \longrightarrow (S^1)^n$ corresponds to an $n\times k$ integer matrix M. The latter gives rise to a continuous f-equivariant map

$$F_M\colon (S^\infty)^k \longrightarrow (S^\infty)^n\,;$$

see Exercises 1.3(b) and 2.4. Since F_M is f-equivariant, it descends to a continuous map

$$\overline{F}_M\colon (S^\infty)^k/(S^1)^k \longrightarrow (S^\infty)^n/(S^1)^n$$

and thus induces a ring homomorphism

$$f^* = \overline{F}_M^*\colon\, H^*\big((S^\infty)^n/(S^1)^n;\mathbb{C}\big) \longrightarrow H^*\big((S^\infty)^k/(S^1)^k;\mathbb{C}\big).$$

By Exercise 1.6, f^* is determined by M and thus by f. By Exercise 1.7, (2.5) holds. □

3 Group cohomology

Let G be a Lie group. A classifying space for G is a contractible CW complex[1] EG on which G acts freely, i.e.

$$g\cdot e \neq e \qquad \forall\; e\in EG,\; g\in G-\mathrm{id}.$$

For example, $\mathbb{C}^*$ acts freely on $E\mathbb{C}^* \equiv \mathbb{C}^\infty - 0$ by complex multiplication as in (1.1); this action restricts to a free action of $S^1\subset\mathbb{C}^*$ on $ES^1\equiv S^\infty$ viewed as a subset of $\mathbb{C}^\infty$. By Exercise 1.2, these spaces $E\mathbb{C}^*$ and ES^1 are indeed contractible.

Let $\mathbb{T}$ denote the n-torus, i.e. either $(S^1)^n$ or $(\mathbb{C}^*)^n$. By the above, we can take $E\mathbb{T}$ to be $(S^\infty)^n$ if $\mathbb{T}=(S^1)^n$ and $(\mathbb{C}^\infty-0)^n$ if $\mathbb{T}=(\mathbb{C}^*)^n$ with the $\mathbb{T}$-action given by

$$(u_1,\ldots,u_n)\cdot(\underline{z}_1,\ldots,\underline{z}_n) = \big(u_1\underline{z}_1,\ldots,u_n\underline{z}_n\big).$$

If G is a subgroup of $\mathbb{T}$, G also acts freely on $E\mathbb{T}$; thus, we can take $EG=E\mathbb{T}$.

The geometric construction of [25] produces a CW complex EG for any given Lie group G; thus, EG always exists. Standard topological arguments imply that EG is unique up to homotopy commuting with the G-action. Thus, the homotopy type of

$$BG \equiv EG/G$$

is well-defined, i.e. depends only on G. In particular, the group cohomology of G, with $\mathbb{C}$-coefficients,

$$H_G^* \equiv H^*(BG;\mathbb{C})$$

is well-defined as well. We do not need the fact that EG is well-defined up to a G-equivariant homotopy in general. However, we will use the fact that our two constructions of $B\mathbb{T}'$ for a torus $\mathbb{T}'$ produce the same groups $H_{\mathbb{T}'}^*$, whether $\mathbb{T}'$ is viewed as a subgroup of a larger torus or on its own.

Exercise 3.1. Suppose $k\le n$, $\mathbb{T}=(S^1)^n$, and $\mathbb{T}'=(S^1)^k$. Show that

[1] For our purposes, *CW complex* can be replaced with *topological space* here

(a) if $\mathbb{T}'$ is identified with the subgroup $\mathbb{T}'\times\{\mathrm{id}\}^{n-k}$ of $\mathbb{T}$, then $(S^\infty)^k/\mathbb{T}'$ and $(S^\infty)^n/\mathbb{T}'$ are homotopy equivalent;

(b) if $f:\mathbb{T}'\longrightarrow\mathbb{T}$ is any injective Lie group homomorphism and $\mathbb{T}'$ acts on $(S^\infty)^n$ via f, then $(S^\infty)^n/_f\mathbb{T}'$ and $(S^\infty)^k/\mathbb{T}'$ are homotopy equivalent.
`Hint:` Use Lemma 2.1 and Exercise 2.2.

Lemma 3.2. *If $\mathbb{T}$ is the n-torus and $D\subset\mathbb{T}$ is a finite subgroup, then*

$$H^i\big(E\mathbb{T}/D;\mathbb{C}\big)=\begin{cases}\mathbb{C}, & \text{if } i=0;\\ 0, & \text{if } i\neq 0.\end{cases}$$

Proof. Let $\pi: E\mathbb{T}\longrightarrow E\mathbb{T}/D$ be the projection map. Since π is a finite-degree regular covering projection (the group D of deck transformations acts transitively on the fibers), the homomorphism

$$\pi^*: H^i\big(E\mathbb{T}/D;\mathbb{C}\big)\longrightarrow H^i(E\mathbb{T};\mathbb{C})^D\equiv\big\{\alpha\in H^i(E\mathbb{T};\mathbb{C}):\ g^*\alpha=\alpha\ \forall\, g\in D\big\} \tag{3.1}$$

is an isomorphism. This can be seen by averaging cocycles (to show surjectivity) and coboundaries (to show injectivity). Since $E\mathbb{T}$ is contractible, the claims follows from (3.1). □

If $\mathbb{T}'\subset G$ are subgroups of $\mathbb{T}$, the inclusion maps between the three groups induce quotient maps

$$B\mathbb{T}'\longrightarrow BG\longrightarrow B\mathbb{T}.$$

These maps in turn induce ring homomorphisms

$$\rho_{G,\mathbb{T}}: H^*_{\mathbb{T}}\longrightarrow H^*_G,\quad \rho_{\mathbb{T}',G}: H^*_G\longrightarrow H^*_{\mathbb{T}'},\quad \rho_{\mathbb{T}',\mathbb{T}}: H^*_{\mathbb{T}}\longrightarrow H^*_{\mathbb{T}'}\quad \text{s.t.}\quad \rho_{\mathbb{T}',\mathbb{T}}=\rho_{\mathbb{T}',G}\circ\rho_{G,\mathbb{T}}.$$

Proposition 3.5 below gives an intrinsic description of the homomorphism $\rho_{\mathbb{T},\mathbb{T}'}$ whenever $\mathbb{T}'$ is a subtorus of $\mathbb{T}$.

Lemma 3.3. *If $\mathbb{T}$ is the n-torus, $G\subset\mathbb{T}$ is a closed subgroup, and $\mathbb{T}'\subset G$ is the identity component of G (the topological component of G containing $\mathrm{id}\in G$), then the homomorphism*

$$\rho_{\mathbb{T}',G}: H^*_G\longrightarrow H^*_{\mathbb{T}'}$$

induced by the inclusion $\mathbb{T}'\longrightarrow G$ is an isomorphism.

Proof. Since $\mathbb{T}'\subset\mathbb{T}$ is a closed connected subgroup, $\mathbb{T}'\approx(S^1)^k$ for some $k\leq n$. By Exercise 2.2, the resulting inclusion $f:(S^1)^k\longrightarrow\mathbb{T}^n$ extends to a Lie group isomorphism $g:\mathbb{T}'\times(S^1)^{n-k}\longrightarrow\mathbb{T}$. Since $\mathbb{T}'\subset G$ is the identity component, g restricts to an isomorphism

$$g:\ \mathbb{T}'\times D\longrightarrow G\subset\mathbb{T}^n$$

for some finite subgroup $D\subset(S^1)^{n-k}$. By Lemma 2.1 and Exercise 2.4, there exists a g-equivariant homeomorphism $F_M:(S^\infty)^n\longrightarrow(S^\infty)^n$. It induces a commutative diagram

$$\begin{array}{ccc} H^*\big((S^\infty)^n/G;\mathbb{C}\big) & \xrightarrow[\approx]{\overline{F}_M^*} & H^*\big((S^\infty)^n/(\mathbb{T}'\times D);\mathbb{C}\big)\\ \Big\downarrow{\scriptstyle\rho_{\mathbb{T}';G}} & & \Big\downarrow{\scriptstyle\rho_{\mathbb{T}';\mathbb{T}'\times D}}\\ H^*\big((S^\infty)^n/_f\mathbb{T}';\mathbb{C}\big) & \xrightarrow{\approx} & H^*\big((S^\infty)^n/_\iota\mathbb{T}';\mathbb{C}\big), \end{array} \tag{3.2}$$

where $\iota\colon (S^1)^k \longrightarrow \mathbb{T}$ is the inclusion as the first k components. Since $D \subset (S^1)^{n-k}$,

$$(S^\infty)^n/(\mathbb{T}' \times D) \approx (S^\infty)^k/\mathbb{T}' \times (S^\infty)^{n-k}/D\,.$$

By Lemma 3.2 and the Kunneth formula, the right side arrow in (3.2),

$$\begin{aligned} H^*\big((S^\infty)^n/(\mathbb{T}' \times D);\mathbb{C}\big) &\approx H^*\big((S^\infty)^k/\mathbb{T}';\mathbb{C}\big) \otimes H^0\big((S^\infty)^{n-k}/D;\mathbb{C}\big) \\ &\longrightarrow H^*\big((S^\infty)^k/\mathbb{T}';\mathbb{C}\big) \otimes H^0\big((S^\infty)^{n-k};\mathbb{C}\big) \approx H^*\big((S^\infty)^n/_\iota\mathbb{T}';\mathbb{C}\big), \end{aligned}$$

is thus an isomorphism. It follows that the left side arrow in (3.2) is also an isomorphism. $\square$

Remark 3.4. The conclusions of Lemmas 3.2 and 3.3 depend on the use of cohomology with coefficients in a field of characteristic 0. For example,

$$H^*\big(ES^\infty/\mathbb{Z}_2;\mathbb{Z}_2\big) = H^*\big(\mathbb{RP}^\infty;\mathbb{Z}_2\big) \approx \mathbb{Z}_2\big[w_1(\gamma_\mathbb{R})\big],$$

where $\gamma_\mathbb{R} \longrightarrow \mathbb{RP}^\infty$ is the real tautological line bundle over $\mathbb{RP}^\infty$.

If $\mathbb{T}$ is an n-torus,

$$B\mathbb{T} \equiv E\mathbb{T}/\mathbb{T} \approx (\mathbb{P}^\infty)^n$$

is the n-fold product of infinite-dimensional complex projective spaces. Thus,

$$H^*_\mathbb{T} \equiv H^*(B\mathbb{T};\mathbb{C}) \approx \mathbb{C}[\alpha_1,\ldots,\alpha_n], \quad \deg\alpha_i = 2, \tag{3.3}$$

where $\alpha_i = \pi_i^* c_1(\gamma^*)$ and $\pi_i : (\mathbb{P}^\infty)^n \longrightarrow \mathbb{P}^\infty$ is the projection onto the i-th component. The next proposition provides a canonical identification of $H^2_\mathbb{T}$ with the dual of the complexified Lie algebra of $\mathbb{T}$,

$$\mathfrak{t}^*_\mathbb{C} \equiv \mathrm{Hom}_\mathbb{C}(\mathfrak{t}_\mathbb{C},\mathbb{C}) \equiv \mathrm{Hom}_\mathbb{C}\big(\mathfrak{t}\otimes_\mathbb{R}\mathbb{C},\mathbb{C}\big) \equiv \mathrm{Hom}_\mathbb{R}\big(\mathfrak{t},\mathbb{C}\big),$$

where $\mathfrak{t} \equiv T_{\mathrm{id}}\mathbb{T}$ is the Lie algebra of $\mathbb{T}$.[2] Thus, each element of $\mathfrak{t}^*_\mathbb{C}$ is a vector space homomorphism $T_{\mathrm{id}}\mathbb{T} \longrightarrow \mathbb{C}$. A Lie group homomorphism (or any smooth map) $f : \mathbb{T}' \longrightarrow \mathbb{T}$ induces vector space homomorphisms

$$\mathrm{d}_{\mathrm{id}}f\colon\ \mathfrak{t}' \equiv T_{\mathrm{id}}\mathbb{T}' \longrightarrow \mathfrak{t} \equiv T_{\mathrm{id}}\mathbb{T}, \qquad \big\{\mathrm{d}_{\mathrm{id}}f\big\}^*\colon\ \mathfrak{t}^*_\mathbb{C} \longrightarrow \mathfrak{t}'^*_\mathbb{C}\,.$$

Proposition 3.5. *For every $\mathbb{T} \equiv (S^1)^n$, there is a natural isomorphism*

$$\Psi_\mathbb{T}\colon H^2_\mathbb{T} \equiv H^2\big(E\mathbb{T}/\mathbb{T};\mathbb{C}\big) \longrightarrow \mathfrak{t}^*_\mathbb{C}$$

with the property that the diagram

$$\begin{CD} H^2_\mathbb{T} @>{f^*}>> H^2_{\mathbb{T}'} \\ @V{\Psi_\mathbb{T}}VV @VV{\Psi_{\mathbb{T}'}}V \\ \mathfrak{t}^*_\mathbb{C} @>{\{\mathrm{d}_{\mathrm{id}}f\}^*}>> \mathfrak{t}'^*_\mathbb{C} \end{CD} \tag{3.4}$$

commutes for every Lie group homomorphism $f\colon \mathbb{T}' \longrightarrow \mathbb{T}$.

[2] If $\mathbb{T} = (\mathbb{C}^*)^n$, $\mathfrak{t}_\mathbb{C}$ should denote the usual Lie algebra of $\mathbb{T}$ for the purposes of Proposition 3.5.

Proof. Let $v_1 \equiv 2\pi\mathfrak{i}$ be a fixed generator of the Lie algebra of S^1, i.e. of $T_1S^1 = \mathfrak{i}\mathbb{R}$. For each integer column vector $\mathbf{c} \equiv (c_1, \ldots, c_n)^\dagger$, let

$$f_{\mathbf{c}} \colon S^1 \longrightarrow S^n, \qquad u \longrightarrow \big(u^{c_1}, \ldots, u^{c_n}\big),$$

be the associated Lie group homomorphism as in Lemma 2.1,

$$f_{\mathbf{c}}^* \colon \; H^*_{\mathbb{T}} \equiv H^*\big((S^\infty)^n/\mathbb{T}; \mathbb{C}\big) \longrightarrow H^*_{S^1} \equiv H^*\big(S^\infty/S^1; \mathbb{C}\big)$$

be the induced ring homomorphism as in Corollary 2.5, and

$$v_{\mathbf{c}} = \big\{\mathrm{d}_1 f_{\mathbf{c}}\big\}(v_1) \in T_{\mathrm{id}}\mathbb{T} = \mathfrak{t}.$$

If in addition $\alpha \in H^2_{\mathbb{T}}$, we define

$$\big\{\Psi_{\mathbb{T}}(\alpha)\big\}(v_{\mathbf{c}}) \in \mathbb{C} \qquad \text{by} \quad \Big(\big\{\Psi_{\mathbb{T}}(\alpha)\big\}(v_{\mathbf{c}})\Big) c_1(\gamma^*) = f_{\mathbf{c}}^*\alpha \in H^2_{S^1}, \tag{3.5}$$

where $\gamma \longrightarrow \mathbb{P}^\infty$ is the tautological complex line bundle. By Exercise 1.5(b), $c_1(\gamma^*)$ generates $H^2_{S^1}$ and the equality in (3.5) determines $\{\Psi_{\mathbb{T}}(\alpha)\}(v_{\mathbf{c}})$. Since $v_{\mathbf{c}} \neq v_{\mathbf{c}'}$ whenever $\mathbf{c} \neq \mathbf{c}'$, $\{\Psi_{\mathbb{T}}(\alpha)\}(v_{\mathbf{c}})$ is determined by $v_{\mathbf{c}} \in \mathfrak{t}$.

By Exercise 1.6, the map

$$\mathbb{Z}^n \longrightarrow \mathbb{C}, \qquad \mathbf{c} \longrightarrow \{\Psi_{\mathbb{T}}(\alpha)\}(v_{\mathbf{c}}),$$

is linear over $\mathbb{Z}$. It follows that $\Psi_{\mathbb{T}}(\alpha)$ is linear on a generating set for $\mathfrak{t}$ and thus extends to a homomorphism $\mathfrak{t} \longrightarrow \mathbb{C}$, i.e. $\Psi_{\mathbb{T}}(\alpha)$ defines an element of $\mathfrak{t}^*_{\mathbb{C}}$. It is immediate from (3.5) that the resulting map

$$\Psi_{\mathbb{T}} \colon H^2_{\mathbb{T}} \longrightarrow \mathfrak{t}^*_{\mathbb{C}}$$

is linear. Let $\mathbf{c}_j \in \mathbb{Z}^n$ denote the j-th unit column vector. By Exercise 1.6,

$$\big\{\Psi_{\mathbb{T}}(\alpha_i)\big\}\big(v_{\mathbf{c}_j}\big) = \delta_{ij} \qquad \forall\, i,j = 1, \ldots, n,$$

and so $\Psi_{\mathbb{T}}$ is injective by (3.3) and thus an isomorphism.

Let $f \colon \mathbb{T}' \equiv (S^1)^k \longrightarrow \mathbb{T}$ be a Lie group homomorphism. For each integer column k-vector $\mathbf{c}'$,

$$\begin{aligned}\Big(\big\{\Psi_{\mathbb{T}'}(f^*\alpha)\big\}(v_{\mathbf{c}'})\Big)c_1(\gamma^*) &\equiv f_{\mathbf{c}'}^*\big(f^*\alpha\big) = \big\{f \circ f_{\mathbf{c}'}\big\}^*\alpha \equiv \Big(\big\{\Psi_{\mathbb{T}}(\alpha)\big\}\big(\mathrm{d}_1\{f \circ f_{\mathbf{c}'}\}(v_1)\big)\Big)c_1(\gamma^*) \\ &= \Big(\big\{\Psi_{\mathbb{T}}(\alpha)\big\}\big(\mathrm{d}_{\mathrm{id}}f\big(\mathrm{d}_1 f_{\mathbf{c}'}(v_1)\big)\big)\Big)c_1(\gamma^*) \equiv \Big(\big\{\{\mathrm{d}_{\mathrm{id}}f\}^*(\Psi_{\mathbb{T}}(\alpha))\big\}(v_{\mathbf{c}'})\Big)c_1(\gamma^*);\end{aligned}$$

the first equality above holds by (2.5) and the second by the chain rule. The equality of the left and right sides above is the commutativity of (3.4). □

Corollary 3.6. *For every $\mathbb{T} \equiv (S^1)^n$, there is a natural isomorphism*

$$\Psi_{\mathbb{T}} \colon H^*_{\mathbb{T}} \longrightarrow \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}$$

with the property that the diagram

$$\begin{array}{ccc} H^*_{\mathbb{T}} & \xrightarrow{\quad f^* \quad} & H^*_{\mathbb{T}'} \\ {\scriptstyle \Psi_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle \Psi_{\mathbb{T}'}} \\ \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}} & \xrightarrow{\mathrm{Sym}^*\{\mathrm{d}_{\mathrm{id}}f\}^*} & \mathrm{Sym}^*\mathfrak{t}'^*_{\mathbb{C}} \end{array} \tag{3.6}$$

commutes for every Lie group homomorphism $f\colon \mathbb{T}' \longrightarrow \mathbb{T}$.

This statement follows immediately from Proposition 3.5 and (3.3). Note that $\mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}}$ is the space of polynomials on the vector space $\mathfrak{t}_{\mathbb{C}}$.

Remark 3.7. In categorical terms, Corollary 2.5 means that the assignments $\mathbb{T} \longrightarrow H^*_{\mathbb{T}}$ and $f \longrightarrow f^*$ define a countervariant functor from the category of parametrized tori and Lie group homomorphisms between them to the category of graded algebras over $\mathbb{C}$ and homomorphisms between them. Another such functor is described by the assignments

$$\mathbb{T} \longrightarrow \mathrm{Sym}^*\mathfrak{t}^*_{\mathbb{C}} \qquad \text{and} \qquad f \longrightarrow \mathrm{Sym}^*\{\mathrm{d}_{\mathrm{id}}f\}^*.$$

Corollary 3.6 means that the isomorphisms $\Psi_{\mathbb{T}}$ define a natural equivalence of these two functors; see [28, Section 28].

A representation ρ of G, i.e. a linear action of G on $\mathbb{C}^k$, induces a vector bundle over BG:

$$V_\rho \equiv EG\times_\rho\mathbb{C}^k \equiv (EG\times\mathbb{C}^k)/\!\sim, \qquad (e,v) \sim (ge, \rho(g)^{-1}v) \quad \forall\,(e,v)\in EG\times\mathbb{C}^k,\ g\in G. \tag{3.7}$$

If ρ is one-dimensional, we will call

$$w_\rho \equiv c_1(V_\rho^*) = -c_1(V_\rho) \in H^*_G \tag{3.8}$$

the weight of ρ.

Exercise 3.8. Show that

(a) the complex line bundle over $B\mathbb{T}$ corresponding to the representation

$$\rho_i\colon \mathbb{T} \longrightarrow \mathbb{C}^*, \qquad \big(u_1,\ldots,u_n\big)\cdot z = u_i z, \tag{3.9}$$

is $\pi_i^*\gamma$ in the notation of (3.3);

(b) the weight of the representation (3.9) is thus α_i.

If G is abelian, every representation ρ of G on $\mathbb{C}^k$ splits uniquely into one-dimensional representations with some weights $\beta_1,\ldots,\beta_k$. We will call $\beta_1,\ldots,\beta_k$ the weights of ρ. In such a case,

$$e(V_\rho^*) = \beta_1\cdot\ldots\cdot\beta_k \in H^*_G. \tag{3.10}$$

We will call the representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$ with weights $\alpha_1,\ldots,\alpha_n$ the standard representation of $\mathbb{T}$. In actual computations later on, it will be convenient to think of $\alpha_1,\ldots,\alpha_n$ as formal complex variables.

4 Equivariant cohomology of topological spaces

If G is an abelian Lie group acting on a topological space M, let

$$B_G M = EG \times_G M \equiv (EG\times M)/\!\sim, \quad \text{where} \quad (e,x) \sim (ge, g^{-1}x) \;\; \forall\,(e,x)\in EG\times M,\ g\in G.$$

The G-equivariant cohomology of M is defined to be

$$H^*_G(M) \equiv H^*(B_G M;\mathbb{C}).$$

Example 4.1. The G-equivariant cohomology of a point pt is the group cohomology of G:

$$B_G\mathrm{pt} \equiv EG\times_G\mathrm{pt} = EG/G \times \mathrm{pt} = BG \quad\Longrightarrow\quad H^*_G(\mathrm{pt}) \equiv H^*(B_G\mathrm{pt};\mathbb{C}) = H^*(BG;\mathbb{C}) \equiv H^*_G.$$

If G acts trivially on M, then $H^*_G(M)$ is simply the tensor product of H^*_G and $H^*(M;\mathbb{C})$:

$$\begin{aligned} g\cdot x = x\ \forall\, g\in G,\, x\in M \quad &\Longrightarrow \quad B_G M \equiv EG\times_G M = EG/G\times M = BG\times M \\ \Longrightarrow \quad H^*_G(M) &\equiv H^*(B_G M;\mathbb{C}) = H^*(BG\times M;\mathbb{C}) \\ &= H^*(BG;\mathbb{C})\otimes H^*(M;\mathbb{C}) \equiv H^*_G\otimes H^*(M;\mathbb{C}). \end{aligned} \tag{4.1}$$

In this case, $H^*_G(M)$ and $H^*_G\otimes H^*(M;\mathbb{C})$ are isomorphic as rings.

Example 4.2. If G acts on itself by group multiplication, the map

$$B_G G\equiv EG\times_G G \longrightarrow EG, \qquad [e,g]\longrightarrow ge,$$

is a well-defined homeomorphism. Since EG is contractible, $H^i_G(G)=0$ for all $i\neq 0$.

If G is a Lie group acting on M, there is a fibration

$$\pi_M\colon B_G M \longrightarrow BG, \qquad [e,x]\longrightarrow [e] \;\; \forall\,(e,x)\in EG\times M, \tag{4.2}$$

with fiber M. It induces an action of H^*_G on $H^*_G(M)$:

$$\alpha\cdot\eta = (\pi_M^*\alpha)\cup\eta \qquad \forall\ \alpha\in H^*_G\equiv H^*(BG;\mathbb{C}),\ \eta\in H^*_G(M)\equiv H^*(B_G M;\mathbb{C}).$$

Thus, $H^*_G(M)$ is an algebra over the ring H^*_G. Since the M-fibration π_M is generally not trivial, $H^*_G(M)$ and $H^*_G\otimes H^*(M;\mathbb{C})$ are usually not isomorphic as rings.

For each $b=[e]\in BG$, the inclusion

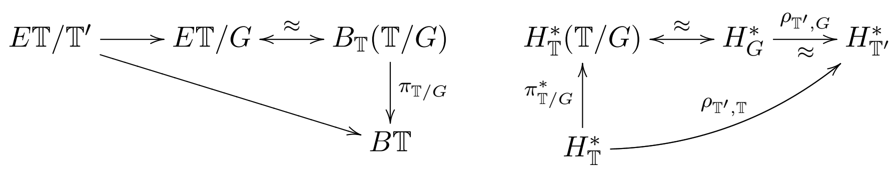
$$\iota_e\colon M\longrightarrow B_G M, \qquad x\longrightarrow [e,x], \tag{4.3}$$

of M as the fiber M_b of π_M over b induces a restriction homomorphism

$$\iota_e^*\colon H^*_G(M)\equiv H^*(B_G M;\mathbb{C})\longrightarrow H^*(M;\mathbb{C})\,. \tag{4.4}$$

It sends $\pi_M^*\alpha$ to zero whenever $\alpha\in H^*_{\mathbb{T}}$ has positive degree. As Example 4.2 illustrates, this homomorphism need not be surjective. It is surjective in the cases of Example 4.1 above and Proposition 5.3 below, with the usual cohomology required from the equivariant cohomology by setting all weights α_i to zero.

Exercise 4.3. Suppose $\mathbb{T}$ is the n-torus, G is a closed subgroup of $\mathbb{T}$, and $\mathbb{T}'$ is the identity component of G. The n-torus $\mathbb{T}$ then acts on the quotient $\mathbb{T}/G$. Show that there are commutative diagrams



and

$$\rho_{\mathbb{T}',G}\big(\alpha\cdot\eta\big)=\rho_{\mathbb{T}',\mathbb{T}}(\alpha)\cup\rho_{\mathbb{T}',G}(\eta)\in H^*_{\mathbb{T}'}\qquad\forall\,\alpha\in H^*_{\mathbb{T}},\ \eta\in H^*_{\mathbb{T}}(\mathbb{T}/G).$$

If G is a Lie group acting on M and S is a subspace of M preserved by G, i.e. $g\cdot x\in S$ for all $x\in S$ and $g\in G$, G also acts on S. The inclusion $S\longrightarrow M$ then induces an inclusion $B_GS\longrightarrow B_GM$ and thus a restriction homomorphism

$$H^*_G(M)\equiv H^*(B_GM;\mathbb{C})\longrightarrow H^*_G(S)\equiv H^*(B_GS;\mathbb{C}),\qquad \eta\longrightarrow \eta|_S.$$

Taking S to be the fixed locus of the G-action,

$$M^G\equiv\big\{x\in M:\ gx=x\ \forall\,g\in G\big\},$$

we obtain a homomorphism of H^*_G-algebras

$$H^*_G(M)\longrightarrow H^*_G(M^G)\approx H^*_G\otimes H^*(M^G;\mathbb{C}),\qquad \eta\longrightarrow \eta|_{M^G}. \tag{4.5}$$

The substance of the Localization Theorem is that this homomorphism is invertible after adjoining the inverses of the nonzero elements of H^*_G, if M is a manifold and $\mathbb{T}$ is a torus. In the case of $\mathbb{P}^{n-1}$ with the standard $\mathbb{T}$-action, (4.5) is an isomorphism even before adjoining these inverses; see Exercise 5.4.

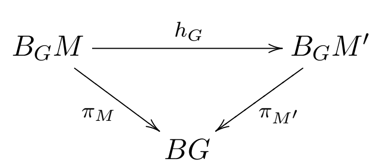
More generally, if G acts on M and M', a G-equivariant continuous map $h: M\longrightarrow M'$ induces a continuous map

$$h_G\colon B_GM\longrightarrow B_GM',\qquad [e,x]\longrightarrow\big[e,h(x)\big], \tag{4.6}$$

and thus a homomorphism

$$h_G^*\colon H^*_G(M')\equiv H^*(B_GM';\mathbb{C})\longrightarrow H^*_G(M)\equiv H^*(B_GM;\mathbb{C}).$$

Since the diagram



commutes, h_G^* commutes with the action of H^*_G, i.e. h^*_G is a homomorphism of H^*_G-modules (in fact, of H^*_G-algebras).

Exercise 4.4. Suppose the n-torus $\mathbb{T}$ acts on a topological space M, G is a closed subgroup of $\mathbb{T}$, and $\mathbb{T}'$ is the identity component of G. Let $h\colon M\longrightarrow\mathbb{T}/G$ be a $\mathbb{T}$-equivariant map. Show that

$$\alpha\cdot\eta = h_{\mathbb{T}}^*\rho_{\mathbb{T}',G}^{-1}\big(\rho_{\mathbb{T}',\mathbb{T}}(\alpha)\big)\cup\eta \qquad \forall\ \alpha\in H_{\mathbb{T}}^*,\ \eta\in H_{\mathbb{T}}^*(M),$$

i.e. the action of $H_{\mathbb{T}}^*$ on $H_{\mathbb{T}}^*(M)$ factors through the natural homomorphism $H_{\mathbb{T}}^*\longrightarrow H_{\mathbb{T}'}^*$.

This observation is a key ingredient in the proof of the Localization Theorem' see Lemma 9.2. Along with Corollary 3.6, it implies that the action of $\mathrm{Sym}^*\mathfrak{t}_{\mathbb{C}}^*$, i.e. of the space of polynomials on the vector space $\mathfrak{t}_{\mathbb{C}}$, on $H_{\mathbb{T}}^*(M)$ factors through the restriction

$$\mathrm{Sym}^*\mathfrak{t}_{\mathbb{C}}^* \longrightarrow \mathrm{Sym}^*\mathfrak{t}_{\mathbb{C}}'^*$$

and an action of $\mathrm{Sym}^*\mathfrak{t}_{\mathbb{C}}'^*$ on $H_{\mathbb{T}}^*(M)$. Note that the homomorphism $\rho_{\mathbb{T}',G}$ is invertible; see Lemma 3.3.

If G is a Lie group acting on M and $\pi_V\colon V\longrightarrow M$ is a vector bundle, a lift of the G-action on M to V is an action of G on V such that

$$\pi_V(g\cdot v) = g\cdot\pi_V(v) \qquad \forall\, g\in G,\ v\in V,$$

and the restriction of this action to each fiber of π_V is linear. Once a lift has been chosen, V is called a G-vector bundle on M. In such a case,

$$B_GV\equiv EG\times_G V \longrightarrow B_GM\equiv EG\times_G M$$

is a vector bundle. Direct sums, tensor products, and other natural transforms of G-vector bundles are again G-vector bundles; so are pullbacks of G-vector bundles by G-equivariant maps. If $V\longrightarrow M$ is oriented as a vector bundle and G is connected, then $B_GV\longrightarrow B_GM$ is also an oriented vector bundle. If this is the case, let

$$\mathbf{e}(V)\equiv e(B_GV)\in H_G^*(M)\equiv H^*(B_GM;\mathbb{C}), \qquad \mathbf{c}(V)\equiv c(B_GV)\in H_G^*(M) \tag{4.7}$$

denote the equivariant Euler class of V and the equivariant Chern class of V if V is a complex vector bundle.

If $V\longrightarrow M$ is a G-vector bundle, the inclusion (4.3) lifts to the inclusion

$$\widetilde{\iota}_e\colon V\longrightarrow B_GV, \qquad v\longrightarrow[e,v],$$

of V as the fiber V_b of $B_GV\longrightarrow BG$ over $b=[e]\in BG$; see Figure 4.1. It induces an isomorphism

$$\widetilde{\iota}_e\colon V\longrightarrow \iota_e^*V$$

of vector bundles over M. Thus,

$$e(V)=\iota_e^*\mathbf{e}(V)\in H^*(M;\mathbb{C}), \qquad c(V)=\iota_e^*\mathbf{c}(V)\in H^*(M;\mathbb{C}), \tag{4.8}$$

i.e. the equivariant Euler and Chern classes of V determine the usual Euler and Chern classes of V via the homomorphism (4.4). Since this homomorphism is not surjective in general, some vector bundles

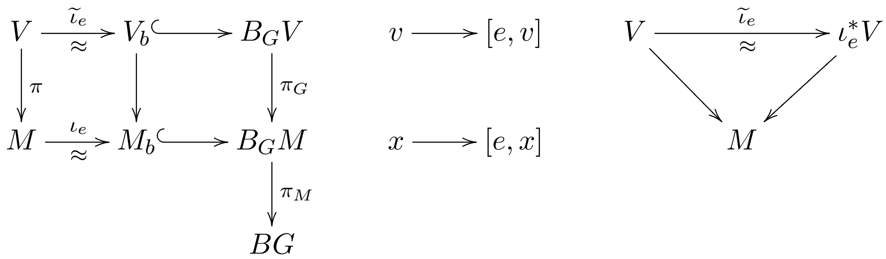


Figure 4.1: The G-equivariant Euler/Chern class encodes the usual Euler/Chern class; $M_b\subset B_GM$ and $V_b\subset B_GV$ denote the fibers of $B_GM\longrightarrow BG$ and $B_GV\longrightarrow BG$ over $b=[e]\in BG$.

over M may not admit any lift of the G-action on M.

Some G-actions have natural lifts. For example, if M is a smooth manifold and $\rho\colon G\longrightarrow \mathrm{Diff}(M)$ is an action of G by diffeomorphisms, then ρ lifts to an action on the tangent bundle of M by

$$g\cdot v = \mathrm{d}_x\{\rho(g)\}(v) \qquad \forall\, g\in G,\, v\in T_xM,\, x\in M.$$

Given one lift of the G-action on M to a complex vector bundle $V\longrightarrow M$ and a one-dimensional representation ρ of G, we can obtain another lift of the G-action on M to V by tensoring V, with its G-action, and $M\times\mathbb{C}_\rho$, with G acting by ρ on $\mathbb{C}$:

$$g\cdot_\rho v = \rho(g)\,(g\cdot v) \qquad \forall\ g\in G,\ v\in V, \tag{4.9}$$

where $\cdot$ is the original G-action on V and $\cdot_g$ is the action twisted by ρ. We denote the G-vector bundle V with the new action by $V(\rho)$.

Exercise 4.5. Let $V\longrightarrow M$ be a G-vector bundle and $\rho\colon G\longrightarrow\mathbb{C}^*$ be a one-dimensional representation. Show that

$$B_G\big(V(\rho)\big) = B_GV\otimes_{\mathbb{C}}\pi_M^*V_\rho \longrightarrow B_GM,$$

where $V_\rho\longrightarrow BG$ is the complex line bundle corresponding to the representation ρ as in (3.7). If $V\longrightarrow M$ is a complex line bundle, conclude that

$$\mathbf{e}\big(V(\rho)\big) = \mathbf{e}(V) - \pi_M^*w_\rho\,,$$

where w_ρ is the weight of the representation ρ as in (3.8).

5 The projective space with the standard action

This section studies the notions introduced in Section 4 in the case of $\mathbb{P}^{n-1}$ with the standard action of $\mathbb{T}\equiv(S^1)^n$,

$$(u_1,\ldots,u_n)\cdot[z_1,\ldots,z_n] = \big[u_1z_1,\ldots,u_nz_n\big] \quad \forall\ (u_1,\ldots,u_n)\in\mathbb{T},\ [z_1,\ldots,z_n]\in\mathbb{P}^{n-1}; \tag{5.1}$$

it is the action induced by the standard representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$:

$$(u_1,\ldots,u_n)\cdot(c_1,\ldots,c_n)=\big(u_1c_1,\ldots,u_nc_n\big)\quad\forall\,(u_1,\ldots,u_n)\in\mathbb{T},\ (c_1,\ldots,c_n)\in\mathbb{C}^n. \tag{5.2}$$

Proposition 5.3, which is central to the applications of Theorem 8.4 in Chapter 4, determines the corresponding topological space $B_{\mathbb{T}}\mathbb{P}^{n-1}$ and equivariant cohomology $H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$. It is also used in Exercise 5.4, which illustrates the main principle behind the Localization Theorem: the fixed locus of a group action G on a manifold M sees the entire equivariant cohomology of M. The conclusions of this section, which are summarized in Figures 5.1 and 5.2, are not used in the proof of Theorem 8.4 and are thus not needed for the main purpose of Chapter 1.

If $\pi: V\longrightarrow M$ is a complex vector bundle of rank n, let $\mathbb{P}V\longrightarrow M$ denote the projectivization of V over M; it consists of complex one-dimensional subspaces ℓ of each of π. This is the $\mathbb{P}^{n-1}$-fiber bundle over M obtained by replacing each fiber of π by its projectivization and using the projectivizations of the transition maps for π to topologize $\mathbb{P}V$. There is a tautological line bundle over $\mathbb{P}V$ described by

$$\widetilde{\gamma}=\big\{(\ell,v)\in\mathbb{P}V\times V:\ v\in\ell\subset\mathbb{P}V\big\}.$$

The projection $\pi\colon\mathbb{P}V\longrightarrow M$ makes $H^*(\mathbb{P}V;\mathbb{C})$ into an algebra over the ring $H^*(M;\mathbb{C})$:

$$\alpha\cdot\eta=(\pi^*\alpha)\cup\eta\qquad\forall\,\alpha\in H^*(M;\mathbb{C}),\ \eta\in H^*(\mathbb{P}V;\mathbb{C}).$$

Exercise 5.1. Let $\pi\colon V\longrightarrow M$ be a complex vector bundle of rank n. Show that

(a) the topological space $\mathbb{P}V$ is well-defined and is a $\mathbb{P}^{n-1}$-fiber bundle over M;

(b) the projection $\pi_1\colon\widetilde{\gamma}\longrightarrow\mathbb{P}V$ induces the structure of a complex line bundle on $\widetilde{\gamma}$.

Lemma 5.2. *If $\pi: V\longrightarrow M$ be a complex vector bundle of rank n, then there exists a natural isomorphism of $H^*(M;\mathbb{C})$-algebras*

$$H^*(\mathbb{P}V;\mathbb{C})\approx H^*(M;\mathbb{C})[x]\big/\big(x^n+c_1(V)x^{n-1}+\ldots+c_n(V)\big), \tag{5.3}$$

where $x=c_1(\widetilde{\gamma}^*)$ is the Chern class of the dual tautological line bundle $\widetilde{\gamma}^*\longrightarrow\mathbb{P}V$.

Proof. Let $\pi: E\longrightarrow B$ be a fiber bundle with fiber F and $\iota_x: F_x\longrightarrow E$ be the inclusion of the fiber over $x\in B$. A cohomology extension of the fiber for π is a homomorphism

$$\Phi\colon H^*(F;\mathbb{C})\longrightarrow H^*(E;\mathbb{C})$$

of graded vector spaces (not algebras) over $\mathbb{C}$ such that the composition map

$$H^*(F;\mathbb{C})\xrightarrow{\ \Phi\ }H^*(E;\mathbb{C})\xrightarrow{\ \iota_x^*\ }H^*(F_x;\mathbb{C})$$

is an isomorphism of vector spaces over $\mathbb{C}$; see [35, p256]. If $H^*(F;\mathbb{C})$ is finitely generated, the map

$$H^*(B;\mathbb{C})\otimes H^*(F;\mathbb{C})\longrightarrow H^*(E;\mathbb{C}),\qquad(\alpha,\mu)\longrightarrow\pi^*\alpha\cup\Phi(\eta),$$

is then an isomorphism of graded modules (but not of algebras) over the ring $H^*(B;\mathbb{C})$; see [35, Theorem 5.7.9].

In our setting, define

$$\Phi\colon H^*(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^*(\mathbb{P}V;\mathbb{C}) \quad \text{by} \quad a\,c_1(\gamma_{n-1}^*)^k \longrightarrow a\,c_1(\widetilde{\gamma}^*)^k \ \ \forall\, a\in\mathbb{C},\ k=0,1,\ldots,n-1, \tag{5.4}$$

where $\gamma_{n-1}\longrightarrow\mathbb{P}^{n-1}$ is the usual tautological line bundle. Since $\widetilde{\gamma}|_{\mathbb{P}V_x}$ is the tautological line bundle $\gamma_{\mathbb{P}V_x}\longrightarrow\mathbb{P}V_x$ and

$$H^*\big(\mathbb{P}V_x;\mathbb{C}\big) \approx \mathbb{C}\big[c_1(\gamma_{\mathbb{P}V_x}^*)\big]/c_1(\gamma_{\mathbb{P}V_x}^*)^n \qquad \forall\, x\in M\,,$$

the composition map

$$H^*(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^*(\mathbb{P}V;\mathbb{C}) \longrightarrow H^*\big(\mathbb{P}V_x;\mathbb{C}\big),$$
$$a\,c_1(\gamma_{n-1}^*)^k \longrightarrow a\,c_1(\gamma_{\mathbb{P}V_x}^*)^k \quad \forall\, a\in\mathbb{C},\ k=0,1,\ldots,n-1,$$

is an isomorphism of vector spaces over $\mathbb{C}$ for all $x\in M$. Thus, (5.4) is a cohomology extension of the fiber for the $\mathbb{P}^{n-1}$-fiber bundle $\pi\colon\mathbb{P}V\longrightarrow M$. By [35, Theorem 5.7.9],

$$H^*(M;\mathbb{C})\big[c_1(\widetilde{\gamma}^*)\big]/c_1(\widetilde{\gamma}^*)^n \longrightarrow H^*(\mathbb{P}V;\mathbb{C}),$$
$$\alpha\,c_1(\gamma_{n-1}^*)^k \longrightarrow \pi^*\alpha\cup c_1(\widetilde{\gamma}^*)^k \quad \forall\, \alpha\in H^*(M;\mathbb{C}),\ k=0,1,\ldots,n-1,$$

is therefore an isomorphism of graded modules over $H^*(M;\mathbb{C})$.

By the previous paragraph, the cohomology $H^*(\mathbb{P}V;\mathbb{C})$ of $\mathbb{P}V$ is a free module over $H^*(M;\mathbb{C})$ with basis $1,c_1(\widetilde{\gamma}^*),\ldots,c_1(\widetilde{\gamma}^*)^{n-1}$. In order to determine the ring structure of $H^*(\mathbb{P}V;\mathbb{C})$, we need to express $c_1(\widetilde{\gamma}^*)^n$ in terms of this basis. Let $\pi\colon\mathbb{P}V\longrightarrow M$ be the bundle projection map. We define a section s of the rank n vector bundle $\widetilde{\gamma}^*\otimes\pi^*V$ over $\mathbb{P}V$ by

$$\big\{s(\ell)\big\}(v) = v \in V_{\pi(\ell)} = (\pi^*V)_\ell \qquad \forall\, \ell\in\mathbb{P}V,\ v\in\widetilde{\gamma}_\ell=\ell.$$

Since this section does not vanish,

$$0 = e\big(\widetilde{\gamma}^*\otimes\pi^*V\big) = x^n+\pi^*c_1(V)x^{n-1}+\ldots+\pi^*c_n(V)$$
$$\equiv x^n+c_1(V)\cdot x^{n-1}+\ldots+c_n(V)\cdot 1\,.$$

This implies that the isomorphism (5.3) respects the ring structures on the two sides and thus is an isomorphism of $H^*(M;\mathbb{C})$-algebras. □

Proposition 5.3. *Let $\mathbb{T}\equiv(S^1)^n$ act on $\mathbb{P}^{n-1}$ in the standard way (5.1) and ρ be the standard representation (5.2) of $\mathbb{T}$ on $\mathbb{C}^n$. Then,*

$$B_{\mathbb{T}}\mathbb{P}^{n-1} = \mathbb{P}V_\rho$$

and there is a canonical ring isomorphism

$$H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \approx \mathbb{C}[\alpha_1,\ldots,\alpha_n,\mathbf{x}]/(\mathbf{x}-\alpha_1)\ldots(\mathbf{x}-\alpha_n), \tag{5.5}$$

with $\mathbf{x}=c_1(\widetilde{\gamma}^*)$. This is an isomorphism of $H_{\mathbb{T}}^*$-algebras under the identification given by (3.3).

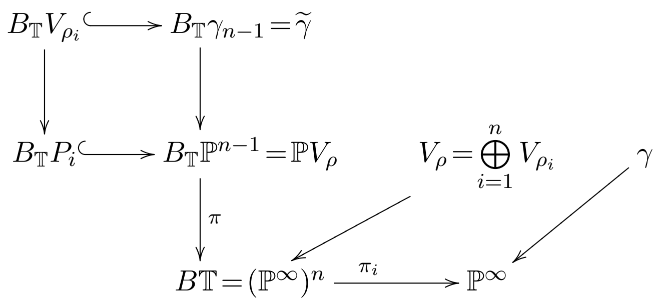


Figure 5.1: The topology of the standard $\mathbb{T} \equiv (S^1)^n$-action on $\mathbb{P}^{n-1}$.

Proof. By definition,

$$B_{\mathbb{T}}\mathbb{P}^{n-1} = E\mathbb{T} \times_{\mathbb{T}} \big((\mathbb{C}^n - 0)/\mathbb{C}^*\big) = \big(E\mathbb{T} \times_\rho \mathbb{C}^n - E\mathbb{T} \times_\rho \{0\}\big)/\mathbb{C}^* = \big(V_\rho - B\mathbb{T}\big) = \mathbb{P}V_\rho\,,$$

where $B\mathbb{T} \subset V_\rho$ represents the zero section of the vector bundle $V_\rho \longrightarrow B\mathbb{T}$. This establishes the first claim.

By Lemma 5.2, the map

$$H^*_{\mathbb{T}}[\mathbf{x}]/\big(\mathbf{x}^n + c_1(V_\rho)\mathbf{x}^{n-1} + \ldots + c_n(V_\rho)\big) \longrightarrow H^*_{\mathbb{T}}(\mathbb{P}^{n-1}), \qquad \alpha\, \mathbf{x}^k \longrightarrow \alpha \cdot c_1(\widetilde{\gamma}^*)^k,$$

is an isomorphism of $H^*_{\mathbb{T}}$-algebras. Since ρ is the direct sum of the representations $\rho_1, \ldots, \rho_n$ in Example 3.8, V_ρ is the direct sum of the line bundles $V_{\rho_1}, \ldots, V_{\rho_n}$ over $B\mathbb{T}$. Since $c_1(V_{\rho_i}) = -\alpha_i$, it follows that

$$\mathbf{x}^n + c_1(V_\rho)\mathbf{x}^{n-1} + \ldots + c_n(V_\rho) = (\mathbf{x} - \alpha_1) \ldots (\mathbf{x} - \alpha_n).$$

This establishes the remaining claims of the proposition. □

Exercise 5.4. The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ has n fixed points:

$$P_1 = [1, 0, \ldots, 0], \qquad P_2 = [0, 1, 0, \ldots, 0], \quad \ldots \quad P_n = [0, \ldots, 0, 1]. \tag{5.6}$$

(a) Show that $\widetilde{\gamma}|_{B_{\mathbb{T}}P_i} = V_{\rho_i}$, where ρ_i is as in (3.9).

(b) Show that the restriction map on the equivariant cohomology induced by the inclusion $P_i \longrightarrow \mathbb{P}^{n-1}$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \approx \mathbb{C}[\alpha_1, \ldots, \alpha_n, \mathbf{x}]/\prod_{k=1}^{k=n}(\mathbf{x} - \alpha_k) \longrightarrow H^*_{\mathbb{T}}(P_i) \approx \mathbb{C}[\alpha_1, \ldots, \alpha_n], \quad \mathbf{x} \longrightarrow \alpha_i. \tag{5.7}$$

(c) Conclude that for all $\eta \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$

$$\eta = 0 \qquad \Longleftrightarrow \qquad \eta|_{P_i} = 0 \;\; \forall\, i = 1, 2, \ldots, n. \tag{5.8}$$

Thus, an element $\eta \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$ is determined by its restrictions $\eta|_{P_i}$ with $i = 1, 2, \ldots, n$.

$$\begin{array}{ccccc} H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) & \longrightarrow & H^*_{\mathbb{T}}(P_i)=H^*_{\mathbb{T}} & \qquad \mathbf{x}=c_1(\widetilde{\gamma}) & \qquad \alpha_i=\pi_i^*c_1(\gamma) \\ \| & & \| & & \\ \mathbb{C}[\alpha_1,\ldots,\alpha_n,\mathbf{x}] & \longrightarrow & \mathbb{C}[\alpha_1,\ldots,\alpha_n] & \qquad \mathbf{x} & \longrightarrow \alpha_i \end{array}$$

Figure 5.2: The cohomology of the standard $\mathbb{T}\equiv(S^1)^n$-action on $\mathbb{P}^{n-1}$.

The standard diagonal $\mathbb{T}$-action on $\mathbb{P}^{n-1}\times\mathbb{C}^n$,

$$(u_1,\ldots,u_n)\cdot\big([z_1,\ldots,z_n],(c_1,\ldots,c_n)\big)=\big([u_1z_1,\ldots,u_nz_n],(u_1c_1,\ldots,u_nc_n)\big),$$

preserves the total space

$$\gamma_{n-1}\equiv\big\{(\ell,v)\in\mathbb{P}^{n-1}\times\mathbb{C}^n:\ v\in\ell\subset\mathbb{C}^n\big\}$$

of the tautological line bundle over $\mathbb{P}^{n-1}$. The standard lift of the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ to γ_{n-1} is the restriction of this action to γ_{n-1}. It induces actions on the line bundles $\gamma_{n-1}^{\otimes c}$ for all $c\in\mathbb{Z}$, which will be called the standard lifts of the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$.

Exercise 5.5. Let $\mathbb{T}$ be the n-torus acting in the standard way on $\mathbb{P}^{n-1}$, ρ be the standard representation of $\mathbb{T}$ on $\mathbb{C}^n$, and $c\in\mathbb{Z}$. Show that

(a) the line bundle $B_{\mathbb{T}}\gamma_{n-1}^{\otimes c}$ for the standard lift of the $\mathbb{T}$-action to $\gamma_{n-1}^{\otimes c}$ is $\widetilde{\gamma}^{\otimes c}\longrightarrow B_{\mathbb{T}}\mathbb{P}^{n-1}$;

(b) its equivariant Euler class is characterized by

$$\mathbf{e}\big(\gamma_{n-1}^{\otimes c}\big)\big|_{P_i}=-c\,\alpha_i\qquad\forall\ i=1,2,\ldots,n;\tag{5.9}$$

(c) under the identification (5.5), the homomorphism (4.4) for $\mathbb{P}^{n-1}$ with the standard $\mathbb{T}$-action is determined by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\longrightarrow H^*(\mathbb{P}^{n-1};\mathbb{C}),\qquad \mathbf{x}\longrightarrow x,$$

where $x=c_1(\gamma_{n-1}^*)$ is the positive generator of $H^*(\mathbb{P}^{n-1};\mathbb{C})$.

Exercise 5.6. Let $\mathbb{T}$ be the n-torus acting in the standard way on $\mathbb{P}^{n-1}$ and $T\mathbb{P}^{n-1}$ and ρ be the standard representation of $\mathbb{T}$ on $\mathbb{C}^n$. Show that

(a) there is an exact sequence of $\mathbb{T}$-vector bundles on $\mathbb{P}^{n-1}$

$$0\longrightarrow\gamma_{n-1}^*\otimes\gamma_{n-1}\longrightarrow\gamma_{n-1}^*\otimes\big(\mathbb{P}^{n-1}\times\mathbb{C}^n_\rho\big)\longrightarrow T\mathbb{P}^{n-1}\longrightarrow 0;$$

(b) the equivariant Euler class of $T\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}$ is characterized by

$$\mathbf{e}\big(T\mathbb{P}^{n-1}\big)\big|_{P_i}=\prod_{k\neq i}(\alpha_i-\alpha_k)\qquad\forall\ i=1,2,\ldots,n.\tag{5.10}$$

6 Computations on manifolds

In this section, we review some classical statements related to topological computations on smooth manifolds. In the next section, we extend the integration homomorphism (6.1) and the cohomology pushforward (6.2) below to the $\mathbb{T}$-equivariant setting. The portion of this section after Exercise 6.2 is used in Section 7 to show that the constructed $\mathbb{T}$-equivariant analogue of (6.2) is well-defined and to establish Corollary 7.3, which is used in the proof of the Localization Theorem in Section 8.

Let M be a compact oriented m-dimensional manifold. Denote its fundamental class in $H_m(M;\mathbb{Z})$ or $H_m(M;\mathbb{C})$ by $[M]$. There is then a well-defined homomorphism

$$\int_M : H^*(M;\mathbb{Z}) \longrightarrow \mathbb{Z}, \qquad [\eta] \longrightarrow \int_M \eta = \big\langle [\eta], [M] \big\rangle, \tag{6.1}$$

of moduli over $\mathbb{Z}$. Furthermore, the homomorphism

$$\mathrm{PD}_M : H^k(M;\mathbb{Z}) \longrightarrow H_{m-k}(M;\mathbb{Z}), \qquad \psi \longrightarrow \psi \cap [M],$$

is an isomorphism in this case; see [26, p276]. We denote its inverse also by PD_M. Since the homomorphism

$$H_k(M;\mathbb{C}) \longrightarrow \big(H^k(M;\mathbb{C})\big)^*, \qquad Z \longrightarrow \langle \cdot, Z \rangle,$$

is an isomorphism by the Universal Coefficient Theorem [28, Theorem 53.5], it follows that the pairing

$$H^*(M;\mathbb{C}) \otimes H^*(M;\mathbb{C}) \longrightarrow \mathbb{C}, \qquad \psi \otimes \eta \longrightarrow \int_M \psi \cup \eta,$$

is nondegenerate. Thus, each element η of $H^*(M;\mathbb{C})$ is uniquely determined by its integral against every element ψ of $H^*(M;\mathbb{C})$.

From now on, we often restrict to the even-degree cohomology $H^{2*}(M;\mathbb{C})$; with the odd-degree cohomology, sign conventions for a number of cohomology operations would need to be specified. We will abbreviate the cup product $\psi \cup \eta$ of two cohomology classes as $\psi\eta$. A crucial relation between the cup and cap products is

$$(\psi\eta) \cap Z = \psi \cap (\eta \cap Z) \qquad \forall\ \psi, \eta \in H^{2*}(M;\mathbb{Z}),\ Z \in H_*(M;\mathbb{Z});$$

see [26, p276]. This implies that

$$\big\langle \psi\eta, Z \big\rangle = \big\langle \psi, \eta \cap Z \big\rangle \qquad \forall\ \psi, \eta \in H^{2*}(M;\mathbb{Z}),\ Z \in H_*(M;\mathbb{Z}).$$

If $f: M \longrightarrow M'$ is a continuous map between two compact oriented manifolds of dimensions $m, m' \in \mathbb{Z}$, the homomorphism

$$f_*: H^*(M;\mathbb{Z}) \longrightarrow H^{*+m'-m}(M';\mathbb{Z}), \qquad \eta \longrightarrow \mathrm{PD}_{M'}\big(f_*(\mathrm{PD}_M \eta)\big), \tag{6.2}$$

is called the cohomology pushforward induced by f. If $m, m' \in 2\mathbb{Z}$, its "restriction" to $H^{2*}(M;\mathbb{C})$ is characterized by the property

$$\int_{M'} \psi(f_*\eta) = \int_M (f^*\psi)\eta \qquad \forall\, \eta\in H^{2*}(M;\mathbb{C}),\ \psi\in H^{2*}(M';\mathbb{C}). \tag{6.3}$$

Under the canonical identification of $H^*(\mathrm{pt};\mathbb{Z})$ with $\mathbb{Z}$, the integration homomorphism (6.1) is the cohomology pushforward homomorphism for the map $M\longrightarrow \mathrm{pt}$.

Exercise 6.1. Let $f: M\longrightarrow M'$ and $g: M'\longrightarrow M''$ be continuous maps between compact oriented manifolds. Show that

$$(g\circ f)_* = g_*\circ f_*\colon\ H^*(M;\mathbb{Z}) \longrightarrow H^{*+m''-m}(M'';\mathbb{Z}). \tag{6.4}$$

Exercise 6.2. Let $f: M\longrightarrow M'$ be a continuous map between two compact oriented even-dimensional manifolds. Show that the "restriction" of the homomorphism (6.2) to $H^{2*}(M;\mathbb{C})$ is characterized by the property (6.3) and that

$$f_*\big((f^*\psi)\eta\big) = \psi(f_*\eta) \qquad \forall\ \eta\in H^{2*}(M;\mathbb{C}),\ \psi\in H^{2*}(M';\mathbb{C}). \tag{6.5}$$

Suppose $\pi: \mathcal{N}\longrightarrow S$ is an oriented vector bundle of rank $\mathfrak{c}$. It carries a unique orientation (or Thom) class $u\in H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z})$ characterized by the property that

$$\big\langle u, [\mathcal{N}_x,\mathcal{N}_x-x]\big\rangle = 1 \qquad \forall\ x\in S, \tag{6.6}$$

where $[\mathcal{N}_x,\mathcal{N}_x-x]\in H_{\mathfrak{c}}(\mathcal{N}_x,\mathcal{N}_x-x;\mathbb{Z})$ is the relative fundamental class; see [26, p110]. The uniqueness part implies that

$$u = \widetilde{f}^*u' \in H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z}) \tag{6.7}$$

for any vector bundle homomorphism $\widetilde{f}: \mathcal{N}\longrightarrow\mathcal{N}'$ covering a continuous map $f: S\longrightarrow S'$ such that the restriction of $\widetilde{f}$ to every fiber of $\mathcal{N}$ is an orientation-preserving isomorphism. By definition,

$$e(\mathcal{N}) = u|_S \in H^{\mathfrak{c}}(S;\mathbb{Z}); \tag{6.8}$$

see [26, p98].

Exercise 6.3. Suppose $\mathcal{N}\longrightarrow S$ and $u\in H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z})$ are as above. Let $u_{\mathfrak{c}}\in H^{\mathfrak{c}}(\mathbb{R}^{\mathfrak{c}},\mathbb{R}^{\mathfrak{c}}-0;\mathbb{Z})$ be a generator. Show that the homomorphism

$$H^{\mathfrak{c}}(\mathbb{R}^{\mathfrak{c}},\mathbb{R}^{\mathfrak{c}}-0;\mathbb{Z}) \longrightarrow H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z}), \qquad u_{\mathfrak{c}}\longrightarrow u,$$

defines a cohomology extension of the fiber for the fiber bundle pair $(\mathcal{N},\mathcal{N}-S)\longrightarrow S$, i.e. the composition

$$H^{\mathfrak{c}}(\mathbb{R}^{\mathfrak{c}},\mathbb{R}^{\mathfrak{c}}-0;\mathbb{Z}) \longrightarrow H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z}) \longrightarrow H^{\mathfrak{c}}(\mathcal{N}_x,\mathcal{N}_x-S_x;\mathbb{Z}),$$

where the last homomorphism is the restriction to a fiber, is an isomorphism for all $x\in S$.

Since $H^*(\mathbb{R}^{\mathfrak{c}},\mathbb{R}^{\mathfrak{c}}-0;\mathbb{Z})$ is one-dimensional, the homomorphism

$$\Phi_{\mathcal{N}}\colon\ H^{*-\mathfrak{c}}(S;\mathbb{Z})\longrightarrow H^{\mathfrak{c}}(\mathcal{N},\mathcal{N}-S;\mathbb{Z}),\qquad \Psi_{\mathcal{N}}(\eta)=(\pi^*\eta)u, \tag{6.9}$$

is an isomorphism by Exercise 6.3 and [35, Theorem 5.7.9]. It is known as the Thom Isomorphism; see also [26, Theorem 10.4].

If M is a smooth oriented manifold and $\iota: S\hookrightarrow M$ is a smooth oriented submanifold, the normal bundle

$$\pi\colon\ \mathcal{N}_M S\equiv TM|_S/TS\longrightarrow S$$

is oriented and is of rank $\mathfrak{c}\equiv\mathrm{codim}_M S$. By the Tubular Neighborhood Theorem [26, Theorem 11.1], $\mathcal{N}_M S$ can be identified with a neighborhood of S in M. Thus, we obtain a diagram of cohomology homomorphisms

$$\begin{array}{ccccc} H^*(\mathcal{N}_M S,\mathcal{N}_M S-S;\mathbb{Z}) & \overset{\approx}{\longleftrightarrow} & H^*(M,M-S;\mathbb{Z}) & \longrightarrow & H^*(M;\mathbb{Z}) \\ \Phi_{\mathcal{N}_M S}\Big\uparrow & & & \overset{\iota_*}{\nearrow} & \\ H^{*-\mathfrak{c}}(S;\mathbb{Z}) & & & & \end{array} \tag{6.10}$$

where the two horizontal arrows are the excision isomorphism and the restriction homomorphism. For $\eta\in H^*(S;\mathbb{Z})$, let

$$(\Phi\eta)_S\in H^*(M,M-S;\mathbb{Z})$$

be the image of $\Phi_{\mathcal{N}_M S}\eta\in H^*(\mathcal{N},\mathcal{N}-S;\mathbb{Z})$ under the excision isomorphism in (6.10) and

$$u_S\equiv(\Phi 1)_S\in H^{\mathfrak{c}}(M,M-S;\mathbb{Z})$$

be the element corresponding to u. By the next lemma, the above diagram commutes. Its proof is based on [26, Exercise 11-C]; a de Rham cohomology argument, applicable only with $\mathbb{R}$ and $\mathbb{C}$-coefficients, appears in the proof of [5, Proposition 6.24(a)].

Lemma 6.4. *If M is a smooth compact oriented manifold and $\iota: S\hookrightarrow M$ is a smooth compact oriented submanifold, then*

$$\big((\Phi\eta)_S\big)|_M\cap[M]=\iota_*\big(\eta\cap[S]\big)\qquad\forall\ \eta\in H^*(S;\mathbb{Z}).$$

Proof. Let $m=\dim M$, $\eta_S\in H^{m-\mathfrak{c}}(S;\mathbb{Z})$ be the orientation class for S, and $\iota_{\mathcal{N}}:\mathcal{N}\longrightarrow M$ be an embedding as a tubular neighborhood. Denote by $[M,M-S]$ and $[\mathcal{N}_M S,\mathcal{N}_M S-S]$ the relative fundamental classes of $(M,M-S)$ and $(\mathcal{N}_M S,\mathcal{N}_M S-S)$, respectively. They are identified under the excision isomorphism, which is given by $\iota_{\mathcal{N}*}$:

$$\iota_{\mathcal{N}*}\big([\mathcal{N}_M S,\mathcal{N}_M S-S]\big)=[M,M-S]. \tag{6.11}$$

Since u is the orientation class for the fiber bundle $(\mathcal{N}_M S,\mathcal{N}_M S-S)$,

$$(\pi^*\eta_S)u\in H^m\big(\mathcal{N}_M S,\mathcal{N}_M S-S;\mathbb{Z}\big)$$

is the orientation class for the total space and so

$$\big\langle\pi^*\eta_S,u\cap[\mathcal{N}_M S,\mathcal{N}_M S-S]\big\rangle=\big\langle(\pi^*\eta_S)u,[\mathcal{N}_M S,\mathcal{N}_M S-S]\big\rangle=1.$$

Thus, the image of $u\cap[\mathcal{N}_M S,\mathcal{N}_M S-S]$ under the homomorphism

$$\pi_*\colon H_*(\mathcal{N};\mathbb{Z})\longrightarrow H_*(S;\mathbb{Z})$$

is $[S]$ and so

$$\begin{split}\iota_{\mathcal{N}*}\big(\Phi_{\mathcal{N}_M S}\eta\cap[\mathcal{N}_M S,\mathcal{N}_M S-S]\big)&=\iota_*\pi_*\big((\pi^*\eta)\cap\big(u\cap[\mathcal{N}_M S,\mathcal{N}_M S-S]\big)\big)\\&=\iota_*\big(\eta\cap\pi_*\big(u\cap[\mathcal{N}_M S,\mathcal{N}_M S-S]\big)\big)=\iota_*\big(\eta\cap[S]\big).\end{split}\tag{6.12}$$

We now combine (6.11) and (6.12) with the commutative diagram

$$\begin{array}{ccccc}H^*(M,M-S)\otimes H_m(M) & \longrightarrow & H^*(M,M-S)\otimes H_m(M,M-S) & \xleftarrow{\approx} & H^*(\mathcal{N},\mathcal{N}-S)\otimes H_m(\mathcal{N},\mathcal{N}-S)\\ \downarrow & & \downarrow\cap & & \downarrow\cap\\ H^*(M)\otimes H_m(M) & \xrightarrow{\cap} & H_{m-*}(M) & \xleftarrow{\iota_{\mathcal{N}*}} & H_{m-*}(\mathcal{N})\end{array}$$

The first horizontal arrow sends $(\Phi\eta)_S\otimes[M]$ to $(\Phi\eta)_S\otimes[M,M-S]$. By the commutativity of the first square, this implies that

$$\big((\Phi\eta)_S\big)|_M\cap[M]=(\Phi\eta)_S\cap[M,M-S].\tag{6.13}$$

By the definition of $(\Phi\eta)_S$ and (6.11), the second horizontal arrow in the first row takes the class $\Phi_{\mathcal{N}_M S}\eta\otimes[\mathcal{N}_M S,\mathcal{N}_M S–S]$ to $(\Phi\eta)_S\otimes[M,M–S]$ as well. Along with (6.12) and (6.13), the commutativity of the right square implies that

$$\big((\Phi\eta)_S\big)|_M\cap[M]=(\Phi\eta)_S\cap[M,M-S]=\iota_*\big(\eta\cap[S]\big),$$

as claimed. □

Corollary 6.5. *If M is a smooth compact oriented manifold and $\iota: S\hookrightarrow M$ is a smooth compact oriented submanifold, then*

$$u_S|_M=\mathrm{PD}_M\big(\iota_*[S]\big)\in H^*(M;\mathbb{Z}).$$

Proof. This is the $\eta=1$ case of Lemma 6.4. □

Corollary 6.6. *If M is a smooth compact oriented manifold and $\iota: S\hookrightarrow M$ is a smooth compact oriented submanifold, then*

$$\iota^*\iota_*\eta=\eta\, e(\mathcal{N}_M S)\in H^*(S;\mathbb{Z})\qquad\forall\ \eta\in H^*(S;\mathbb{Z}).$$

If in addition $\iota'\colon S'\hookrightarrow M$ is a subspace of M disjoint from S, then

$$\iota'^*\iota_*\eta=0\in H^*(S';\mathbb{Z})\qquad\forall\ \eta\in H^*(S;\mathbb{Z}).$$

Proof. The excision isomorphism on cohomology going in the opposite direction to the second arrow in the first row in the diagram in the proof of Lemma 6.4 is $\iota_{\mathcal{N}}^*$. Thus, by (6.2) and Lemma 6.4,

$$\iota^*\iota_*\eta=\iota^*\big((\Phi\eta)_S\big)=\big(\iota_{\mathcal{N}}^*\big((\Phi\eta)_S\big)\big)\big|_S=\big((\pi^*\eta)u\big)\big|_S=(\pi^*\eta)|_S\,(u|_S)=\eta\,(u|_S).$$

The first claim now follows from (6.8). Since $\iota_*\eta$ is the restriction of a class in $H^*(M,M-S;\mathbb{Z})$ to $H^*(M)$ and $S'\subset M-S$, the restriction of $\iota_*\eta$ to S vanishes; this establishes the second claim. □

Corollary 6.7. *Let M, M' be smooth compact oriented manifolds, $S\subset M$ and $S'\subset M'$ be smooth compact oriented submanifolds, and $f\colon (M,S)\longrightarrow(M',S')$ be a smooth map. If the homomorphism*

$$\mathrm{d}f\colon \mathcal{N}_M S\equiv\frac{TM|_S}{TS}\longrightarrow\mathcal{N}_{M'}S'\equiv\frac{TM'|_{S'}}{TX'} \tag{6.14}$$

induced by the differential of f is an orientation-preserving diffeomorphism, then

$$\mathrm{PD}_M\big(\iota_*[S]\big)=f^*\,\mathrm{PD}_{M'}\big(\iota_*[S']\big)\in H^*(M;\mathbb{Z}).$$

Proof. Since (6.14) is an isomorphism, we can identify $\mathcal{N}_M S$ with a tubular neighborhood of S in M and $\mathcal{N}_{M'}S'$ with a tubular neighborhood of S' in M' so that the diagram

$$\begin{array}{ccccc}
H^*(\mathcal{N}_{M'}S',\mathcal{N}_{M'}S'-S';\mathbb{Z}) & \xleftarrow{\approx}\!\!\rightarrow & H^*(M',M'-S;\mathbb{C}) & \longrightarrow & H^*(M';\mathbb{Z})\\
\downarrow{\scriptstyle\{\mathrm{d}f\}^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*}\\
H^*(\mathcal{N}_M S,\mathcal{N}_M S-S;\mathbb{Z}) & \xleftarrow{\approx}\!\!\rightarrow & H^*(M,M-S;\mathbb{Z}) & \longrightarrow & H^*(M;\mathbb{Z})
\end{array}$$

commutes; such an identification can be chosen by pulling back a metric from M'. By (6.7), $\{\mathrm{d}f\}^*$ sends the orientation class u' for $\mathcal{N}'$ to the orientation class u for $\mathcal{N}$. The claim thus follows from the commutativity of this diagram and Corollary 6.5. $\square$

7 Equivariant pushforward

In this section, we extend the integration homomorphism (6.1) and the cohomology pushforward (6.2) to the $\mathbb{T}$-equivariant setting. These homomorphisms are used in the proof of the Localization Theorem 8.4. We conclude with Corollary 7.3 which describes the latter homomorphism for embeddings of $\mathbb{T}$-invariant submanifolds and explains the dominator in (8.3); the latter can be viewed as a cycle version of Theorem 8.4 and implies its statement.

If $\mathbb{T}$ acts on a compact oriented even-dimensional manifold M, the fibers of the fiber bundle (4.2) are compact oriented even-dimensional manifolds. Thus, there is a well-defined integration-along-the-fiber homomorphism

$$\int_M\colon H^{2*}_{\mathbb{T}}(M)\equiv H^{2*}(B_{\mathbb{T}}M;\mathbb{C})\longrightarrow H^*_{\mathbb{T}}\equiv H^*(B\mathbb{T};\mathbb{C}). \tag{7.1}$$

If $Z\subset B\mathbb{T}$ is a compact oriented manifold, $B_{\mathbb{T}}M|_Z\equiv\pi_M^{-1}(Z)$ is also a compact oriented manifold and

$$\Big\langle\int_M\eta,[Z]\Big\rangle=\big\langle\eta,\big[B_{\mathbb{T}}M|_Z\big]\big\rangle\qquad\forall\,\eta\in H^{2*}(B_{\mathbb{T}}M;\mathbb{C}), \tag{7.2}$$

where $[Z]$ and $[B_{\mathbb{T}}M|_Z]$ are the homology classes in $B\mathbb{T}$ and $B_{\mathbb{T}}M$ determined by Z and $B_{\mathbb{T}}M|_Z$, respectively. The cycle $[B_{\mathbb{T}}M|_Z]$ can be defined for any cycle $[Z]$ in $B\mathbb{T}$ by pulling back the fiber bundle (4.2) by the singular simplicies making up a cycle representing $[Z]$. The equality in (7.2) for all cycles $[Z]$ in $B\mathbb{T}$ then defines $\int_M\eta$ as an element of $H^*_{\mathbb{T}}$. It can also be used to give a cochain-level description of the homomorphism (7.1). If $\alpha\in H^*_{\mathbb{T}}$,

$$\begin{aligned}\Big\langle\int_M\alpha\cdot\eta,[Z]\Big\rangle&=\Big\langle\int_M(\pi_M^*\alpha)\eta,[Z]\Big\rangle=\big\langle(\pi_M^*\alpha)\eta,\big[B_{\mathbb{T}}M|_Z\big]\big\rangle=\big\langle\eta,(\pi_M^*\alpha)\cap\big[B_{\mathbb{T}}M|_Z\big]\big\rangle\\
&=\big\langle\eta,\big[B_{\mathbb{T}}M|_{\alpha\cap Z}\big]\big\rangle=\Big\langle\int_M\eta,\alpha\cap[Z]\Big\rangle=\Big\langle\alpha\int_M\eta,[Z]\Big\rangle\end{aligned}$$

for all $[Z]\in H_*(B\mathbb{T};\mathbb{C})$. Thus,

$$\int_M \alpha\cdot\eta = \alpha\int_M \eta \qquad \forall\ \alpha\in H^*_{\mathbb{T}},\ \eta\in H^{2*}_{\mathbb{T}}(M),$$

i.e. (7.1) is a homomorphism of modules over the ring $H^*_{\mathbb{T}}$. The next proposition extends the cohomology pushforward (6.2) to the equivariant setting.

Proposition 7.1. *If $f: M\longrightarrow M'$ is a $\mathbb{T}$-equivariant continuous map between two compact oriented manifolds of dimensions $m,m'\in 2\mathbb{Z}$, then there exists a unique $\mathbb{T}$-equivariant cohomology pushforward homomorphism*

$$f_*: H^{2*}_{\mathbb{T}}(M)\longrightarrow H^{2*}_{\mathbb{T}}(M') \tag{7.3}$$

characterized by the property that

$$\int_{M'}\psi(f_*\eta) = \int_M (f^*_{\mathbb{T}}\psi)\eta \qquad \forall\ \eta\in H^{2*}_{\mathbb{T}}(M),\ \psi\in H^{2*}_{\mathbb{T}}(M'), \tag{7.4}$$

with $f_{\mathbb{T}}: B_{\mathbb{T}}M\longrightarrow B_{\mathbb{T}}M'$ as in (4.6).

An implicit claim of this proposition is that the pairing

$$H^{2*}_{\mathbb{T}}(M')\otimes H^{2*}_{\mathbb{T}}(M')\longrightarrow H^*_{\mathbb{T}}, \qquad \eta_1\otimes\eta_2\longrightarrow \int_{M'}\eta_1\eta_2\,,$$

is nondegenerate; this is shown below. Under the canonical identification of $H^*_{\mathbb{T}}(\mathrm{pt})$ with $H^*_{\mathbb{T}}$, the $\mathbb{T}$-equivariant integration homomorphism (7.1) is the $\mathbb{T}$-equivariant cohomology pushforward homomorphism for the $\mathbb{T}$-equivariant map $M\longrightarrow\mathrm{pt}$. Similarly to Exercise 6.2, we find that

$$f_*\big((f^*\psi)\eta\big) = \psi(f_*\eta) \qquad \forall\ \eta\in H^{2*}_{\mathbb{T}}(M),\ \psi\in H^{2*}_{\mathbb{T}}(M'). \tag{7.5}$$

If $g: M'\longrightarrow M''$ is also a $\mathbb{T}$-equivariant continuous map between two compact oriented even-dimensional manifolds, then

$$(g\circ f)_* = g_*\circ f_*:\ H^{2*}_{\mathbb{T}}(M)\longrightarrow H^{2*}_{\mathbb{T}}(M'')\,. \tag{7.6}$$

The $\mathbb{T}$-equivariant cohomology pushforward homomorphism

$$\iota_*: H^{2*}_{\mathbb{T}}(M^{\mathbb{T}}) = H^*_{\mathbb{T}}\otimes H^{2*}(M^{\mathbb{T}};\mathbb{C})\longrightarrow H^{2*}_{\mathbb{T}}(M)$$

for the inclusion of the $\mathbb{T}$-fixed locus $M^{\mathbb{T}}\longrightarrow M$ enters in the proof of the Localization Theorem 8.4; see (8.3).

We will construct the homomorphism (7.3) using (6.2) and a sequence of finite-dimensional approximations to $E\mathbb{T}$ and $B_{\mathbb{T}}M$. For each $r\in\mathbb{Z}^+\cup\{\infty\}$, let

$$E_r\mathbb{T} = \big(S^{2r+1}\big)^n \subset \big(\mathbb{C}^{r+1}\big)^n, \qquad B_r\mathbb{T} = E_r\mathbb{T}/\mathbb{T}, \quad\text{and}\quad B_rM = E_r\mathbb{T}\times_{\mathbb{T}}M;$$

thus, $B_\infty\mathbb{T} = B\mathbb{T}$ and $B_\infty M = B_{\mathbb{T}}M$. Let $\pi_{M;r}: B_rM\longrightarrow B_r\mathbb{T}$ be the M-fiber bundle obtained by restricting the fiber bundle (4.2) and

$$\iota_{r',r}: B_rM\longrightarrow B_{r'}M \quad r'\ge r, \qquad \iota_r: B_rM\longrightarrow B_{\mathbb{T}}M$$

be the inclusion maps. The $\mathbb{T}$-equivariant inclusion maps $E_r\mathbb{T}\longrightarrow E_{r'}\mathbb{T}$ induce homomorphisms of the homotopy exact sequences for the $\mathbb{T}$-fibrations $E_r\mathbb{T}\times M\longrightarrow B_rM$:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \pi_s(\mathbb{T}) & \longrightarrow & \pi_s(E_r\mathbb{T})\times\pi_s(M) & \longrightarrow & \pi_s(B_rM) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \pi_s(\mathbb{T}) & \longrightarrow & \pi_s(E_{r'}\mathbb{T})\times\pi_s(M) & \longrightarrow & \pi_s(B_{r'}M) & \longrightarrow & \ldots \end{array}$$

Since $\pi_s(S^{2r+1})$ is trivial for $s\le 2r$, the first two vertical arrows above are isomorphisms for $s\le 2r$. By the Five Lemma [28, Lemma 24.3], the homomorphism

$$\pi_s(B_rM)\longrightarrow\pi_s(B_{r'}M)$$

induced by the inclusion $\iota_{r',r}$ is thus an isomorphism if $s\le 2r$ and so $\pi_s(B_{r'}M,B_rM)$ is trivial for $s\le 2r$. By the Relative Hurewicz Theorem [35, Proposition 7.5.1] and the Universal Coefficient Theorem [28, Theorem 53.1], the last property implies that

$$H_s\big(B_{r'}M,B_rM;\mathbb{Z}\big),H^s\big(B_{r'}M,B_rM;\mathbb{C}\big)=0 \qquad \forall\ s\le 2r\le 2r'.$$

Thus, the restriction map

$$\iota_{r',r}^*\colon H^s\big(B_{r'}M;\mathbb{C}\big)\longrightarrow H^s\big(B_rM;\mathbb{C}\big) \tag{7.7}$$

is an isomorphism for $s<2r$. In particular, every element of $H^s_{\mathbb{T}}(M)$ is determined by its restriction to B_rM if $s<2r$.

If $f:M\longrightarrow M'$ is a continuous $\mathbb{T}$-equivariant map between two compact oriented manifolds of dimensions $m,m'\in 2\mathbb{Z}$, respectively, we define

$$\begin{gathered} f_*\colon H^{2*}_{\mathbb{T}}(M)\longrightarrow H^{2*}_{\mathbb{T}}(M') \qquad \text{by} \qquad \iota_r^*(f_*\eta)=f_{r*}(\iota_r^*\eta)\\ \text{if}\quad \eta\in H^{2s}(B_{\mathbb{T}}M;\mathbb{C}),\ 2s<2r,2r+m-m', \end{gathered} \tag{7.8}$$

where $f_r:B_rM\longrightarrow B_rM'$ is the map induced by f. We need to verify that $f_*\eta$ in (7.8) does not depend of the choice of sufficiently large $r\in\mathbb{Z}^+$, i.e.

$$\iota_{r',r}^*\big(f_{r'*}\eta\big)=f_{r*}\big(\iota_{r',r}^*\eta\big)\in H^{2s}(B_rM';\mathbb{C}) \quad \forall\ \eta\in H^{2s}\big(B_{r'}M;\mathbb{C}\big),\ 2s<2r,2r+m-m',\ r\le r'. \tag{7.9}$$

We also need to verify that $f_*\eta$ defined by (7.8) is the unique element of $H^{2*}_{\mathbb{T}}(M')$ satisfying (7.4) for all $\psi\in H^{2*}_{\mathbb{T}}(M')$ and that the resulting homomorphism f_* commutes with scalar multiplication by $H^*_{\mathbb{T}}$. Since ι_r^* is an isomorphism for all $r\in\mathbb{Z}^+$ sufficiently large, the last two properties are equivalent to

$$\pi_{M';r*}\big(\psi(f_{r*}\eta)\big)=\pi_{M;r*}\big((f_r^*\psi)\eta\big) \qquad \forall\ \psi\in H^{2*}(B_rM';\mathbb{C}),\ \eta\in H^{2*}(B_rM;\mathbb{C}), \tag{7.10}$$

$$f_{r*}\big((\pi_{M;r}^*\alpha)\eta\big)=(\pi_{M';r}^*\alpha)(f_{r*}\eta) \qquad \forall\ \alpha\in H^{2*}(B_r\mathbb{T};\mathbb{C}),\ \eta\in H^{2*}(B_rM;\mathbb{C}), \tag{7.11}$$

for some r sufficiently large (assuming (7.8) is a well-defined homomorphism). Since $\pi_{M;r}=\pi_{M';r}\circ f_r$ and the usual cohomology pushforward satisfies (6.4), (7.10) and (7.11) are implied by

$$\psi(f_{r*}\eta)=f_{r*}\big((f_r^*\psi)\eta\big) \qquad \text{and} \qquad f_{r*}\big(\big(f_r^*(\pi_{M';r}^*\alpha)\big)\eta\big)=(\pi_{M';r}^*\alpha)(f_{r*}\eta),$$

respectively; both of these properties are special cases of (6.5).

Since (7.7) is an isomorphism for all $r' \geq r$ sufficiently large and the cohomology pushforward f_{r*} is specified by (6.3), (7.9) is equivalent to

$$\int_{B_r M'} \big(\iota_{r',r}^*\psi\big)\big(\iota_{r',r}^*(f_{r'*}\eta)\big) = \int_{B_r M} \big(f_r^*\iota_{r',r}^*\psi\big)\big(\iota_{r',r}^*\eta\big) \qquad \forall\ \psi \in H^{2*}\big(B_{r'}M';\mathbb{C}\big).$$

Since $\iota_{r',r} \circ f_r = f_{r'} \circ \iota_{r',r}$, this equality is in turn equivalent to

$$\int_{B_{r'}M'} \psi(f_{r'*}\eta)\iota_{r',r*}(1) = \int_{B_{r'}M} (f_{r'}^*\psi)\eta\, \iota_{r',r*}(1);$$

see (6.3). Applying (6.3) to the left-hand side of this identity and (6.2) to both sides, we find that it is equivalent to

$$\int_{B_{r'}M} f_{r'}^*\big(\psi \mathrm{PD}_{B_{r'}M'}\big(\iota_{r',r*}(B_r M')\big)\big)\eta = \int_{B_{r'}M} (f_{r'}^*\psi)\mathrm{PD}_{B_{r'}M}\big(\iota_{r',r*}(B_r M)\big)\eta,$$

which is implied by

$$f_{r'}^*\, \mathrm{PD}_{B_{r'}M}\big(\iota_{r',r*}(B_r M)\big) = \mathrm{PD}_{B_{r'}M}\big(\iota_{r',r*}(B_r M)\big). \tag{7.12}$$

The differential of $\pi_{M;r'} : (B_{r'}M, B_r M) \longrightarrow (B_{r'}\mathbb{T}, B_r\mathbb{T})$ induces an orientation-preserving vector bundle isomorphism

$$\mathrm{d}\pi_{M;r'} : \mathcal{N}_{B_{r'}M}(B_r M) \longrightarrow \pi_{M;r'}^*\mathcal{N}_{B_{r'}\mathbb{T}}(B_r\mathbb{T})$$

between the normal bundles of $B_r M$ in $B_{r'}M$ and of $B_r\mathbb{T}$ in $B_{r'}\mathbb{T}$. Since $\pi_{M;r} = \pi_{M';r} \circ f_r$, the differential of $f_{r'} : (B_{r'}M, B_r M) \longrightarrow (B_{r'}M', B_r M')$ thus induces an orientation-preserving vector bundle isomorphism

$$\mathrm{d}f_{r'} : \mathcal{N}_{B_{r'}M}(B_r M) \longrightarrow \pi_{M;r'}^*\mathcal{N}_{B_{r'}M'}(B_r M').$$

Therefore, (7.12) follows from Corollary 6.7.

Remark 7.2. The assumptions of Corollary 6.7 require the map

$$f_{r'} : (B_{r'}M, B_r M) \longrightarrow (B_{r'}M', B_r M')$$

to be smooth. However, its proof depends only on choosing tubular neighborhood structures for $B_r M$ in $B_{r'}M$ and $B_r M'$ in $B_{r'}M'$ compatibly. In this case, such structures are induced by pulling back a tubular neighborhood structure for $B_r\mathbb{T}$ in $B_{r'}\mathbb{T}$. For our purposes, it would be sufficient to restrict to the smooth category though.

Suppose $\mathbb{T}$ acts smoothly on a smooth compact oriented even-dimensional manifold M. Let $\iota_S : S \hookrightarrow M$ be the inclusion of a smooth compact oriented even-dimensional submanifold preserved by the $\mathbb{T}$-action. The differentials of the diffeomorphisms of (M,S) corresponding to the $\mathbb{T}$-action lift the restriction of the $\mathbb{T}$-action on S to a fiberwise linear action on $\mathcal{N}_M S$. Thus, $\mathcal{N}_M S \longrightarrow S$ is canonically a $\mathbb{T}$-vector bundle and has a well-defined $\mathbb{T}$-equivariant Euler class

$$\mathbf{e}\big(\mathcal{N}_M S\big) \in H_{\mathbb{T}}^{2*}(S);$$

see (4.7).

Corollary 7.3. *Suppose $\mathbb{T}$ acts smoothly on a smooth compact oriented even-dimensional manifold M and $\iota_S\colon S\longrightarrow M$ is the inclusion of a smooth compact oriented even-dimensional submanifold preserved by the $\mathbb{T}$-action. Then,*

$$\iota_{S;\mathbb{T}}^*\big(\iota_{S*}\eta\big)=\eta\,\mathbf{e}(\mathcal{N}_MS)\in H_{\mathbb{T}}^*(S)\qquad\forall\ \eta\in H_{\mathbb{T}}^*(S).$$

Proof. For each $r\in\mathbb{Z}^+$, let

$$\pi_r\colon\ B_r\mathcal{N}=E_r\mathbb{T}\times_{\mathbb{T}}(\mathcal{N}_MS)\longrightarrow B_rS$$

and $\iota_{S;r}\colon B_rS\longrightarrow B_rM$ be the inclusion map. The map

$$\begin{gathered}\mathcal{N}_{B_rM}(B_rS)\equiv\frac{T(E_r\mathbb{T}\times_{\mathbb{T}}M)|_{B_rS}}{T(E_r\mathbb{T}\times_{\mathbb{T}}S)}\longrightarrow B_r\mathcal{N},\\ [0,v]\longrightarrow\big[e,v+T_xS\big]\qquad\forall\ (0,v)\in T_{(e,x)}(E_r\mathbb{T}\times M),\ x\in S,\end{gathered}$$

is a well-defined vector bundle isomorphism. Since the vector bundle π_r is the restriction of $B_{\mathbb{T}}(\mathcal{N}_MS)$ to $B_rS\subset B_{\mathbb{T}}S$, it follows that

$$e\big(\mathcal{N}_{B_rM}(B_rS)\big)=\iota_r^*\,\mathbf{e}(\mathcal{N}_MS)\in H^*(B_rS;\mathbb{C})\qquad\forall\ r\in\mathbb{Z}^+.\tag{7.13}$$

Since the homomorphism (7.7) is an isomorphism whenever $s\le r\le r'$, it is sufficient to show that

$$\iota_r^*\iota_{S;\mathbb{T}}^*\big(\iota_{S*}\eta\big)=\iota_r^*\big(\eta\,\mathbf{e}(\mathcal{N}_MS)\big)\tag{7.14}$$

for some $r\in\mathbb{Z}^+$ sufficiently large. Since $\iota_{S;\mathbb{T}}\circ\iota_r=\iota_r\circ\iota_{S;r}$, (7.8) and (7.13) imply that (7.14) is equivalent to

$$\iota_{S;r}^*\iota_{S;r*}\big(\iota_r^*\eta\big)=(\iota_r^*\eta)\,e\big(\mathcal{N}_{B_rM}(B_rS)\big)\,.$$

This identity holds by Corollary 6.6. □

Exercise 7.4. Suppose $\mathbb{T}$ acts smoothly on a smooth compact oriented even-dimensional manifold M. Let $\iota\colon S\longrightarrow M$ be a smooth compact oriented even-dimensional submanifold preserved by the $\mathbb{T}$-action and $\iota'\colon S'\longrightarrow M$ be a subspace preserved by the $\mathbb{T}$-action which is disjoint from S. Show that

$$\iota_{\mathbb{T}}'^*\big(\iota_*\eta\big)=0\in H_{\mathbb{T}}^*(S')\qquad\forall\ \eta\in H_{\mathbb{T}}^*(S).$$

Exercise 7.5. Suppose $\mathbb{T}$ acts smoothly on a smooth compact oriented even-dimensional manifold M and $\iota_S\colon S\longrightarrow M$ is the inclusion of a smooth compact oriented even-dimensional submanifold preserved by the $\mathbb{T}$-action. Let $\mathcal{N}=\mathcal{N}_MS$. Show that the diagram

$$\begin{array}{ccccc} H^*(B_{\mathbb{T}}\mathcal{N},B_{\mathbb{T}}\mathcal{N}-B_{\mathbb{T}}S) & \stackrel{\approx}{\longleftrightarrow} & H^*(B_{\mathbb{T}}M,B_{\mathbb{T}}M-B_{\mathbb{T}}S) & \longrightarrow & H_{\mathbb{T}}^*(M)\\ \Phi_{\mathcal{N}}\big\uparrow & & & \nearrow_{\iota_*} & \\ H_{\mathbb{T}}^*(S) & & & & \end{array}$$

commutes.

8 Localization theorem: statement and examples

The Atiyah-Bott Localization Theorem reduces the computation of many integrals on a compact oriented manifold (evaluation of top cohomology classes against the fundamental class) with a torus action to integrals over the fixed locus

$$M^{\mathbb{T}} \equiv \big\{x\in M:\ g\cdot x = x\ \forall\, g\in\mathbb{T}\big\}.$$

The topological components F_i of $M^{\mathbb{T}}$ are smooth manifolds which are generally much simpler than the entire space. Each contributes a rational function on $\mathfrak{t}_{\mathbb{C}}$, i.e. in the variables $\alpha_1,\ldots,\alpha_n$ in the notation of (3.3). Once these fractions are added together, the denominators cancel and we end up with a *number.*

Exercise 8.1. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact manifold M. Show that

(a) the manifold M admits a $\mathbb{T}$-invariant Riemannian metric;

(b) if γ is a geodesic with respect to the Levi-Civita connection of such a metric, then so is $g\cdot\gamma$ for every $g\in\mathbb{T}$.

The $\mathbb{T}$-action on M lifts via the differentials to a $\mathbb{T}$-action on TM. For each $x\in M^{\mathbb{T}}$, this lift sends T_xM linearly to itself and thus corresponds to a representation of $\mathbb{T}^n$ on T_xM. Each such representation splits into the trivial representation on the subspace

$$\big(T_xM\big)^{\mathbb{T}} \equiv \big\{v\in T_xM:\ \mathrm{d}_xg(v) = v\ \forall\, g\in\mathbb{T}\big\}$$

and into two-dimensional nontrivial representations. After choosing an orientation on the target of each two-dimensional representation, it becomes identified with a representation of $\mathbb{T}^n$ on $\mathbb{C}$ of the form

$$(u_1,\ldots,u_n)\cdot c = u_1^{a_1}\ldots u_n^{a_n}c$$

for some $(a_1,\ldots,a_n)\in\mathbb{Z}^n-\mathbf{0}$. The choice of orientation on the two-dimensional representation determines the sign of this tuple; conversely, the choice of the sign determines the orientation. Since the action of $\mathbb{T}$ is smooth, the combinatorial data of this decomposition, i.e. the dimension of $(T_xM)^{\mathbb{T}}$ and the sets of tuples $(a_1,\ldots,a_n)\in\mathbb{Z}^n-\mathbf{0}$ up to sign, do not depend on the choice of x in a topological component F_i of $M^{\mathbb{T}}$, i.e. they are determined by F_i. The subspace of $TM|_{F_i}$ corresponding to a fixed value of $\pm(a_1,\ldots,a_n)$ forms a $\mathbb{T}$-vector bundle over F_i; this vector bundle is oriented by fixing the sign of the tuple. Thus, the choice of the sign for each tuple $\pm(a_1,\ldots,a_n)$ determines an orientation on the vector bundle $TM|_{F_i}/(TM)^{\mathbb{T}}|_{F_i}$ over F_i. Since $TM|_{F_i}$ is oriented, these choices determine an orientation on $(TM)^{\mathbb{T}}|_{F_i}$. By the next lemma, $F_i\subset(T_xM)^{\mathbb{T}}$ is a compact orientable manifold with tangent bundle $(TM)^{\mathbb{T}}|_{F_i}$.

Remark 8.2. The vector subbundle of $TM|_{F_i}$ corresponding to a fixed value of $\pm(a_1,\ldots,a_n)$ may not in general split into rank 2 subbundles, but often does.

Lemma 8.3. *Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact manifold M. Then each topological component F_i of the $\mathbb{T}$-fixed locus $M^{\mathbb{T}}$ is a smooth submanifold of M with the tangent bundle described by*

$$T_xF_i = \big(T_xM\big)^{\mathbb{T}} \qquad \forall\ x\in F_i. \tag{8.1}$$

Proof. Let $\exp : TM \longrightarrow M$ be the exponential map with respect to the Levi-Civia connection of $\mathbb{T}$-invariant metric on M provided by Exercise 8.1(a); its restriction to a neighborhood of each x in T_xM is a diffeomorphism onto a neighborhood of x in M. For $x\in F_i$ and $v\in(T_xM)^{\mathbb{T}}$, let $\gamma_v : \mathbb{R}\longrightarrow M$ be the geodesic with $\gamma_v(0)=x$ and $\gamma_v'(0)=v$. By Exercise 8.1(b), $g\cdot\gamma_v$ is also a geodesic. Since

$$\begin{aligned}\{g\cdot\gamma_v\}(0) &\equiv g\cdot\gamma_v(0) = g\cdot x = x\in F_i\subset M \qquad \text{and}\\ \{g\cdot\gamma_v\}'(0) &= \{\mathrm{d}_x g\}\big(\gamma_v'(0)\big) = \{\mathrm{d}_x g\}(v) = v\in\big(T_xM\big)^{\mathbb{T}}\subset T_xM,\end{aligned}$$

$g\cdot\gamma_v=\gamma_v$ and $\gamma_v(t)\in M^{\mathbb{T}}$ for $t\in\mathbb{R}$. On the other hand, if $v\notin\{T_xM\}^{\mathbb{T}}$, the same argument shows that $\{g\cdot\gamma_v\}'(0)\neq v$ and so $\{g\cdot\gamma_v\}(t)\neq\gamma_v(t)$ for all $t\in\mathbb{R}^*$ sufficiently small. Thus, the restriction of exp to a neighborhood of x in $(T_xM)^{\mathbb{T}}$ is a homeomorphism onto a neighborhood of x in F_i and presents F_i locally as a coordinate slice in M. Thus, $F_i\subset M$ is a smooth submanifold of M with the tangent bundle given by (8.1). □

Once the $\mathbb{T}$-vector bundle $\mathcal{N}_MF_i\longrightarrow F_i$ is oriented, there is a well-defined equivariant Euler class

$$\mathbf{e}(\mathcal{N}_MF_i)\in H^*_{\mathbb{T}}(F_i).$$

Since $\mathbb{T}$ acts trivially on F_i, there is a decomposition

$$\mathbf{e}(\mathcal{N}_MF_i) = \beta_0\otimes 1 + \sum_{\ell=1}^{\ell=N}\beta_\ell\otimes\eta_l, \qquad \text{where} \qquad \beta_\ell\in H^*_{\mathbb{T}},\ \eta_\ell\in H^{>0}(F_i;\mathbb{C});$$

see Example 4.1. If $x\in F_i$ is any point,

$$\beta_0 = \mathbf{e}(\mathcal{N}_MF_i)\big|_x = \mathbf{e}(\mathcal{N}_MF_i|_x)\in H^*_{\mathbb{T}}(x) = H^*_{\mathbb{T}}.$$

Since $\mathcal{N}_MF_i|_x$ splits into non-trivial irreducible representations of $\mathbb{T}$, the Euler class of $\mathcal{N}_MF_i|_x$, i.e. the product of the negative weights of these representations, is nonzero; see (3.10). Thus, $\beta_0\neq 0$.

We denote by $\mathcal{H}^*_{\mathbb{T}}$ the field of fractions of $H^*_{\mathbb{T}}$. By (3.3),

$$\mathcal{H}^*_{\mathbb{T}}\approx\mathbb{C}(\alpha_1,\ldots,\alpha_n).$$

If $\mathbb{T}$ acts on M, let

$$\mathcal{H}^*_{\mathbb{T}}(M) = H^*_{\mathbb{T}}(M)\otimes_{H^*_{\mathbb{T}}}\mathcal{H}^*_{\mathbb{T}}.$$

If $\mathbb{T}$ acts trivially on F, as is the case for every topological component F_i of $M^{\mathbb{T}}$, then

$$\mathcal{H}^*_{\mathbb{T}}(F) = \mathcal{H}^*_{\mathbb{T}}\otimes H^*(F;\mathbb{C})$$

as vector spaces over the field $\mathcal{H}^*_{\mathbb{T}}$. By the previous paragraph, $\mathbf{e}(\mathcal{N}_MF_i)$ is invertible in $\mathcal{H}^*_{\mathbb{T}}(F_i)$:

$$\begin{aligned}\mathbf{e}(\mathcal{N}_MF_i)^{-1} &= \big(\beta_0^{-1}\otimes 1\big)\bigg(1+\sum_{r=1}^{\infty}(-1)^r\big(\beta_0^{-r}\otimes 1\big)\bigg(\sum_{\ell=1}^{\ell=N}\beta_\ell\otimes\eta_l\bigg)^r\bigg)\\ &= \big(\beta_0^{-1}\otimes 1\big)\bigg(1+\sum_{r=1}^{\dim F_i}(-1)^r\big(\beta_0^{-r}\otimes 1\big)\bigg(\sum_{\ell=1}^{\ell=N}\beta_\ell\otimes\eta_l\bigg)^r\bigg).\end{aligned}$$

If $f\colon M \longrightarrow M'$ is a $\mathbb{T}$-equivariant map, then

$$f_*\colon H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}(M')$$

is a homomorphism of $H^*_{\mathbb{T}}$-modules. Thus, f induces a homomorphism

$$f_*\colon \mathcal{H}^*_{\mathbb{T}}(M) \longrightarrow \mathcal{H}^*_{\mathbb{T}}(M')$$

of vector spaces over $\mathcal{H}^*_{\mathbb{T}}$ which is characterized by (7.4) with $H^*_{\mathbb{T}}$ replaced by $\mathcal{H}^*_{\mathbb{T}}$.

Theorem 8.4 (basic version)**.** *If the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact oriented even-dimensional manifold M, the fixed locus $M^{\mathbb{T}}$ is a disjoint union of smooth compact oriented submanifolds F_i of M. Furthermore, the equivariant Euler class of the normal bundle $\mathcal{N}_M F_i$ of F_i in M is well-defined in $H^*_{\mathbb{T}}(F_i)$ and invertible in $\mathcal{H}^*_{\mathbb{T}}(F_i)$. Finally,*

$$\int_M \psi = \sum_{F_i} \int_{F_i} \frac{\psi|_{F_i}}{\mathbf{e}(\mathcal{N}_M F_i)} \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\ \psi \in \mathcal{H}^*_{\mathbb{T}}(M), \tag{8.2}$$

where the sum is taken over all components F_i of $M^{\mathbb{T}}$.

Theorem 8.5 (cycle version)**.** *If the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact oriented even-dimensional manifold M, the fixed locus $M^{\mathbb{T}}$ is a disjoint union of smooth compact oriented submanifolds F_i of M. Furthermore, the equivariant Euler class of the normal bundle $\mathcal{N}_M F_i$ of F_i in M is well-defined in $H^*_{\mathbb{T}}(F_i)$ and invertible in $\mathcal{H}^*_{\mathbb{T}}(F_i)$. Finally,*

$$\psi = \iota_{F*}(\tilde{\iota}_F^*\psi) = \sum_{F_i} \iota_{F_i*}\left(\frac{\iota_{F_i}^*\psi}{\mathbf{e}(\mathcal{N}_M F_i)}\right) \qquad \forall\ \psi \in H^*_{\mathbb{T}}(M), \tag{8.3}$$

where the sum is taken over all components F_i of $M^{\mathbb{T}}$.

Remark 8.6. The requirement on the dimension of M to be even is not necessary. It is included because we have defined the cohomology pushforward only for even-dimensional manifolds in order to streamline the presentation. Quite often, this theorem is applied to complex manifolds, in which case this restriction is not an issue. The proof of the injectivity of the restriction homomorphism (8.5) in Section 9 does not depend on the dimension of M being even; essentially the same argument shows that this homomorphism is also surjective. If ι_{S*} for the embedding of a $\mathbb{T}$-invariant submanifold $S \subset M$ is defined by the diagram in Exercise 7.5, then the proof that the composition (8.4) is the identity applies as well, independently of the dimension of M.

Applying π_{M*} to both sides of (8.3) and using (7.4) with $f = \iota_{F_i}$ and $\psi = 1$, we obtain (8.2).

With notation as in Theorem 8.5, define

$$\Phi\colon \mathcal{H}^*_{\mathbb{T}}(M^{\mathbb{T}}) \longrightarrow \mathcal{H}^*_{\mathbb{T}}(M), \qquad \Phi(\eta) = \sum_{F_i} \iota_{F_i*}\left(\frac{\eta|_{F_i}}{\mathbf{e}(\mathcal{N}_M F_i)}\right).$$

By Corollary 7.3 and Exercise 7.4,

$$\iota_{F_j}^*\big(\Phi(\eta)\big) = \begin{cases} \eta, & \text{if } i=j; \\ 0, & \text{if } i\neq j; \end{cases} \qquad \forall\ \eta \in \mathcal{H}^*_{\mathbb{T}}(F_i) \subset \mathcal{H}^*_{\mathbb{T}}(F).$$

Thus,

$$\iota_{M^{\mathbb{T}}}^* \circ \Phi = \mathrm{id}\colon \mathcal{H}_{\mathbb{T}}^*(M^{\mathbb{T}}) \longrightarrow \mathcal{H}_{\mathbb{T}}^*(M^{\mathbb{T}}). \tag{8.4}$$

In order to establish (8.3), it is therefore sufficient to show that

$$\iota_{M^{\mathbb{T}}}^*\colon \mathcal{H}_{\mathbb{T}}^*(M) \longrightarrow \mathcal{H}_{\mathbb{T}}^*(M^{\mathbb{T}}) \tag{8.5}$$

is injective. This is done in Section 9.

Theorem 8.4 is typically applied as follows. Let $V \longrightarrow M$ be a rank m real oriented vector bundle over a compact oriented manifold M of (even) dimension m; quite often, both may be complex. In many geometric situation, it is desirable to determine the integer $\langle e(V), M\rangle$. Suppose $\mathbb{T} \equiv (S^1)^n$ acts smoothly on M and this action lifts to an action on V. As in Figure 4.1, there are embeddings

$$\iota_e\colon M \longrightarrow B_{\mathbb{T}}M \qquad \text{and} \qquad \widetilde{\iota}_e\colon V \longrightarrow B_{\mathbb{T}}V$$

as fibers over $[e] \in B\mathbb{T}$. By (4.8) and the naturality of the pushforward,

$$\langle e(V), M\rangle = \int_M \iota_e^* \mathbf{e}(V) = \iota_{[e]}^* \int_M \mathbf{e}(V) \in H^0([e];\mathbb{C}) = \mathbb{C}. \tag{8.6}$$

The last integral above is an element of

$$H^0(B\mathbb{T};\mathbb{C}) \subset H_{\mathbb{T}}^* \approx \mathbb{C}[\alpha_1, \ldots, \alpha_n]$$

and so is a constant function in $\alpha_1, \ldots, \alpha_n$. Thus, the sum of the rational functions in $\alpha_1, \ldots, \alpha_n$ resulting from applying (8.2) with $\psi = \mathbf{e}(V)$ reduces to a number; this number is $\langle e(V), M\rangle$. While the output of this computation does not depend on the choice of the lift of the $\mathbb{T}$-action on M to an action on V, some choices might drastically simplify the resulting sums. This is illustrated below in the case of $\gamma^{*\oplus(n-1)} \longrightarrow \mathbb{P}^{n-1}$ and turns out to be particularly handy in confirming the Aspinwall-Morrison multiple cover formula in Chapter 3.

Theorem 8.4 is completely straightforward to use if $\mathbb{T}$ acts with isolated fixed points, i.e. $M^{\mathbb{T}}$ is a finite union of one-point sets F_i. In such a case, $\mathcal{N}_M F_i = TM|_{F_i}$ and each term on the right-hand side of (8.2) is an element of $\mathcal{H}_{\mathbb{T}}^*$, i.e. a rational function in $\alpha_1, \ldots, \alpha_n$. For example, we can then immediately compute the Euler characteristic of M:

$$\chi(M) = \int_M \mathbf{e}(TM) = \sum_{x \in M^{\mathbb{T}}} \int_x \frac{\mathbf{e}(TM)|_x}{\mathbf{e}(T_x M)} = \sum_{x \in M^{\mathbb{T}}} 1 = |M^{\mathbb{T}}|.$$

Corollary 8.7. *If the n-torus $\mathbb{T}$ is acting smoothly on a smooth compact oriented manifold M and there are only finitely many $\mathbb{T}$-fixed points in M, then the Euler characteristic $\chi(M)$ of M is the number of $\mathbb{T}$-fixed points. In particular, a smooth compact oriented manifold of negative Euler characteristic admits no smooth torus action with only isolated fixed points.*

For example, $(S^1)^n$ acts on $S^{2n-1} \subset \mathbb{C}^n$ by complex multiplication without fixed points and thus $\chi(S^{2n-1}) = 0$. Since the action of $(S^1)^n$ on $S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$ by complex multiplication has two fixed points, $\chi(S^{2n}) = 2$. Since the standard action of $(S^1)^n$ on $\mathbb{P}^{n-1}$ has n fixed points, $\chi(\mathbb{P}^{n-1}) = n$.

Exercise 8.8. Find the Euler characteristic of the Grassmannian $\mathbb{G}(k,n)$ of k-planes in $\mathbb{C}^n$.

Example 8.9. Suppose the n-torus is acting in the standard way on $\mathbb{P}^{n-1}$. Let $P_i \in \mathbb{P}^{n-1}$ be the i-th fixed point as in (5.6). By Proposition 5.3 and Exercise 5.5(a),

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \approx \mathbb{C}[\alpha_1,\ldots,\alpha_n,\mathbf{x}]/(\mathbf{x}-\alpha_1)\ldots(\mathbf{x}-\alpha_n),$$

with $\mathbf{x}\equiv\mathbf{e}(\gamma^*)$ denoting the equivariant Euler class of the line bundle $\gamma^* \longrightarrow \mathbb{P}^{n-1}$ with the standard lift of the $\mathbb{T}$-action on $\mathbb{P}^{n-1}$. By Exercise 5.4(b), $\mathbf{x}|_{P_i}=\alpha_i$. By (8.6),

$$1 = \langle c_1(\gamma^*)^{n-1},\mathbb{P}^{n-1}\rangle = \int_{\mathbb{P}^{n-1}} \mathbf{e}\big(\gamma^{*\oplus(n-1)}\big) = \sum_{i=1}^{n} \frac{\mathbf{x}^{n-1}|_{P_i}}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} = \sum_{i=1}^{n} \frac{\alpha_i^{n-1}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\,;$$

the last equality holds by (5.10). At the end of this section, we use the Residue Theorem on S^2 to show directly that the sum of fractions on the right-hand side above is indeed 1.

Example 8.10. A more efficient way of computing the number in Example 8.9 via Theorem 8.4 is to twist the standard lift of the $\mathbb{T}$-action to γ^* as in (4.9) by a one-dimensional representation ρ_i as in (3.9) and to use a different ρ_i for each of the $n-1$ factors. We denote the resulting $\mathbb{T}$-vector bundle by $\gamma^*(\alpha_i)$. By Exercise 4.5 and Exercise 3.8(b), the equivariant Euler class of this line bundle is

$$\mathbf{e}\big(\gamma^*(\alpha_i)\big) = \mathbf{e}(\gamma^*) - w_{\rho_i} = \mathbf{x} - \alpha_i\,,$$

but the usual Euler class does not change. Applying Theorem 8.4 as in Example 8.9, we now find that

$$\langle c_1(\gamma^*)^{n-1},\mathbb{P}^{n-1}\rangle = \int_{\mathbb{P}^{n-1}} \prod_{k=1}^{n-1} \mathbf{e}\big(\gamma^*(\alpha_k)\big) = \sum_{i=1}^{n} \frac{\prod\limits_{k=1}^{n-1}(\mathbf{x}-\alpha_k)|_{P_i}}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} = \sum_{i=1}^{n} \frac{\prod\limits_{k=1}^{n-1}(\alpha_i-\alpha_k)}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)} = \sum_{i=1}^{n-1} 0 + 1 = 1.$$

Exercise 8.11. Suppose the n-torus is acting in the standard way on $\mathbb{P}^{n-1}$. Let $P_i \in \mathbb{P}^{n-1}$ be as in Example 8.9. With the notation as in (3.3), let

$$\phi_i = \prod_{k\neq i}(x-\alpha_k) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}), \qquad i=1,2,\ldots,n. \tag{8.7}$$

Show that

$$\int_{\mathbb{P}^{n-1}} \psi\phi_i = \int_{P_i} \psi|_{P_i} \equiv \psi|_{P_i} \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\ \psi\in\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}),\ i=1,2,\ldots,n, \tag{8.8}$$

i.e. ϕ_i is the equivariant Poincare dual of P_i in $\mathbb{P}^{n-1}$.

It does not happen often that a $\mathbb{T}$-action on M has only isolated fixed points. Nevertheless, as will be demonstrated in Chapters 3 and 4, even if the fixed loci are not isolated points, they are still much simpler than the entire space, and Theorem 8.4 still provides a method for computing seemingly unmanageable integrals.

We conclude this section with a direct verification of the identity in Example 8.9 via the Residue Theorem. If $f=f(z)$ is a meromorphic function on $\mathbb{C}$, we denote by $\underset{z=z_0}{\mathfrak{R}} f$ the residue of $f\mathrm{d}z$ at $z=z_0$:

$$\underset{z=z_0}{\mathfrak{R}} f \equiv \underset{z=z_0}{\mathfrak{R}} f\mathrm{d}z \equiv \frac{1}{2\pi\mathfrak{i}}\oint f(z)\mathrm{d}z,$$

where the integral is taken over a simple closed positively oriented loop around z_0 enclosing no other pole of $f\mathrm{d}z$. The value of the residue of a 1-form is independent of the choice of a local holomorphic coordinate. The $\Sigma=S^2$ case of the following statement will be used a number of times in the proof of Theorem 19.1 in Chapter 4.

Exercise 8.12 (Residue Theorem)**.** If η is a meromorphic 1-form on a compact Riemann surface Σ, show that

$$\sum_{z_0\in\Sigma}\underset{z=z_0}{\mathfrak{R}}\eta=0.$$

For example, we can apply this statement to the 1-form

$$\eta\equiv\frac{z^{n-1}\mathrm{d}z}{\prod\limits_{k=1}^{n}(z-\alpha_k)}$$

on S^2, viewing $\alpha_1,\ldots,\alpha_n$ as generic complex numbers. The poles of η on $\mathbb{C}^*\subset S^2$ are at $z=\alpha_1,\ldots,\alpha_n$. Since

$$\eta=\frac{\mathrm{d}z}{z-\alpha_i}\cdot\frac{z^{n-1}}{\prod\limits_{k\neq i}(z-\alpha_k)}$$

and the value of the last fraction at $z=\alpha_i$ is in $\mathbb{C}^*$,

$$\underset{z=\alpha_i}{\mathfrak{R}}\left\{\frac{z^{n-1}}{\prod\limits_{k=1}^{n}(z-\alpha_k)}\right\}\equiv\underset{z=\alpha_i}{\mathfrak{R}}\eta=\frac{\alpha_i^{n-1}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}.$$

In order to study the behavior of η around $z=\infty$, we substitute $w=1/z$. Since

$$\eta=-\frac{\mathrm{d}w}{w}\cdot\frac{1}{\prod\limits_{i=1}^{n}(1-\alpha_i w)},$$

we find that the residue of η at $z=\infty$ is -1. Thus,

$$0=\sum_{z_0\in S^2}\underset{z=z_0}{\mathfrak{R}}\eta=\sum_{i=1}^{n}\frac{\alpha_i^{n-1}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}-1.$$

This gives a direct proof of the identity in Example 8.9.

In Chapter 4, we use Theorem 8.4 to relate integrals over moduli spaces of stable maps to integrals over $\mathbb{P}^{n-1}$. We then use the Residue Theorem on S^2, similarly to the previous paragraph, to study properties of the latter.

9 Localization theorem: proof

In order to establish Theorems 8.4 and 8.5, it remains to show that the restriction homomorphism (8.5) is injective. For this, it is sufficient to show that there exists $\alpha\in H^*_{\mathbb{T}}-0$ such that

$$\alpha\cdot\eta=0 \qquad \forall\,\eta\in\ker\big\{\iota_{\mathbb{T}}^*\colon H^*_{\mathbb{T}}(M)\longrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}})\big\}. \tag{9.1}$$

This will be done using Mayer-Vietoris and Exercise 4.4. Essentially the same argument can be used to show that this homomorphism is also surjective (which is not necessary to do for our purposes).

Exercise 9.1. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a smooth manifold M. Use Exercise 8.1 and Lemma 8.3 to show there exist neighborhoods $\mathcal{U}_1,\mathcal{U}_2$ of $M^{\mathbb{T}}$ in M preserved by the $\mathbb{T}$-action such that $\overline{\mathcal{U}_1}\subset\mathcal{U}_2$ and $\mathcal{U}_2$ $\mathbb{T}$-equivariantly retracts onto $M^{\mathbb{T}}$.

By Exercise 9.1, the restriction homomorphism $H^*_{\mathbb{T}}(\mathcal{U}_2)\longrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}})$ is an isomorphism. Thus, it is sufficient to establish (9.1) with $M^{\mathbb{T}}$ replaced by $\mathcal{U}_2$. By Mayer-Vietoris applied to

$$B_{\mathbb{T}}M=B_{\mathbb{T}}\mathcal{U}_2\cup B_{\mathbb{T}}(M-\overline{\mathcal{U}}_1),$$

there is an exact sequence

$$H^{s-1}_{\mathbb{T}}\big(\mathcal{U}_2-\overline{\mathcal{U}_1}\big)\longrightarrow H^s_{\mathbb{T}}(M)\longrightarrow H^s_{\mathbb{T}}(\mathcal{U}_2)\oplus H^s_{\mathbb{T}}(M-\overline{\mathcal{U}_1}) \tag{9.2}$$

of modules over $H^*_{\mathbb{T}}$; the second arrow above consists of restriction homomorphisms. Thus, it is sufficient to show that

$$\alpha\cdot\eta=0 \qquad \forall\,\eta\in H^*_{\mathbb{T}}(M-\overline{\mathcal{U}_1}),H^*_{\mathbb{T}}\big(\mathcal{U}_2-\overline{\mathcal{U}_1}\big) \tag{9.3}$$

for some $\alpha\in H^*_{\mathbb{T}}-0$; see the proof of Lemma 9.2 and Remark 9.3 below.

Lemma 9.2. *Suppose $\mathbb{T}$ acts continuously on a topological space X, $\{\mathcal{U}_i\}_{i\le N}$ is an open cover of X by $\mathbb{T}$-invariant subsets, and $f_i\colon\mathcal{U}_i\longrightarrow\mathbb{T}/G_i$ is a $\mathbb{T}$-equivariant continuous map for some proper compact subgroups $G_i\subset\mathbb{T}$. Then there exists $\alpha\in H^*_{\mathbb{T}}-0$ such that*

$$\alpha\cdot\eta=0 \qquad \forall\,\eta\in H^*_{\mathbb{T}}(X). \tag{9.4}$$

Proof. Let $\mathbb{T}_i\subsetneq\mathbb{T}$ denote the identity component of G_i. By Exercise 4.4, there exist a homomorphism $h_i\colon H^*_{\mathbb{T}}\longrightarrow H^*_{\mathbb{T}_i}$ of graded algebras over $\mathbb{C}$ and an action $\cdot_i$ of $H^*_{\mathbb{T}_i}$ on $H^*_{\mathbb{T}}(\mathcal{U}_i)$ such that

$$\alpha\cdot\eta=h_i(\alpha)\cdot_i\eta \qquad \forall\,\eta\in H^*_{\mathbb{T}}(\mathcal{U}_i),\ \alpha\in H^*_{\mathbb{T}}. \tag{9.5}$$

Since $\mathbb{T}_i\subsetneq\mathbb{T}$, h_i is not injective. By (9.5), any nonzero element α of $\ker h_i$ satisfies (9.4) with $X=\mathcal{U}_i$. Thus, the claim holds for $N=1$.

Suppose $N\ge2$ and the claim of the lemma holds with N replaced by $N-1$. Let $\mathcal{U}\subset X$ denote the union of the open subsets $\mathcal{U}_1,\ldots,\mathcal{U}_{N-1}\subset X$. By Mayer-Vietoris, there is an exact sequence

$$H^{*-1}_{\mathbb{T}}(\mathcal{U}\cap\mathcal{U}_N)\xrightarrow{\ \delta\ }H^*_{\mathbb{T}}(X)\xrightarrow{\ r\ }H^*_{\mathbb{T}}(\mathcal{U})\oplus H^*_{\mathbb{T}}(\mathcal{U}_N) \tag{9.6}$$

of $H^*_{\mathbb{T}}$-modules. By the induction assumption, there exist $\alpha_\cap,\alpha_{\mathcal{U}},\alpha_N\in H^*_{\mathbb{T}}-0$ such that

$$\alpha_\cap\cdot\eta_\cap=0\ \ \forall\,\eta_\cap\in H^*_{\mathbb{T}}(\mathcal{U}\cap\mathcal{U}_N),\quad \alpha_{\mathcal{U}}\cdot\eta_{\mathcal{U}}=0\ \ \forall\,\eta_{\mathcal{U}}\in H^*_{\mathbb{T}}(\mathcal{U}),\quad \alpha_N\cdot\eta_N=0\ \ \forall\,\eta_N\in H^*_{\mathbb{T}}(\mathcal{U}_N).$$

Let $\alpha\!=\!\alpha_\cap\alpha_{\mathcal U}\alpha_N$; since $H^*_{\mathbb T}$ is an integral domain, $\alpha\!\neq\!0$. If $\eta\!\in\! H^*_{\mathbb T}(X)$, then

$$r\big(\alpha_{\mathcal U}\alpha_N\cdot\eta\big)=\alpha_{\mathcal U}\alpha_N\cdot r(\eta)=0$$

and so $\alpha_{\mathcal U}\alpha_N\cdot\eta\!=\!\delta\mu$ for some $\mu\!\in\! H^*_{\mathbb T}(\mathcal U\cap\mathcal U_N)$. Thus,

$$\alpha\cdot\eta=\alpha_\cap\cdot\delta\mu=\delta(\alpha_\cap\mu)=\delta(0)=0,$$

which establishes (9.4). □

Remark 9.3. We can also take $\alpha\!=\!(\alpha_{\mathcal U}\alpha_N)^2$ in the proof of Lemma 9.2. Let $\pi\colon B_{\mathbb T}X\longrightarrow B\mathbb T$ be the bundle projection. Since

$$r\big(\pi^*(\alpha_{\mathcal U}\alpha_N)\big)=r\big(\alpha_{\mathcal U}\alpha_N\cdot1\big)=\alpha_{\mathcal U}\alpha_N\cdot r(1)=0,$$

the Mayer-Vietoris exact sequence implies that $\pi^*(\alpha_{\mathcal U}\alpha_N)\!=\!\delta\mu$ for some $\mu\!\in\! H^*_{\mathbb T}(\mathcal U\cap\mathcal U_N)$. Thus,

$$\begin{aligned}\alpha\cdot\eta=\pi^*\big((\alpha_{\mathcal U}\alpha_N)^2\big)\eta=\big(\pi^*(\alpha_{\mathcal U}\alpha_N)\delta\mu\big)\eta=\delta\big(\pi^*(\alpha_{\mathcal U}\alpha_N)\mu\big)\eta\\ =\delta\big(\big(\pi^*(\alpha_{\mathcal U}\alpha_N)|_{\mathcal U}\big)\big|_{\mathcal U\cap\mathcal U_N}\mu\big)\eta=\delta\big(0|_{\mathcal U\cap\mathcal U_N}\mu\big)\eta=0;\end{aligned}$$

the third equality above uses [35, 5.6.12], while the fifth is the restriction of the previous equation to $H^*_{\mathbb T}(\mathcal U)$. Therefore, we do not need to be concerned with $H^*_{\mathbb T}$ of the intersection in the case of the exact sequence (9.6). In the more formal terminology of [2], this translates into the statement of Exercise 9.9.

Exercise 9.4. Suppose the n-torus $\mathbb T$ is acting smoothly on a smooth manifold M, $x\!\in\! M$, and

$$\mathbb T_x\equiv\big\{g\!\in\!\mathbb T\colon\ g\cdot x=x\big\}\subset\mathbb T\qquad\text{and}\qquad\mathbb Tx\equiv\big\{g\cdot x\colon\ g\!\in\!\mathbb T\big\}\subset M$$

are the stabilizer and the orbit of x, respectively. Show that

(a) the subspace $\mathbb Tx$ of M is a compact submanifold, which is $\mathbb T$-equivariantly diffeomorphic to $\mathbb T/\mathbb T_x$;

(b) there exist a $\mathbb T$-invariant neighborhood $\mathcal U_x$ of $\mathbb Tx$ in M (i.e. $g\cdot y\in\mathcal U_x$ for all $g\!\in\!\mathbb T$, $y\!\in\!\mathcal U_x$) and a continuous $\mathbb T$-equivariant map $f_x\colon\mathcal U_x\longrightarrow\mathbb T/\mathbb T_x$.

Let $\mathcal U_1$ and $\mathcal U_2$ be as in (9.2). By Exercise 9.4(b), there exist an open cover $\{\mathcal U_i'\}_{i\le N}$ of $M-\overline{\mathcal U_1}$ by $\mathbb T$-invariant subsets and continuous $\mathbb T$-equivariant maps $f_i\colon\mathcal U_i'\longrightarrow\mathbb T/G_i$ for some proper compact subgroups $G_i\subset\mathbb T$. By Lemma 9.2, this implies that there exists $\alpha\in H^*_{\mathbb T}-0$ so that (9.3) holds for $H^*_{\mathbb T}(M-\overline{\mathcal U_1})$. By the same reasoning, there exists $\alpha\!\in\! H^*_{\mathbb T}-0$ so that (9.3) holds for $H^*_{\mathbb T}(\mathcal U_2-\overline{\mathcal U_1})$. This establishes (9.1) and concludes the proof of Theorems 8.4 and 8.5.

Example 9.5. The fixed point set F for the S^1-action on S^2 by rotations around the z-axis consists of the north and south poles. The action of S^1 on the cylinder S^2-F is free and

$$H^*_{\mathbb T}(S^2-F)\equiv H^*\big(E\mathbb T\times_{\mathbb T}(S^2-F);\mathbb C\big)\approx H^*\big((S^2-F)/\mathbb T;\mathbb C\big)=H^*\big(I^\circ;\mathbb C\big),$$

where I° is the open interval $(-1,1)$. The action of $H^{>0}_{\mathbb T}\subset H^*_{\mathbb T}$ on $H^*_{\mathbb T}(S^2-F)$ is trivial and thus

$$\mathcal H^*_{\mathbb T}(S^2-F)\equiv H^*_{\mathbb T}(S^2-F)\otimes_{H^*_{\mathbb T}}\mathcal H^*_{\mathbb T}=0,$$

as follows from the previous paragraph.

The main part of the proof Theorem 8.5 in [2] is formulated in terms of supports of modules. We conclude this section by reviewing this formulation. The existence of $\alpha \in H^*_{\mathbb{T}} - 0$ satisfying (9.3) is equivalent to the supports of the $H^*_{\mathbb{T}}$-modules $H^*_{\mathbb{T}}(M - \overline{\mathcal{U}_1})$ and $H^*_{\mathbb{T}}(\mathcal{U}_2 - \overline{\mathcal{U}_1})$ being proper subspaces of the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathbb{T}$. Exercise 9.7 below is thus a more sophisticated formula for the first paragraph of this section. Exercise 9.8(b) is a reformulation of the first part of the proof of Lemma 9.2 above.

Let $\mathbb{T}$ be the n-torus. By Corollary 3.6, each element f of $H^*_{\mathbb{T}} = \mathrm{Sym}^* \mathfrak{t}^*_{\mathbb{C}}$ is a polynomial on $\mathfrak{t}_{\mathbb{C}}$. If $\mathcal{K} \subset H^*_{\mathbb{T}}$, let

$$V_{\mathcal{K}} = \bigcap_{f \in \mathcal{K}} f^{-1}(0) \subset \mathfrak{t}_{\mathbb{C}}.$$

Exercise 9.6. For $\mathcal{K}_1, \mathcal{K}_2 \subset H^*_{\mathbb{T}}$, define

$$\mathcal{K}_1 \cdot \mathcal{K}_2 = \big\{ f_1 \cdot f_2 \colon \, f_1 \in \mathcal{K}_1, \, f_2 \in \mathcal{K}_2 \big\}.$$

If $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subset H^*_{\mathbb{T}}$ and $\mathcal{K}_1 \cdot \mathcal{K}_2 \subset \mathcal{K}$, show that $V_{\mathcal{K}} \subset V_{\mathcal{K}_1} \cup V_{\mathcal{K}_2}$.

If W is a module over $H^*_{\mathbb{T}}$, we define the annihilator of W to be

$$\mathcal{K}(W) = \big\{ f \in H^*_{\mathbb{T}} \colon \, f \cdot w = 0 \;\forall\, w \in W \big\} \subset H^*_{\mathbb{T}}.$$

The support of a module W over $H^*_{\mathbb{T}}$ is defined by

$$\mathrm{Supp}(W) = V_{\mathcal{K}(W)} \equiv \bigcap_{f \in \mathcal{K}(W)} f^{-1}(0) \subset \mathfrak{t}_{\mathbb{C}}.$$

If $W = \{0\}$, $\mathcal{K}(W) = H^*_{\mathbb{T}}$ and $\mathrm{Supp}(\{0\}) = \emptyset$. If $W \neq \{0\}$ is a torsion-free module over $H^*_{\mathbb{T}}$, i.e.

$$f \cdot w \neq 0 \qquad \forall\, f \in H^*_{\mathbb{T}} - 0, \; w \in W - 0,$$

then $\mathcal{K}(W) = \{0\}$ and $\mathrm{Supp}(W) = \mathfrak{t}_{\mathbb{C}}$. If $\mathbb{T}$ acts on a topological space, $H^*_{\mathbb{T}}(M)$ is a module over $H^*_{\mathbb{T}}$ and thus the support of $H^*_{\mathbb{T}}(M)$ is a subspace of $\mathfrak{t}_{\mathbb{C}}$.

Exercise 9.7. Suppose $h \colon W \longrightarrow W'$ is a homomorphism of $H^*_{\mathbb{T}}$-modules such that

$$\mathrm{Supp}(\ker h), \mathrm{Supp}(\mathrm{coker}\, h) \subsetneq \mathfrak{t}_{\mathbb{C}}.$$

Show that the induced homomorphism

$$W \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}} \longrightarrow W' \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}$$

is an isomorphism.

Exercise 9.8. Show that

(a) if $W_1 \longrightarrow W \longrightarrow W_2$ is an exact sequence of $H^*_{\mathbb{T}}$-modules, then

$$\mathrm{Supp}(W) \subset \mathrm{Supp}(W_1) \cup \mathrm{Supp}(W_2).$$

(b) if $\mathfrak{t}'_{\mathbb{C}}$ is a vector subspace of $\mathfrak{t}_{\mathbb{C}}$ and W is a module over $H^*_{\mathbb{T}}$ such that the action of $H^*_{\mathbb{T}}$ on W is the composition of the restriction homomorphism

$$H^*_{\mathbb{T}} \approx \mathrm{Sym}^* \mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathrm{Sym}^* \mathfrak{t}'^*_{\mathbb{C}} \tag{9.7}$$

with an action of $\mathrm{Sym}^* \mathfrak{t}'^*_{\mathbb{C}}$ on W, then $\mathrm{Supp}(W) \subset \mathfrak{t}'_{\mathbb{C}}$.

Exercise 9.9. Suppose $\mathbb{T}$ acts on $X = \mathcal{U}_1 \cup \mathcal{U}_2$ preserving the open subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of X (i.e. $g \cdot x \in \mathcal{U}_i$ if $g \in \mathbb{T}$ and $x \in \mathcal{U}_i$). Show that

$$\mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_1 \cup \mathcal{U}_2)\big) \subset \mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_1)\big) \cup \mathrm{Supp}\big(H^*_{\mathbb{T}}(\mathcal{U}_2)\big).$$

Exercise 9.10. Suppose the n-torus $\mathbb{T}$ is acting smoothly on a compact manifold M. Show that

$$\mathrm{Supp}\big(H^*_{\mathbb{T}}(M - M^{\mathbb{T}})\big) \subsetneq \mathfrak{t}_{\mathbb{C}}\,.$$

`Hint:` use Exercises 9.1, 9.4, 4.4, 9.8(b), and 9.9.

Chapter 2

Overview of Gromov-Witten Theory

The aims of this section are

(1) to describe the moduli space setup to which the Atiyah-Bott Localization Theorem 8.4 is applied in [12, 21] to confirm the mirror symmetry prediction of [6] for the genus 0 Gromov-Witten invariants of the quintic threefold $X_5 \subset \mathbb{P}^4$,

(2) to explain what this setup has to do with counting complex curves in general and with Gromov-Witten invariants in particular.

As the latter is not necessary for understanding the arguments of [12, 21], we start with the former.

The theory of Gromov-Witten invariants, i.e. certain counts of (pseudo-)holomorphic curves, originates in the introduction of pseudo-holomorphic curves techniques into symplectic topology in [14]. Its foundations were further developed on the symplectic/analytic side in [23, 32, 20, 10] and on the algebro-geometric side in [11, 3]. Connections of these counts with string theory and a rich algebraic structure for them were indicated in [39]. As Gromov-Witten invariants are invariants of symplectic manifolds, we will describe them from the symplectic viewpoint before doing so from the algebraic viewpoint. The latter fits better with computations, as in Chapter 4, though.

10 Stable curves and maps to projective spaces

A nodal curve Σ is a compact connected complex curve whose singular points are nodes. Such a curve can be obtained from a compact Riemann surface $\widetilde{\Sigma}$, i.e. a smooth complex curve, by identifying pairs of distinct points. In other words, if Σ is a nodal curve, there exist a compact Riemann surface $\widetilde{\Sigma}$ and a finite set

$$S \equiv \big\{(x_1, x_1'), \ldots, (x_m, x_m')\big\} \subset \widetilde{\Sigma} \times \widetilde{\Sigma}$$

such that $x_i \neq x_j$ and $x_i' \neq x_j'$ if $i \neq j$, $x_i \neq x_j'$ for all i and j, and

$$\Sigma = \widetilde{\Sigma}/\sim, \qquad \text{where} \quad x_i \sim x_i' \quad \forall\, i = 1, 2, \ldots, m. \tag{10.1}$$

The quotient map

$$q \colon \widetilde{\Sigma} \longrightarrow \Sigma$$

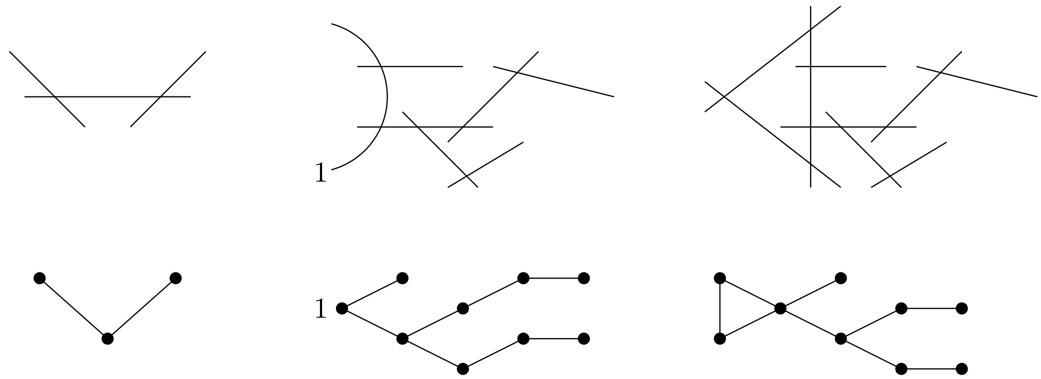


Figure 10.1: Examples of nodal curves of genus 0 and genus 1 and of their dual graphs. The label 1 indicates that the genus of the corresponding irreducible component is 1; all other irreducible components are of genus 0.

is called the normalization of Σ. It is unique up to isomorphism, i.e. if $q' : \widetilde{\Sigma}' \longrightarrow \Sigma$ is another normalization, there exists a bi-holomorphic map $\widetilde{\sigma}\colon \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma}'$ so that the diagram

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\ \widetilde{\sigma}\ } & \widetilde{\Sigma}' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & \Sigma & \end{array}$$

commutes. The domain $\widetilde{\Sigma}$ of q is often called the normalization of Σ as well. We denote by $\Sigma^* \subset \Sigma$ the open subspace of smooth points of Σ, i.e. the complement of the nodes $q(x_i) = q(x_i')$ of Σ.

The combinatorial structure of a nodal curve Σ can be encoded by the dual graph $\Gamma(\Sigma)$. Each vertex in $\Gamma(\Sigma)$ corresponds to a connected component $\widetilde{\Sigma}_k$ of $\widetilde{\Sigma}$ or equivalently to an irreducible component of Σ. Each edge in $\Gamma(\Sigma)$ corresponds to a pair $(x_i, x_i') \in S$ or equivalently to a node of Σ. There can be multiple edges between the same pair of vertices as well as edges with both ends on the same vertex. We label each vertex by the genus $g(\widetilde{\Sigma}_k)$ of the corresponding connected component of $\widetilde{\Sigma}$, if $g(\widetilde{\Sigma}_k) \neq 0$. Examples of nodal curves with the corresponding graphs appear in Figure 10.1.

The arithmetic genus g_a of a nodal curve Σ is the usual genus of the smooth complex curve obtained from Σ by replacing a small neighborhood of each node (which is homeomorphic to two disks in $\mathbb{C}$ joined at the centers) by a cylinder (the two spaces have the same boundary); see Section 12. Alternatively,

$$g_a = \dim_{\mathbb{C}} H^1(\Sigma; \mathcal{O}_\Sigma) = \dim_{\mathbb{C}} H^0(\Sigma; \omega_\Sigma),$$

where $\mathcal{O}_\Sigma \longrightarrow \Sigma$ is the structure sheaf of Σ and $\omega_\Sigma \longrightarrow \Sigma$ is the dualizing sheaf of Σ. The vector space $H^0(\Sigma; \omega_\Sigma)$ consists of meromorphic 1-forms on $\widetilde{\Sigma}$ that are holomorphic outside of all $x_i, x_i' \in S$ and have at most simple poles at $x_i, x_i' \in S$ with the residues adding up to 0 for each $i = 1, \ldots, m$. A nodal curve Σ is called rational if $g_a = 0$. Thus, if Σ is rational, it is a tree of spheres; see the left diagrams in

Figure 10.1. If Σ is of genus 1, it consists of a principal (irreducible) component (or components) Σ_P, which is either a smooth torus or a loop of $N_P \geq 1$ spheres, and trees of spheres descendant from Σ_P; see the remaining diagrams in Figure 10.1.

An isomorphism $\sigma : \Sigma_1 \longrightarrow \Sigma_2$ between two nodal curves is a homeomorphism which restricts to a holomorphic map $\Sigma_1^* \longrightarrow \Sigma_2^*$ between the subspaces of smooth points. An automorphism of a nodal curve Σ is an isomorphism of Σ with itself. For example, if Σ is represented by the first diagram in Figure 10.1, the group $\mathrm{Aut}(\Sigma)$ of automorphisms of Σ has two connected components. The identity component of $\mathrm{Aut}(\Sigma)$ can be identified with

$$\mathrm{PSL}_2(\infty) \times \mathbb{C}^* \times \mathrm{PSL}_2(0), \qquad \text{where} \quad \mathrm{PSL}_2(z_0) = \big\{\sigma \in \mathrm{PSL}_2 \colon \sigma(z_0) = z_0\big\}.$$

The elements of the other component of $\mathrm{Aut}(\Sigma)$ interchange the two side spheres. A nodal curve Σ is called stable if the group $\mathrm{Aut}(\Sigma)$ is finite. This implies that $g_a(\Sigma) \geq 2$; see Exercise 10.4 below.

Exercise 10.1. Let Σ be a nodal curve with m nodes as above and $\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_N$ be the connected components of its normalization $\widetilde{\Sigma}$. Show that

$$g_a(\Sigma) - 1 = \sum_{k=1}^{k=N} \big(g(\widetilde{\Sigma}_k) - 1\big) + m = \sum_{v \in \mathrm{Ver}} \big(g_v - 1\big) + |\mathrm{Edg}|,$$

where Ver and Edg are the sets of vertices and edges, respectively, in the dual graph $\Gamma(\Sigma)$ of Σ and $g_v \in \mathbb{Z}^{\geq 0}$ is the genus label of the vertex $v \in \mathrm{Ver}$.

Exercise 10.2. Let Σ_1 and Σ_2 be nodal curves with normalizations $q_1 \colon \widetilde{\Sigma}_1 \longrightarrow \Sigma_1$ and $q_2 \colon \widetilde{\Sigma}_2 \longrightarrow \Sigma_2$, respectively. Show that an isomorphism $\sigma \colon \Sigma_1 \longrightarrow \Sigma_2$ corresponds to a bi-holomorphic map $\widetilde{\sigma} \colon \widetilde{\Sigma}_1 \longrightarrow \widetilde{\Sigma}_2$ that descends to a map $\Sigma_1 \longrightarrow \Sigma_2$, i.e. so that there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\Sigma}_1 & \xrightarrow{\ \widetilde{\sigma}\ } & \widetilde{\Sigma}_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ \Sigma_1 & \xrightarrow{\ \sigma\ } & \Sigma_2. \end{array}$$

Exercise 10.3. Let Σ_1 and Σ_2 be isomorphic nodal curves. Show that $g_a(\Sigma_1) = g_a(\Sigma_2)$ and that the dual graphs $\Gamma(\Sigma_1)$ and $\Gamma(\Sigma_2)$ with their vertex labels are the same.

Exercise 10.4. Let Σ be a nodal curve and $q \colon \widetilde{\Sigma} \longrightarrow \Sigma$ be its normalization. Show that Σ is stable if and only if the cardinality of the set

$$\widetilde{\Sigma}_k^\bullet \equiv \widetilde{\Sigma}_k - q^{-1}(\Sigma^*)$$

is at least 3 for every connected component $\widetilde{\Sigma}_k \subset \widetilde{\Sigma}$ with $g(\widetilde{\Sigma}_k) = 0$ and at least 1 for every connected component $\widetilde{\Sigma}_k \subset \widetilde{\Sigma}$ with $g(\widetilde{\Sigma}_k) = 1$.

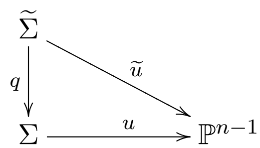
Let Σ be a nodal curve. A continuous map $u \colon \Sigma \longrightarrow \mathbb{P}^{n-1}$ is holomorphic if $u|_{\Sigma^*}$ is holomorphic. The genus $g(u)$ of such a map is the arithmetic genus $g_a(\Sigma)$ of Σ. The degree $\deg(u) \in \mathbb{Z}$ of $u \colon \Sigma \longrightarrow \mathbb{P}^{n-1}$ is defined by

$$u_*[\Sigma] = \deg(u)\, \ell \in H_2(\mathbb{P}^{n-1}; \mathbb{Z}),$$

where $\ell \in H_2(\mathbb{P}^{n-1};\mathbb{Z})$ is the standard generator, i.e. the homology class of a linearly embedded $\mathbb{P}^1 \subset \mathbb{P}^{n-1}$. An isomorphism $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2$ between nodal curves is an isomorphism between two maps $u_1 : \Sigma_1 \longrightarrow \mathbb{P}^{n-1}$ and $u_2 : \Sigma_2 \longrightarrow \mathbb{P}^{n-1}$ if $u_1 = u_2 \circ \sigma$. An automorphism of a map $u : \Sigma \longrightarrow \mathbb{P}^{n-1}$ is an isomorphism of u with itself. We denote by $\mathrm{Aut}(u)$ the group of automorphisms of u. A genus g degree d nodal map to $\mathbb{P}^{n-1}$ is stable if $\mathrm{Aut}(u)$ is a finite group.

Exercise 10.5. Let Σ be a nodal curve with normalization $q : \widetilde{\Sigma} \longrightarrow \Sigma$ and $\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_N$ be the connected components of $\widetilde{\Sigma}$. Show that

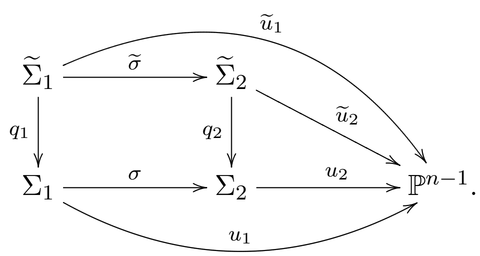
(a) a holomorphic map $u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}$ corresponds to a holomorphic map $\widetilde{u}\colon \widetilde{\Sigma} \longrightarrow \mathbb{P}^{n-1}$ that descends to a map $\Sigma \longrightarrow \mathbb{P}^{n-1}$, i.e. so that there is a commutative diagram



(b) $\deg(\widetilde{u}|_{\widetilde{\Sigma}_k}) \in \mathbb{Z}^{\geq 0}$ for every $k = 1, \ldots, N$ and $\deg(\widetilde{u}|_{\widetilde{\Sigma}_k}) = 0$ if and only if $\widetilde{u}|_{\widetilde{\Sigma}_k}$ is constant;

(c) $\deg(u) = \deg(\widetilde{u}|_{\widetilde{\Sigma}_1}) + \ldots + \deg(\widetilde{u}|_{\widetilde{\Sigma}_N})$.

Exercise 10.6. Let $u_1 : \Sigma_1 \longrightarrow \mathbb{P}^{n-1}$ and $u_2 : \Sigma_2 \longrightarrow \mathbb{P}^{n-1}$ be holomorphic maps, $q_1 : \widetilde{\Sigma}_1 \longrightarrow \Sigma_1$ and $q_2\colon \widetilde{\Sigma}_2 \longrightarrow \Sigma_2$ be the normalizations of Σ_1 and Σ_2, respectively, and $\widetilde{u}_1 : \widetilde{\Sigma}_1 \longrightarrow \mathbb{P}^{n-1}$ and $\widetilde{u}_2\colon \widetilde{\Sigma}_2 \longrightarrow \mathbb{P}^{n-1}$ be the lifts of u_1 and u_2 as in Exercise 10.5. Show that an isomorphism $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2$ between u_1 and u_2 corresponds to a bi-holomorphic map $\widetilde{\sigma}\colon \widetilde{\Sigma}_1 \longrightarrow \widetilde{\Sigma}_2$ as in Exercise 10.2 so that the diagram



commutes. Conclude that isomorphic maps have the same genus and degree.

Exercise 10.7. Let $u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}$ be a holomorphic map, $q : \widetilde{\Sigma} \longrightarrow \Sigma$ be the normalization of Σ, $\widetilde{\Sigma}_k^\bullet \subset \widetilde{\Sigma}_k$ be as in Exercise 10.4, and $\widetilde{u}\colon \widetilde{\Sigma} \longrightarrow \mathbb{P}^{n-1}$ be the the lift of u as in Exercise 10.5. Show that u is stable if and only if $\deg(\widetilde{u}|_{\widetilde{\Sigma}_k}) \neq 0$ for every connected component $\widetilde{\Sigma}_k \subset \widetilde{\Sigma}$ with $g(\widetilde{\Sigma}_k) = 0$ and $|\widetilde{\Sigma}_k^\bullet| < 3$ and every connected component $\widetilde{\Sigma}_k \subset \widetilde{\Sigma}$ with $g(\widetilde{\Sigma}_k) = 1$ and $\widetilde{\Sigma}_k^\bullet = \emptyset$.

Exercise 10.8. Let $d \in \mathbb{Z}^+$. Show that the automorphism group of the map

$$u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}, \qquad [x,y] \longrightarrow \big[x^d, y^d, 0, \ldots, 0\big],$$

is the subgroup of S^1 consisting of the d-th roots of unity with the action on $\mathbb{P}^1$ given by

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad \zeta \cdot [x,y] = [x, \zeta y],$$

for all $\zeta \in S^1$.

11 Quantum Lefschetz Hyperplane Principle

The classical Lefschetz Hyperplane Theorem [13, p156] relates the usual cohomology of a complex hypersurface which is the zero set of a transverse holomorphic section of an ample/positive holomorphic line bundle to the cohomology of the ambient space. Roughly speaking, the quantum Lefschetz Hyperplane Principle is intended to do the same for the GW-invariants of the two spaces and in particular for the quantum cohomology; the latter is determined by the genus 0 GW-invariants with 3 insertions.

Given $a\in\mathbb{Z}^+$, let

$$\pi\colon \mathcal{L}\equiv\gamma^{*\otimes a}=\mathcal{O}_{\mathbb{P}^{n-1}}(a)\longrightarrow\mathbb{P}^{n-1} \tag{11.1}$$

denote the a-th exterior power of the dual of the tautological line bundle $\gamma\longrightarrow\mathbb{P}^{n-1}$. A holomorphic section s of $\mathcal{L}$ corresponds to a degree a homogeneous polynomial in n variables. If s is transverse to the zero set in $\mathcal{L}$, then

$$X_a\equiv s^{-1}(0)\subset\mathbb{P}^{n-1}$$

is a smooth hypersurface with

$$c_1(X_a)=(n-a)c_1(\gamma^*)\big|_{X_a}\in H^2(X_a;\mathbb{Z}).$$

It is Calabi-Yau if $a=n$ and Fano if $a<n$. In this section, we describe a relation between GW-invariants of X_a and the GW-invariants of $\mathbb{P}^{n-1}$ in the special case $a=n=5$ and $g=0$. The first restriction is dropped in Section 14.

For $g,d\in\mathbb{Z}^{\ge0}$, we denote by $\mathfrak{M}_g(\mathbb{P}^{n-1},d)$ the set of isomorphism (or equivalence) classes of stable degree d holomorphic maps $u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}$ from genus g smooth complex curves Σ and by

$$\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)\supset\mathfrak{M}_g(\mathbb{P}^{n-1},d)$$

the set of isomorphism classes of all stable degree d holomorphic maps $u:\Sigma\longrightarrow\mathbb{P}^{n-1}$. The set $\mathfrak{M}_g(\mathbb{P}^{n-1},d)$ carries a natural topology induced from the space of C^∞-maps from a smooth genus g surface to $\mathbb{P}^{n-1}$. The same is the case for every stratum $\mathfrak{M}_\Gamma(\mathbb{P}^{n-1},d)$ of $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ consisting of equivalence classes of holomorphic maps $u:\Sigma\longrightarrow\mathbb{P}^{n-1}$ from genus g nodal curves Σ with a fixed dual graph $\Gamma(\Sigma)=\Gamma$. It is a fundamental discovery of [14] that these topologies extend to a natural topology on $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ with respect to which $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ is a compact and Hausdorff space. The subspace

$$\mathfrak{M}_g(\mathbb{P}^{n-1},d)\subset\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$$

is open with respect to this topology, but is not dense if $g\ge1$; see Exercise 11.1 below. The set $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ with the topology arising from [14] is called the moduli space of stable genus g degree d maps to $\mathbb{P}^{n-1}$. As shown in [11], $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ is a separated and proper Deligne-Mumford stack of finite type over $\mathbb{C}$ and carries a canonical two-term obstruction theory; see Section 15 for more details.

Exercise 11.1. Use Exercises 10.5 and 10.7 to show that

$$\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},0),\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},0)=\emptyset,\qquad\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},1)=\mathfrak{M}_0(\mathbb{P}^{n-1},1).$$

Show also that

$$\mathfrak{M}_g(\mathbb{P}^{n-1},1)=\emptyset,\quad\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},1)\neq\emptyset\qquad\forall\ g\in\mathbb{Z}^+.$$

Exercise 11.2. A linearly embedded subspace $\mathbb{P}^1 \subset \mathbb{P}^{n-1}$ is the projectivization of a two-dimensional linear subspace $P \subset \mathbb{C}^n$. Thus, the set of such subspaces of $\mathbb{P}^{n-1}$ is canonically identified with the Grassmannian $\mathbb{G}(2,n)$. Show that the map

$$\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},1)=\mathfrak{M}_0(\mathbb{P}^{n-1},1) \longrightarrow \mathbb{G}(2,n), \qquad \big[u:\mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}\big] \longrightarrow u(\mathbb{P}^1), \tag{11.2}$$

is a homeomorphism.

The above moduli spaces are particularly nice in the $g=0$ case. Every degree d holomorphic map $u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}$ is of the form

$$u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}, \qquad [x,y] \longrightarrow \big[p_1(x,y),\ldots,p_n(x,y)\big], \tag{11.3}$$

for some degree d homogeneous polynomials $p_1,\ldots,p_n$ on $\mathbb{C}^2$ with no common linear factor. Since the tuples $(p_1,\ldots,p_n)$ and $(\lambda p_1,\ldots,\lambda p_n)$ with $\lambda \in \mathbb{C}^*$ determine the same map in (11.3), the space of such maps is a dense open subspace $W_{n,d}$ of $\mathbb{P}^{n(d+1)-1}$. Since $\mathrm{Aut}(u)$ is a finite group for all u as in (11.3),

$$\mathfrak{M}_0(\mathbb{P}^{n-1},d) \equiv W_{n,d}/\mathrm{PSL}_2$$

is thus a complex orbifold of dimension $n(d+1)-4$, i.e. it is a Hausdorff topological space so that each point has a neighborhood homeomorphic to the quotient of an open subspace of $\mathbb{C}^{n(d+1)-4}$ by a finite group action and all overlap maps are holomorphic. From the symplectic viewpoint, the entire space $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},d)$ is an oriented topological orbifold stratified by smooth orbifolds of even dimensions; see Section 14. From the algebro-geometric viewpoint, $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},d)$ is a nonsingular irreducible Deligne-Mumford stack; see Section 15.

Exercise 11.3. Show that the stratum $\mathfrak{M}_\Gamma(\mathbb{P}^{n-1},1)$ of $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},1)$ consisting of equivalence classes of maps $u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}$ from curves Σ with a fixed dual graph $\Gamma(\Sigma)=\Gamma$ is a complex sub-orbifold of $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},1)$ of codimension equal to the number $|\mathrm{Edg}|$ of edges in Γ.

Example 11.4. Let $n \geq 2$. The domain Σ of an element $[u:\Sigma \longrightarrow \mathbb{P}^{n-1}]$ of $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},2)$ either is $\mathbb{P}^1$ or consists of two copies of $\mathbb{P}^1$ with one point in the first copy identified with some point in the second copy. The dual graphs of the two possible domains are shown on the left side of Figure 11.1, with the vertex labels in this case indicating the degree of the restriction of u to the corresponding irreducible component of Σ. If $\Sigma=\mathbb{P}^1$, u is either a parametrization of a smooth conic (degree 2) curves in $\mathbb{P}^{n-1}$, in which case the equivalence class $[u]$ of u is determined by its image, or a double cover of a line in $\mathbb{P}^{n-1}$, in which case $[u]$ is determined by the two branch points in its image. If Σ is a wedge of two copies of $\mathbb{P}^1$, the image of u is either two distinct lines, in which case $[u]$ is determined by its image, or a single line, in which case $[u]$ is determined by the image of the node of Σ in the image of u. The above four possibilities are illustrated by the four diagrams on the right side of Figure 11.1. As a topological space, $\overline{\mathfrak{M}}_0(\mathbb{P}^2,2)$ is the same as the classical space of complete conics in $\mathbb{P}^2$ described in [19, Chapter 3]. The latter is a smooth subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ of complex dimension 5. However, the elements $[u]$ of $\overline{\mathfrak{M}}_0(\mathbb{P}^2,2)$ whose image is a line in $\mathbb{P}^{n-1}$ come with the extra data that $\mathrm{Aut}(u)=\mathbb{Z}_2$, i.e. these elements are "half-points" in $\overline{\mathfrak{M}}_0(\mathbb{P}^2,2)$. For $n>3$, the classical space of complete conics in $\mathbb{P}^{n-1}$ is a fiber bundle over the Grassmannian $\mathbb{G}(3,n)$ and includes the extra data of a plane $\mathbb{P}^2 \subset \mathbb{P}^{n-1}$ containing the conic. This plane is uniquely determined by conics as in the middle of Figure 11.1, but not by double lines. Thus, there is a "forgetful" map from the classical space of complete conics to $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},2)$, which is a homeomorphism outside of the preimage of a subspace of codimension $n-2$.

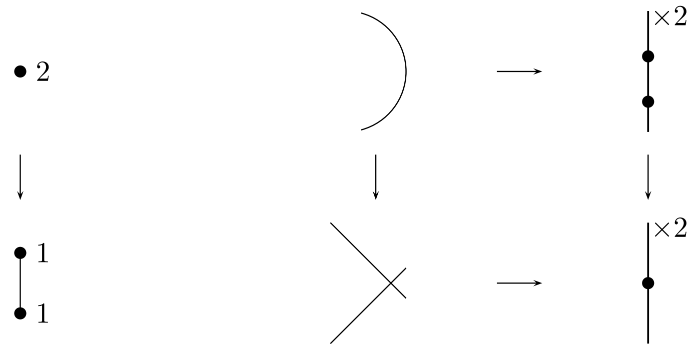


Figure 11.1: The strata and substrata of $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},2)$ with $n\geq 3$. The former are described by the dual graph of the domain Σ of the map $u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}$ with the vertex labels indicating the degree of u on the corresponding components of Σ. The latter are described by the type of the image $u(\Sigma)\subset\mathbb{P}^{n-1}$, either a smooth conic, a union of two distinct lines, or a double line with either one or two special points. The arrows indicate the possible limits between the strata and the substrata.

The total space $\mathcal{L}$ of the line bundle (11.1) is a complex manifold and the homomorphism

$$\pi_*\colon H_2(\mathcal{L};\mathbb{Z})\longrightarrow H_2(\mathbb{P}^{n-1};\mathbb{Z})$$

is an isomorphism. Thus, we can define the notion of stable genus g degree d holomorphic map $\widetilde{u}\colon\Sigma\longrightarrow\mathcal{L}$ just as in Section 10, replacing $\mathbb{P}^{n-1}$ with $\mathcal{L}$. We denote by

$$\mathfrak{M}_g(\mathcal{L},d)\subset\overline{\mathfrak{M}}_g(\mathcal{L},d)$$

the set of isomorphism (or equivalence) classes of stable degree d holomorphic maps $u\colon\Sigma\longrightarrow\mathcal{L}$ from genus g smooth complex curves Σ and the set of isomorphism classes of all stable degree d holomorphic maps $u\colon\Sigma\longrightarrow\mathcal{L}$, respectively. Every stratum $\mathfrak{M}_\Gamma(\mathcal{L},d)$ of $\overline{\mathfrak{M}}_g(\mathcal{L},d)$ consisting of equivalence classes of holomorphic maps $u\colon\Sigma\longrightarrow\mathcal{L}$ from genus g nodal curves Σ with a fixed dual graph $\Gamma(\Sigma)=\Gamma$ again carries a natural topology induced from the space of C^∞-maps to $\mathcal{L}$. By [14], these topologies extend to a natural topology on $\overline{\mathfrak{M}}_g(\mathcal{L},d)$ with respect to which $\overline{\mathfrak{M}}_g(\mathcal{L},d)$ is Hausdorff, though generally not compact (because $\mathcal{L}$ is not compact). The inclusion $\iota:\mathbb{P}^{n-1}\longrightarrow\mathcal{L}$ of the zero section induces the inclusion

$$\widehat{\iota}\colon\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)\longrightarrow\overline{\mathfrak{M}}_g(\mathcal{L},d),\qquad \widehat{\iota}\big([u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}]\big)=[\iota\circ u\colon\Sigma\longrightarrow\mathcal{L}];\tag{11.4}$$

it is an embedding with respect to the natural topologies of [14] on the two moduli spaces.

If $\widetilde{u}\colon\Sigma\longrightarrow\mathcal{L}$ is non-constant holomorphic map, then so is the map

$$u=\pi\circ\widetilde{u}\colon\Sigma\longrightarrow\mathbb{P}^{n-1},$$

because $\widetilde{u}(\Sigma)$ cannot be contained in a single fiber. This implies that u is a stable map if $\widetilde{u}$ is. Thus, the map

$$\widehat{\pi}\colon\overline{\mathfrak{M}}_g(\mathcal{L},d)\longrightarrow\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d),\qquad \widehat{\pi}\big([\widetilde{u}\colon\Sigma\longrightarrow\mathcal{L}]\big)=[\pi\circ\widetilde{u}\colon\Sigma\longrightarrow\mathbb{P}^{n-1}],\tag{11.5}$$

is well-defined; it is continuous with respect to the natural topologies of [14] on the two moduli spaces. Since $u=\pi\circ\widetilde{u}$ if and only if

$$\widetilde{u}(z)\in\mathcal{L}_{u(z)}\equiv(u^*\mathcal{L})_z \qquad \forall\, z\in\Sigma,$$

it follows that

$$\widehat{\pi}^{-1}\big([u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}]\big)\approx H^0(\Sigma;u^*\mathcal{L})/\mathrm{Aut}(\Sigma).$$

Thus, the fiber of $\widehat{\pi}$ over $[u]$ is a vector orbi-space, i.e. a quotient of complex vector space by a finite group action, with $\widehat{\iota}([u])$ corresponding to the element 0 in each fiber. In summary,

$$\widehat{\pi}\colon\ \mathcal{V}_{g,d}\equiv\overline{\mathfrak{M}}_g(\mathcal{L},d)\longrightarrow\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d) \tag{11.6}$$

is a "bundle" of vector orbi-spaces with zero section $\widehat{\iota}$. However, it is generally not a vector orbi-bundle if $g\in\mathbb{Z}^+$ because the dimensions of the fibers may vary, as illustrated by Exercise 11.7 below. An algebro-geometric description of (11.6) as a push-forward sheaf appears in Section 15.

Exercise 11.5. Let Σ be a smooth connected complex curve of genus $g\in\mathbb{Z}^+$ and $\mathcal{L}\longrightarrow\mathbb{P}^{n-1}$ be as in (11.1) with $a\in\mathbb{Z}^+$. Use the Kodaira-Serre duality [13, p153] to show that

$$H^1(\Sigma;u^*\mathcal{L})=0$$

for every holomorphic map $u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}$ with $\deg(u)>2g-2$. Show also that the evaluation map

$$H^0(\Sigma;u^*\mathcal{L})\longrightarrow\mathcal{L}_{u(z)},\qquad \xi\longrightarrow\xi(z),$$

is surjective for every $z\in\Sigma$ and every holomorphic map $u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}$ with $\deg(u)>2g-1$.

Exercise 11.6. Let $u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}$ be a representative of an element of $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},d)$ with $d\in\mathbb{Z}^+$ and $\mathcal{L}\longrightarrow\mathbb{P}^{n-1}$ be as in (11.1) with $a\in\mathbb{Z}^+$. Show that

$$\dim_{\mathbb{C}}H^0(\Sigma;u^*\mathcal{L})=ad+1,\qquad H^1(\Sigma;u^*\mathcal{L})=\{0\}.$$

Exercise 11.7. Let $u:\Sigma\longrightarrow\mathbb{P}^{n-1}$ be a representative of an element of $\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},d)$ with $d\in\mathbb{Z}^+$, $\Sigma_P\subset\Sigma$ be the principal component as defined in Section 10, and $\mathcal{L}\longrightarrow\mathbb{P}^{n-1}$ be as in (11.1) with $a\in\mathbb{Z}^+$. Show that

$$\dim_{\mathbb{C}}H^0(\Sigma;u^*\mathcal{L})=\begin{cases}ad, & \text{if }\deg(u|_{\Sigma_P})\neq0;\\ ad+1, & \text{if }\deg(u|_{\Sigma_P})=0.\end{cases}$$

Exercise 11.8. Let $\gamma_2\longrightarrow\mathbb{G}(2,n)$ denote the tautological two-plane bundle. Show that under the identification (11.2) the bundle (11.6), with $(g,d)=(0,1)$, corresponds to the vector bundle

$$\mathrm{Sym}^a\gamma_2^*\longrightarrow\mathbb{G}(2,n) \tag{11.7}$$

of rank $a+1$.

A holomorphic section s of (11.1) induces a section $\widehat{s}$ of (11.6) defined by

$$\widehat{s}\colon\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)\longrightarrow\overline{\mathfrak{M}}_g(\mathcal{L},d),\qquad \widehat{s}\big([u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}]\big)=[s\circ u\colon\Sigma\longrightarrow\mathcal{L}]. \tag{11.8}$$

An element $[u\colon\Sigma\longrightarrow\mathbb{P}^{n-1}]$ of $\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)$ lies in the zero set of this section, i.e. in the image of the map $\widehat{\iota}$ in (11.4), if and only if

$$s\big(u(z)\big)=0\in\mathcal{L}_{u(z)}\quad\forall\, z\in\Sigma,$$

$$
\begin{array}{ccc}
 & & \mathcal{L}=\mathcal{O}_{\mathbb{P}^{n-1}}(a) \\
 & & s\uparrow\ \downarrow\pi \\
X_a=s^{-1}(0) & \hookrightarrow & \mathbb{P}^{n-1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & \mathcal{V}_{g,d}=\overline{\mathfrak{M}}_g(\mathcal{L},d) \\
 & & \widehat{s}\uparrow\ \downarrow\widehat{\pi} \\
\overline{\mathfrak{M}}_g(X_a,d)=\widehat{s}^{-1}(0) & \hookrightarrow & \overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)
\end{array}
$$

$$\widehat{\pi}\big([\widetilde{u}\!:\Sigma_g\longrightarrow\mathcal{L}]\big) = [\pi\circ\widetilde{u}\!:\Sigma_g\longrightarrow\mathbb{P}^{n-1}], \qquad \widehat{s}\big([u\!:\Sigma_g\longrightarrow\mathbb{P}^{n-1}]\big) = [s\circ u\!:\Sigma_g\longrightarrow\mathcal{L}]$$

Figure 11.2: From classical to quantum Lefschetz Hyperplane Principle.

i.e. if and only if $u(z)\in X_a$ for all $z\in\Sigma$. Thus, $[u]\in\widehat{s}^{-1}(0)$ if and only if $\mathrm{Im}(u)\subset X_a$ and so

$$\widehat{s}^{-1}(0) = \big\{[u]\in\overline{\mathfrak{M}}_g(\mathbb{P}^{n-1},d)\colon\ \mathrm{Im}(u)\subset X_a\big\} \equiv \overline{\mathfrak{M}}_g(X_a,d) \tag{11.9}$$

is the moduli space of stable genus g degree d maps into $X_a\subset\mathbb{P}^{n-1}$. This situation is summarized in Figure 11.2.

Since the GW-invariants of X_a arise from the moduli spaces $\overline{\mathfrak{M}}_g(X_a,d)$, (11.9) suggests that they should be encoded in some way by the "bundles" of vector orbi-spaces in (11.6). This has been shown to be the case for $g=0,1$. The situation is particularly nice if $g=0$. In this case, (11.6) is indeed a vector orbi-bundle, as Exercise 11.6 suggests, and the Euler class of this bundle relates the GW-invariants of X_a to the GW-invariants of $\mathbb{P}^{n-1}$; see (11.10) for the $n=a=5$ case and Section 14 for the general case.

The most interesting GW-invariants of the quintic threefold $X_5\subset\mathbb{P}^4$, i.e. for $n=a=5$ in the above notation, are the rational numbers $N_{g,d}$ obtained as the degrees of virtual fundamental classes of the moduli spaces $\overline{\mathfrak{M}}_g(X_5,d)$. While these numbers are formally and intrinsically defined in Section 14, what is needed for the purposes of the proof of mirror symmetry for the genus 0 GW-invariants in Chapter 4 is that

$$N_{0,d} = \big\langle e(\mathcal{V}_{0,d}),\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},d)\big\rangle \equiv \int_{\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1},d)} e(\mathcal{V}_{0,d})\,. \tag{11.10}$$

Exercise 11.9. Show that the section of (11.7) corresponding to the section (11.8) in the $(g,d)=(0,1)$ case is transverse to the zero set for a generic choice of s.

Exercise 11.10. Use (11.10) and Exercise 11.8 to show that $N_{0,1}$ is the number of lines on a generic quintic hypersurface $X_5\subset\mathbb{P}^4$ and that $N_{0,1}=2875$.

Even in the $g=0$ case, the section (11.8) need not be transverse for $d\geq2$ for any choice of s. For example,

$$\overline{\mathfrak{M}}_0(X_5,d) = \widehat{s}^{-1}(0)$$

contains the degree d covers $\mathbb{P}^1\longrightarrow\mathbb{P}^1$ of the 2875 lines in the quintic hypersurface $X_5\subset\mathbb{P}^4$, if s is generic. These spaces are $(2d{-}2)$-dimensional over $\mathbb{C}$ and so $\widehat{s}$ is not transverse to the zero set if $d\geq2$ (because the expected dimension of the zero set of $\widehat{s}$ is 0). By the Aspinwall-Morrison formula [1], each space of such multiple covers contributes $1/d^3$ to $N_{0,d}$. A direct proof of this fact, based on the symplectic definition of $N_{0,d}$ as in Section 14, appears in [38]. A quick, but indirect, proof is given in Chapter 3 using the Localization Theorem 8.4.

For the purposes of the proof of mirror symmetry for the $g=0$ GW-invariants $N_{0,d}$ of the quintic threefold $X_5 \subset \mathbb{P}^4$, (11.10) can be taken as the definition of $N_{0,d}$. We have also described everything necessary to determine the fixed loci of the relevant torus action on $\overline{\mathfrak{M}}_0(\mathbb{P}^{n-1}, d)$ and the restrictions of $e(\mathcal{V}_{0,d})$ to these loci. In order to describe the normal bundles to these loci, we define stable marked curves and maps, as well as their moduli spaces, in Sections 12 and 13. The remainder of Chapter 2 is not necessary for the purposes of Chapter 4.

12 Stable marked curves

A prestable genus g k-marked curve is a tuple $(\Sigma, y_1, \ldots, y_k)$, where Σ is a nodal curve of genus g and $y_1, \ldots, y_k \in \Sigma$ are distinct smooth points of Σ. Two such tuples $(\Sigma, y_1, \ldots, y_k)$ and $(\Sigma', y_1', \ldots, y_k')$ are equivalent if there exists an isomorphism $\sigma: \Sigma \longrightarrow \Sigma'$ of nodal curves such that

$$\sigma(y_i) = y_i' \qquad \forall\, i = 1, 2, \ldots, k.$$

We denote by $\mathrm{Aut}(\Sigma, y_1, \ldots, y_k)$ the group of automorphisms, i.e. self-equivalences, of $(\Sigma, y_1, \ldots, y_k)$. A prestable genus g k-marked curve $(\Sigma, y_1, \ldots, y_k)$ is called stable if the group $\mathrm{Aut}(\Sigma, y_1, \ldots, y_k)$ is finite. We denote by $\overline{\mathcal{M}}_{g,k}$ the set of equivalence classes of stable genus g k-marked curves; this set is topologized below.

Exercise 12.1. Let $(\Sigma, y_1, \ldots, y_k)$ be a prestable genus 0 k-marked curve. Show that

(a) if $\Sigma = \mathbb{P}^1$, then $(\mathbb{P}^1, y_1, \ldots, y_k)$ is stable if and only if $k \geq 3$;

(b) $(\Sigma, y_1, \ldots, y_k)$ is stable if and only if every (irreducible) component of Σ contains at least 3 special (marked or singular) points of Σ;

(c) if $(\Sigma, y_1, \ldots, y_k)$, its group of automorphisms is trivial.

Exercise 12.2. Show that

(a) a smooth prestable genus 1 k-marked curve $(E, y_1, \ldots, y_k)$ is stable if and only if $k \geq 1$;

(b) a prestable genus 1 k-marked curve $(\Sigma, y_1, \ldots, y_k)$ is stable if and only if every genus 0 component of Σ contains at least 3 special (marked or singular) points of Σ and the genus 1 component of Σ (if there is one) contains at least 1 special point.

Exercise 12.3. Let Σ be a smooth complex curve that admits infinitely many holomorphic automorphisms. Show that

(a) the holomorphic bundle $T\Sigma \longrightarrow \Sigma$ admits a nonzero holomorphic section (but it may vanish at some points);

(b) the Euler characteristic of Σ, is non-negative.

Conclude that every smooth complex curve of genus at least 2 is stable.

By Exercises 12.1-12.3, the set $\overline{\mathcal{M}}_{g,k}$ is non-empty as long as $g,k\geq 0$ and $2g+k\geq 3$.

Example 12.4. If (Σ, y_1, y_2, y_3) is a stable genus 0 3-marked curve, Σ consists of one component, i.e. $\Sigma\approx\mathbb{P}^1$. Since for any two triples, (y_1,y_2,y_3) and (y_1',y_2',y_3'), of distinct points on $\mathbb{P}^1$, there exists an automorphism $\sigma\in\mathrm{PSL}_2$ of $\mathbb{P}^1$ such that $\sigma(y_i)=y_i'$ for $i=1,2,3$, $\overline{\mathcal{M}}_{0,3}$ consists of a single point: the equivalence class of $\mathbb{P}^1$ with a choice of 3 distinct points.

Example 12.5. If $(\Sigma, y_1, y_2, y_3, y_4)$ is a stable genus 0 4-marked curve, Σ consists of either one component or two components. In the latter case, each of the components carries exactly two of the marked points y_1,y_2,y_3,y_4. Similarly to Example 12.4, each of the three pairings of the points y_1,y_2,y_3,y_4 determines a unique element in $\overline{\mathcal{M}}_{0,4}$. If $\Sigma\approx\mathbb{P}^1$, the only invariant of the distinct points (y_1,y_2,y_3,y_4) on $\mathbb{P}^1$ is the cross ratio:

$$\mathrm{CR}(y_1,y_2,y_3,y_4)\equiv\frac{y_1-y_3}{y_1-y_4}:\frac{y_2-y_3}{y_2-y_4}\,, \tag{12.1}$$

where we view y_1,y_2,y_3,y_4 as elements of $\mathbb{C}\sqcup\{\infty\}$. In particular, the map

$$\mathrm{CR}\colon \big\{(y_1,y_2,y_3,y_4)\in(\mathbb{P}^1)^4\colon\ y_i\neq y_j \text{ if } i\neq j\big\}/\sim\longrightarrow\mathbb{C}-\{0,1\},$$

is a well-defined bijection. Alternatively, each equivalence class $[\mathbb{P}^1,y_1,y_2,y_3,y_4]$ has a representative of the form $[\mathbb{P}^1,\infty,0,1,y]$ with $y\in\mathbb{C}-\{0,1\}$. Thus, $\overline{\mathcal{M}}_{0,4}$ can be naturally identified with $\mathbb{C}-\{0,1\}$ along with 3 points (for the two-component curves), i.e. with $\mathbb{P}^1$.

Example 12.6. If (Σ,y_1) is a stable genus 1 1-marked curve, Σ is either a smooth torus or a sphere with two points identified; y_1 is a smooth point of Σ. Similarly to Example 12.5, the latter determines a unique element in $\overline{\mathcal{M}}_{1,1}$. If Σ is a smooth torus, (Σ,y_1) can be identified with $(\mathbb{C}/\Lambda,0)$ for some lattice $\Lambda\subset\mathbb{C}$ (i.e. Λ is a discreet subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$). Furthermore, $(\mathbb{C}/\Lambda',0)$ is equivalent to $(\mathbb{C}/\Lambda,0)$ if and only if $\Lambda'=\lambda\Lambda$ for some $\lambda\in\mathbb{C}^*$. In particular, the map

$$\mathbb{H}\equiv\big\{z\in\mathbb{C}\colon \mathrm{Im}(z)>0\big\}\longrightarrow\mathcal{M}_{1,1},\qquad \tau\longrightarrow\big(\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau),0\big), \tag{12.2}$$

where $\mathcal{M}_{1,1}\subset\overline{\mathcal{M}}_{1,1}$ is the locus of smooth curves, is surjective. Since the lattices $\mathbb{Z}\oplus\mathbb{Z}\tau$ and $\mathbb{Z}\oplus\mathbb{Z}\tau'$, with $\tau,\tau'\in\mathbb{H}$ are the same if and only if

$$\tau'=\frac{a\tau+b}{c\tau+d}\qquad\text{for some}\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2\mathbb{Z},$$

the map (12.2) induces a bijection

$$\mathbb{H}/\mathrm{PSL}_2\mathbb{Z}\longrightarrow\mathcal{M}_{1,1},\qquad\text{where}\quad \mathrm{PSL}_2\mathbb{Z}=\mathrm{SL}_2\mathbb{Z}/\{\pm\mathbb{I}\}.$$

There is also a bijective map, the j-invariant,

$$\mathsf{j}\colon\mathbb{H}/\mathrm{PSL}_2\mathbb{Z}\longrightarrow\mathbb{C};$$

see [34, Section VII-3.3]. Thus, $\overline{\mathcal{M}}_{1,1}$ can be naturally identified with $\mathbb{C}$ along with 1 point (for the singular curve), i.e. with $\mathbb{P}^1$. However, all elements of $\overline{\mathcal{M}}_{1,1}$ have non-trivial automorphisms.

Definition 12.7. *Let $(\Sigma, y_1, \ldots, y_k)$ be a prestable marked curve. A* smooth family of deformations *of $(\Sigma, y_1, \ldots, y_k)$ is a tuple $(\pi, s_1, \ldots, s_k)$, where $\pi: \mathcal{W} \longrightarrow \Delta$ and $s_i: \Delta \longrightarrow \mathcal{W}$ are holomorphic maps between a complex manifold $\mathcal{W}$ and a neighborhood Δ of 0 in $\mathbb{C}^r$ for some $r \in \mathbb{Z}^+$, such that*

$$\pi \circ s_i = \mathrm{id}_\Delta, \quad \pi^{-1}(0) = \Sigma, \quad s_i(0) = y_i \ \ \forall\, i = 1, 2, \ldots, k, \quad s_i(\Delta) \cap s_j(\Delta) = \emptyset \ \ \forall\ i \neq j,$$

and π is a submersion along $\Sigma^* \subset \Sigma$.

Definition 12.8. *Let $(\Sigma, y_1, \ldots, y_k)$ be a prestable marked curve. A sequence of prestable curves $(\Sigma_r, y_{r;1}, \ldots, y_{r;k})$* converges to *$(\Sigma, y_1, \ldots, y_k)$ if there exist a tuple $(\pi, s_1, \ldots, s_k)$ as in Definition 12.7, a sequence $\upsilon_r \in \Delta$ converging to 0, and $r^* \in \mathbb{Z}^+$ such that*

$$\pi^{-1}(\upsilon_r) = \Sigma_r \quad \textit{and} \quad s_i(\upsilon_r) = y_{r;i} \ \ \forall\, i = 1, 2, \ldots, k, \qquad \forall\ r \geq r^*.$$

Example 12.9. For $m \in \mathbb{Z}^+$, denote by $\mathbb{P}^1_m$ a chain of m copies of $\mathbb{P}^1$. Let

$$\pi_1 : \mathcal{W}_1 \equiv \mathbb{C} \times \mathbb{P}^1 \equiv \mathbb{C} \times \mathbb{P}^1_1 \longrightarrow \mathbb{C}$$

be the projection to the first component, $\mathcal{W}_2$ be the blowup of $\mathcal{W}_1$ at a point of $\pi_1^{-1}(0)$, $\mathcal{W}_3$ be the blowup of $\mathcal{W}_2$ at a smooth point in the preimage $\mathbb{P}^1_2$ of the induced projection $\pi_2 : \mathcal{W}_2 \longrightarrow \mathbb{C}$, $\mathcal{W}_4$ be the blowup of $\mathcal{W}_3$ at a smooth point on an outer component in the preimage $\mathbb{P}^1_3$ of the induced projection $\pi_3 : \mathcal{W}_3 \longrightarrow \mathbb{C}$, and so on. Thus, $\pi_m : \mathcal{W}_m \longrightarrow \mathbb{C}$ is a family of deformations of $\mathbb{P}^1_m$ with all smooth fibers being $\mathbb{P}^1 = \mathbb{P}^1_1$. According to Definition 12.8, this means that the sequence $\Sigma_r = \mathbb{P}^1$ converges to $\mathbb{P}^1_m$ for each $m \in \mathbb{Z}^+$. By choosing the blowup points on different components, we can produce a fibration $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ with the central fiber being any pre-specified genus 0 complex curve. Thus, the sequence $\Sigma_r = \mathbb{P}^1$ converges to every genus 0 complex curve. However, none of these curves is stable.

Exercise 12.10. Let x_1, x_2, x_3, x_4 be the four points in $\mathbb{P}^2$ given by

$$x_1 = [1,0,0], \quad x_2 = [0,1,0], \quad x_3 = [0,0,1], \quad x_4 = [1,1,1].$$

(a) Show that any conic (degree 2 curve) in $\mathbb{P}^2$ passing through x_1, x_2, x_3, x_4 is of the form

$$\mathcal{C}_{A,B} = \big\{ [Z_1, Z_2, Z_3] \in \mathbb{P}^2 : (A - B) Z_1 Z_2 - A Z_2 Z_3 + B Z_1 Z_3 = 0 \big\}$$

for some $(A, B) \in \mathbb{C}^2 - 0$.

(b) Show that $\mathcal{C}_{A,B}$ is isomorphic to $\mathbb{P}^1$ if $[A, B] \neq [0,1], [1,0], [1,1]$ and in such a case the cross ratio of x_1, x_2, x_3, x_4 on $\mathcal{C}_{A,B}$ is given by

$$\mathrm{CR}_{\mathcal{C}_{A,B}}(x_1, x_2, x_3, x_4) = \frac{B}{A}.$$

(c) Conclude that the projection on the first component

$$\pi : \mathcal{U} \equiv \big\{ \big([A, B]; [Z_1, Z_2, Z_3]\big) \in \mathbb{P}^1 \times \mathbb{P}^2 : (A - B) Z_1 Z_2 - A Z_2 Z_3 + B Z_1 Z_3 = 0 \big\} \longrightarrow \mathbb{P}^1$$

provides a smooth family of deformations for every element of $\overline{\mathcal{M}}_{0,4}$; see Figure 12.1.

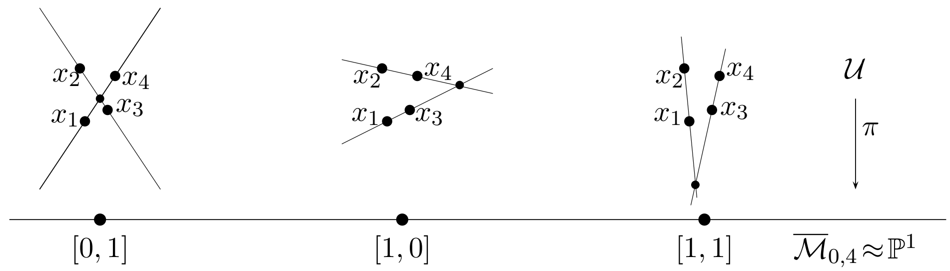


Figure 12.1: The family $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,4}$, with the 3 special fibers shown; all other fibers are isomorphic to $\mathbb{P}^1$.

Example 12.11. Let $x_1, x_2, \ldots, x_8$ be eight points in general position $\mathbb{P}^2$. Then the space $\mathcal{D} \subset \mathbb{P}^9$ of cubics (degree 3 curves) in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, and the projection on the first component

$$\pi : \mathcal{U} \equiv \big\{ (\mathcal{C}; [Z_1, Z_2, Z_3]) \in \mathcal{D} \times \mathbb{P}^2 : [Z_1, Z_2, Z_3] \in \mathcal{C} \big\} \longrightarrow \mathcal{D}$$

provides a smooth family of deformations for every element of $\overline{\mathcal{M}}_{1,1}$. A general fiber of π is a smooth curve of genus 1. Since the number of rational cubics through 8 general points in $\mathbb{P}^2$ is 12 (see [41, Section 2] for example), π has 12 singular fibers (each is a sphere with two points identified). On the other hand, for all $[\Sigma, y_1] \in \overline{\mathcal{M}}_{1,1}$, except for two elements of $\mathcal{M}_{1,1}$, the group of automorphisms of (Σ, y_1) is of order 2 (see Example 12.6 above and [34, Section VII-1.1]). Thus, the map

$$\mathcal{D} \longrightarrow \overline{\mathcal{M}}_{1,1}, \qquad \mathcal{C} \longrightarrow [\mathcal{C}, x_1],$$

is generically 12-to-1.

The set $\overline{\mathcal{M}}_{g,k}$ with the induced convergence topology of Definition 12.8 is called the Deligne-Mumford moduli space of genus g k-marked curves. It is Hausdorff and compact. It is a complex orbifold of dimension $3g-3+k$ and is a manifold if $g=0$. In the algebro-geometric language, $\overline{\mathcal{M}}_{g,k}$ is a separated, proper, nonsingular Deligne-Mumford stack. A direct analytic construction of $\overline{\mathcal{M}}_{g,k}$ as a complex orbifold is the subject of [31].

Example 12.9, which can be adapted to arbitrary genus g and number k of marked points, illustrates the significance of the stability requirement on the elements of $\overline{\mathcal{M}}_{g,k}$. It implies that the set $\overline{\mathcal{M}}'_{g,k}$ of the equivalence classes of all genus g k-marked curves (not necessarily stable) is not even $T1$ (the one-point sets are not closed) in the topology of Definition 12.8. Thus, the notion of stable nodal curves is precisely right: it adds enough curves to the smooth ones to achieve compactness without ruining Hausdorffness. The bigger moduli space $\overline{\mathcal{M}}'_{g,k}$ is an Artin stack and is useful for some geometric considerations related to the moduli space of stable maps; see [37, Section 3], for example.

13 Stable marked maps

It is a seminal observation of [14] that the convergence topology on $\overline{\mathcal{M}}_{g,k}$ can be usefully extended to topologies on spaces of maps, especially (pseudo-) holomorphic maps, from marked nodal curves to manifolds. From this point of view, the target manifold is the point in the case of $\overline{\mathcal{M}}_{g,k}$. In this section, we define this topology. In the following sections, we discuss how it gives rise to invariants of symplectic manifolds, now called Gromov-Witten invariants, and describe the linearization of the Cauchy-Riemann equation, which is fundamental to analyzing these invariants.

Let X be a (smooth) manifold and $g,k\in\mathbb{Z}^{\ge0}$. A prestable genus g k-marked nodal map to X is a tuple $(\Sigma,y_1,\ldots,y_k;u)$, where $(\Sigma,y_1,\ldots,y_k)$ is a prestable genus g k-marked nodal curve and $u\colon\Sigma\longrightarrow X$ is a continuous map. Such a map is called smooth (resp. L^p_1) if the composition

$$\widetilde{u}\equiv u\circ q\colon \widetilde{\Sigma}\longrightarrow X,$$

where $q:\widetilde{\Sigma}\longrightarrow\Sigma$ is the normalization of Σ, is smooth (resp. L^p_1). If J is an almost complex structure on X, i.e. a complex structure on the fibers of the vector bundle $TX\longrightarrow X$, a nodal map as above is called J-holomorphic if $\widetilde{u}$ is J-holomorphic, i.e. if it satisfies the Cauchy-Riemann equation corresponding to J:

$$\bar\partial_{J,\widetilde{\mathfrak{j}}}\equiv\frac{1}{2}\big(\mathrm{d}\widetilde{u}+J\circ\mathrm{d}\widetilde{u}\circ\widetilde{\mathfrak{j}}\big)=0,$$

where $\widetilde{\mathfrak{j}}$ is the complex structure on $\widetilde{\Sigma}$. The degree of a nodal map $(\Sigma,y_1,\ldots,y_k;u)$ is the homology class $A\in H_2(X;\mathbb{Z})$ such that

$$A=u_*[\Sigma]=\widetilde{u}_*[\widetilde{\Sigma}]\in H_2(X;\mathbb{Z}).$$

Two nodal maps $(\Sigma,y_1,\ldots,y_k;u)$ and $(\Sigma',y_1',\ldots,y_k';u')$ to X are equivalent if there exists an isomorphism

$$\sigma\colon(\Sigma,y_1,\ldots,y_k)\longrightarrow(\Sigma',y_1',\ldots,y_k')$$

of marked nodal curves such that $u=u'\circ\sigma$. Equivalent nodal maps have the same genus and the same degree.

We denote by $\mathrm{Aut}(\Sigma,y_1,\ldots,y_k;u)$ the group of automorphisms, i.e. self-equivalences, of a nodal map $(\Sigma,y_1,\ldots,y_k;u)$ to X. Such a map is called stable if its automorphism group is finite. For $g,k\in\mathbb{Z}^{\ge0}$ and $A\in H_2(X;\mathbb{Z})$, we denote by $\overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)$ the set of equivalence classes of stable smooth genus g k-marked degree A maps to X. It is topologized by the convergence property of Definition 13.1 below. If in addition J is an almost complex structure on X, let

$$\overline{\mathfrak{M}}_{g,k}(X,A;J)\subset\overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)$$

denote the subset of equivalence classes of J-holomorphic maps.

Definition 13.1 (Gromov's Convergence Topology)**.** *Let X be a Riemannian manifold and $\mathbf{u}\equiv(\Sigma,y_1,\ldots,y_k;u)$ be a smooth marked nodal map to X. A sequence of smooth nodal maps $(\Sigma_r,y_{r;1},\ldots,y_{r;k};u_r)$ to X converges to $\mathbf{u}$ if there exist a tuple $(\pi,s_1,\ldots,s_k)$, a sequence $\upsilon_r\in\Delta$ converging to 0, and $r^*\in\mathbb{Z}^+$ as in Definition 12.7 such that $u_r\longrightarrow u$ in the C^∞-topology on the compact subset of $\Sigma^*\subset\Sigma$ and*

$$\lim_{r'\longrightarrow\infty}\lim_{r\longrightarrow\infty}\int_{\Sigma_{\upsilon_r}\cap\mathcal{W}_{r'}}\big|\mathrm{d}u_r\big|^2=0 \tag{13.1}$$

for every sequence of open subsets $\mathcal{W}_1\supset\mathcal{W}_2\supset\ldots$ of $\mathcal{W}$ so that $\bigcap_{r'=1}^{\infty}\mathcal{W}_{r'}$ is a single point.

Since $\pi\colon \mathcal{W}\longrightarrow\Delta$ is a submersion along $\Sigma^*\subset\pi^{-1}(0)$, for every open subset $U\subset\Sigma^*$ with $\overline{U}\subset\Sigma^*$ there exist a neighborhood $\Delta_U\subset\Delta$ of 0 and a diffeomorphism

$$\psi_U\colon \Delta_U\times U\longrightarrow\mathcal{W} \qquad \text{s.t.} \quad \pi\big(\psi_U(\upsilon,z)\big)=\upsilon \quad \forall\ (\upsilon,z)\in\Delta_U\times U\,.$$

The first convergence requirement in Definition 13.1 means that the smooth function

$$u_r\circ\psi_U(\upsilon_r,\cdot)\colon U\longrightarrow X$$

and all its derivatives converge to the smooth function $u|_U$ uniformly on compact subsets of U for every open subset $U\subset\Sigma^*$ with $\overline{U}\subset\Sigma^*$. The convergence property for each U does not depend on the choice of ψ_U.

The order of the limits in (13.1) is important; if it were switched, this requirement would have been automatically satisfied. If the first convergence requirement of Definition 13.1 is satisfied and $\mathcal{W}_1\supset\mathcal{W}_2\supset\ldots$ are open subsets of $\mathcal{W}$ such that $\bigcap\limits_{r'=1}^{\infty}\mathcal{W}_{r'}$ is a single point $z_0\in\Sigma^*$, then

$$\lim_{r'\longrightarrow\infty}\lim_{r\longrightarrow\infty}\int_{\Sigma_{\upsilon_r}\cap\mathcal{W}_{r'}}\big|\mathrm{d}u_r\big|^2=\lim_{r'\longrightarrow\infty}\int_{\Sigma\cap\mathcal{W}_{r'}}\big|\mathrm{d}u\big|^2=0.$$

Thus, the second convergence requirement of Definition 13.1 is aimed at controlling the behavior of $\mathrm{d}u$ near the nodes of Σ. If it is satisfied and $\mathcal{W}_{r'}$ is a sequence of shrinking neighborhoods of the nodes, then

$$\begin{split}\int_{\Sigma}\big|\mathrm{d}u\big|^2&=\lim_{r'\longrightarrow\infty}\int_{\Sigma-\mathcal{W}_{r'}}\big|\mathrm{d}u\big|^2=\lim_{r'\longrightarrow\infty}\lim_{r\longrightarrow\infty}\int_{\Sigma_{\upsilon_r}-\mathcal{W}_{r'}}\big|\mathrm{d}u_r\big|^2\\&=\lim_{r'\longrightarrow\infty}\lim_{r\longrightarrow\infty}\int_{\Sigma_r}\big|\mathrm{d}u_r\big|^2-\lim_{r'\longrightarrow\infty}\lim_{r\longrightarrow\infty}\int_{\Sigma_{\upsilon_r}\cap\mathcal{W}_{r'}}\big|\mathrm{d}u_r\big|^2=\lim_{r\longrightarrow\infty}\int_{\Sigma_r}\big|\mathrm{d}u_r\big|^2.\end{split}\tag{13.2}$$

Thus, the energy of the limiting map u,

$$E(u)\equiv\frac{1}{2}\int_{\Sigma}\big|\mathrm{d}u\big|^2,\tag{13.3}$$

is the limit of the energies $E(u_r)$ of the maps u_r, i.e. no energy is lost (or gained) in the limit.

Example 13.2. As can be seen from (13.2), the first convergence requirement of Definition 13.1 prevents any energy gain in the limit. However, it does not prevent a loss of energy. For example, let $\pi_m\colon\mathcal{W}_m\longrightarrow\mathbb{C}$ and $\mathbb{P}^1_m=\pi_m^{-1}(0)$ be as in Example 12.9. Choose a holomorphic map $u\colon\mathbb{P}^1_3\longrightarrow\mathbb{P}^1$ so that the two nodes of $\mathbb{P}^1_3$ are mapped to the same point in $\mathbb{P}^1$. Any such map can be deformed to a family of maps $u_\upsilon:\Sigma_\upsilon\longrightarrow\mathbb{P}^1$ from the nearby fibers of π_3 which converge to u in the sense of Definition 13.1 as $\upsilon\longrightarrow0$. From the points of view of Sections 14 and 15, such a family exists because any genus 0 to $\mathbb{P}^1$ is regular or unobstructed. Alternatively, every (Hurwitz) cover $\mathbb{P}^1\longrightarrow\mathbb{P}^1$ is determined by the branch points in the target and some additional discrete data; deforming the map u to u_υ corresponds to moving some of the branch points apart. The maps u_υ induce a continuous map $\widetilde{u}\colon\mathcal{W}_3\longrightarrow\mathbb{P}^1$. Blowing down the middle sphere in $\mathbb{P}^1_3$, we obtain $\mathcal{W}_2$. Since the two nodes of $\mathbb{P}^1_3$ are mapped by u to the same point in $\mathbb{P}^1$, the induced map u' on $\mathbb{P}^1_2$ is still continuous at the node.

Since the smooth fibers of π_2 are the same as the smooth fibers of π_3, each u_υ can be viewed as a map from $\pi_2^{-1}(\upsilon)$. The maps u_υ still converge to u' outside of the node as $\upsilon \longrightarrow 0$. However, the energy is lost in the limit in this case, since the energy of u' is smaller than the energy of u. Relatedly, the map $\widetilde{u}'\colon \mathcal{W}_2 \longrightarrow \mathbb{P}^1$ is not continuous at the node of $\mathbb{P}^1_2 = \pi_2^{-1}(0)$, even though its restriction to $\pi_2^{-1}(0)$ is continuous.

If X is a compact, any two Riemannian metrics on X are uniformly comparable (i.e. the norms with respect to the two metrics differ by uniformly bounded factors). Thus, the notion of convergence of Definition 13.1 for maps to compact targets does not on the choice of Riemannian metric on the target X.

By Sobolev's Embedding Theorem, the L^p_1-Sobolev norm on the space of smooth functions from a surface Σ to a manifold X dominates the C^0-norm for $p>2$. In other words, if $u, u_r : \Sigma \longrightarrow X$ are smooth functions and

$$\lim_{r\longrightarrow\infty}\left(\int_\Sigma d_g\big(u(z),u_r(z)\big)^p+\int_\Sigma d_g\big(\nabla u(z),\nabla u_r(z)\big)^p\right)=0,$$

where $d_g(\cdot,\cdot)$ is the distance functions X and TX with respect to the Riemannian metric, then

$$\lim_{r\longrightarrow\infty} d_g\big(u(z),u_r(z)\big)=0 \qquad \forall\, z\in\Sigma.$$

This is generally not the case for $p=2$; see [24, Lemma 10.4.1]. One of the fundamental observations of [14] is that it is the case when restricting to J-holomorphic maps.

Let ω be a symplectic form on X. An almost complex structure J on X is called ω-tame if

$$\omega(v,Jv)>0 \qquad \forall\, v\in T_xX,\ v\neq 0,\ x\in X.$$

The triple (X,ω,J) is then called an almost Kahler manifold. For such a manifold,

$$g_{\omega,J}(v,w)\equiv\frac{1}{2}\big(\omega(v,Jw)-\omega(Jv,w)\big) \qquad \forall\, v,w\in T_xX,\ x\in X, \tag{13.4}$$

is a Riemannian metric on X such that

$$g_{\omega,J}(Jv,Jw)=g_{\omega,J}(v,w) \qquad \forall\, v,w\in T_xX,\ x\in X.$$

For a J-holomorphic map $u\colon\Sigma\longrightarrow X$, the energy (13.3) of u with respect to the metric $g_{\omega,J}$ in (13.4) is determined by the degree $u_*[\Sigma]$ of u; see Exercise 13.4. This is an important ingredient in the proof of [14] that the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ of stable genus g k-marked J-holomorphic maps of degree A with the topology of Definition 13.1 is compact if X is compact, as well as Hausdorff.

For many symplectic manifolds (X,ω), the genus 0 moduli space $\overline{\mathfrak{M}}_{0,k}(X,A;J)$ is stratified by smooth oriented manifolds of the expected even dimensions for a generic choice of ω-compatible almost complex structure J. In these cases, which include $\mathbb{P}^{n-1}$, other Fano manifolds, and Calabi-Yau threefolds, invariants of (X,ω) can be defined by counting the elements of the main stratum of $\overline{\mathfrak{M}}_{0,k}(X,A;J)$ that are sent by the evaluation maps of Exercise 13.3 to various cycles in X^k, as done in [24, Chapter 7].

The general principle behind this construction can be extended by positive genus g via the notion of (J,ν)-maps, as done in [32, 33] and recalled in Section 14 below.

Invariants for general compact symplectic manifolds are defined by the virtual fundamental constructions of [20] and [10]. They provide two approaches to extracting invariants from coherent systems of local versions of the (J,ν)-type maps of [32, 33]. The approach of [20] makes use of the configuration space $\overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)$, but with an L^p_1-version of the topology of Definition 13.1. This topology insures that all evaluation maps are continuous and that the relevant linearizations of the Cauchy-Riemann equation have locally uniform elliptic estimates, even as the domains of the map change; see [20, Section 3]. The space $\overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)$ with the topology of [20] is Hausdorff. This insures that a regularization of $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ constructed in [20] as a small deformation of this moduli space in $\overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)$ is Hausdorff and determines a rational homology class. The latter can be used to define invariants of (X,ω) by pulling back classes from X_k by the evaluation maps of Exercise 13.3. Because of elliptic regularity considerations, the restriction of the topology of [20] to $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ agrees with the topology of Definition 13.1. On the other hand, a similar homology class in [10] is assembled in a more abstract way; this requires some care with verifying that the glued space is Hausdorff. An analogue of the constructions of [20, 10] in the algebraic category is the subject of [3] and produces a class in the Chow group of $\overline{\mathfrak{M}}_{g,k}(X,A;J)$. These developments imply that Gromov's moduli space of stable J-holomorphic maps is precisely right: compact (if the target X is compact), Hausdorff, and virtually smooth (in a sense made precise in both symplectic topology and algebraic geometry).

Exercise 13.3. Let X be a Riemannian manifold, $g,k\in\mathbb{Z}^{\ge 0}$, $A\in H_2(X;\mathbb{Z})$, and $i=1,\ldots,k$. Show that the i-th evaluation morphism,

$$\mathrm{ev}_1\colon \overline{\mathfrak{X}}^{\infty}_{g,k}(X,A)\longrightarrow X, \qquad [\Sigma,y_1,\ldots,y_k;u]\longrightarrow u(y_i),$$

is well-defined on the equivalence classes of nodal maps and is continuous with respect to the topology of Definition 13.1.

Exercise 13.4. Let (X,ω,J) be an almost Kahler manifold and $u\colon\Sigma\longrightarrow X$ be a J-holomorphic map. Show that the energy (13.3) of u with respect to the metric $g_{\omega,J}$ in (13.4) is given by

$$E_{\omega,J}(u)=\big\langle \omega, u_*[\Sigma]\big\rangle\,. \tag{13.5}$$

14 A symplectic perspective

15 An algebro-geometric perspective

Chapter 3

Localization on Moduli Spaces of Stable Maps

16 Moduli spaces of rational maps to $\mathbb{P}^1$

Chapter 4

Mirror Symmetry

In this section we state and prove a mirror symmetry formula for the genus 0 Gromov-Witten invariants of a quintic 3-fold, as well as similar formulas for other projective hypersurfaces. The presentation generally follows [15, Chapters 29,30], but we do not treat the Fano cases separately (until the final step) or renormalize the power series involved. Following a suggestion of D. Zagier in a related setting, we also work with power series in $q=e^t$, instead of power series in t and e^t.

17 Statement for a quintic

There are a number of related mathematical formulations of mirror symmetry for a quintic 3-fold X_5, i.e. a degree 5 hypersurface in $\mathbb{P}^4$; see [47, Appendix B] for a comparison. They originate in the astounding prediction of [6] that relates "counts" of curves in X_5 (later re-interpreted as certain combinations of Gromov-Witten invariants) to the geometry of a one-dimensional mirror family of X_5. Such a relation was completely unexpected mathematically and is still mysterious; it has been proved, but not really explained. The 3-fold X_5 is special from the point of view of string theory because it is Calabi-Yau, i.e.

$$c_1(X) \equiv c_1(TX) = 0 \in H^2(X;\mathbb{Z}) \approx \mathbb{Z}. \tag{17.1}$$

Exercise 17.1. Verify the two statements contained in (17.1).

Mirror symmetry formulas express GW-invariants of X_5 in terms of the hypergeometric series

$$F(w,q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod\limits_{r=1}^{r=5d}(5w+r)}{\prod\limits_{r=1}^{r=d}(w+r)^5}\,. \tag{17.2}$$

This is a power series in q with coefficients in rational functions in w that are regular at $w=0$. In particular, F admits a Taylor expansion around $w=0$:

$$F(w,q) = I_0(q)\big(1+J(q)w+O(w^2)\big), \qquad \text{where} \tag{17.3}$$

$$I_0(q) = F(0,q) = 1+\sum_{d=1}^{\infty} q^d\frac{(5d)!}{(d!)^5}\,, \qquad J(q) = \frac{1}{I_0(q)}\sum_{d=1}^{\infty} q^d\bigg(\frac{(5d)!}{(d!)^5}\sum_{r=d+1}^{5d}\frac{5}{r}\bigg). \tag{17.4}$$

Hypergeometric functions, like $F(x,q)$, play a prominent role in complex geometry, including in the study of moduli spaces of complex manifolds (and varieties).

Exercise 17.2. Show that there exists $\tilde{J}\in Q\cdot\mathbb{Q}[[Q]]$ such that

$$q=\big(q\,e^{J(q)}\big)\cdot e^{\tilde{J}(q\,e^{J(q)})} \qquad \text{and} \qquad Q=\big(Q\,e^{\tilde{J}(Q)}\big)\cdot e^{J(Q\,e^{\tilde{J}(Q)})}.$$

In other words, the transformations

$$q\longrightarrow q\,e^{J(q)} \qquad \text{and} \qquad Q\longrightarrow Q\,e^{\tilde{J}(Q)}, \tag{17.5}$$

are mutual inverses.

The maps (17.5) are called *mirror symmetry transformations.*

If $R(w,q)\in\mathbb{Q}(w)[[q]]$ is any power series in q with coefficients in rational functions in w that are regular at $w=0$, denote by

$$[\![R(w,q)]\!]_{w;m}\in\mathbb{Q}[[q]]$$

the coefficient of w^m in the Taylor expansion of $R(w,q)$ at $w=0$:

$$R(w,q)\equiv\sum_{m=0}^{\infty}w^m[\![R(w,q)]\!]_{w^m}.$$

Theorem 17.3. *If N_d is the genus 0 degree d GW-invariant of X_5,*

$$\sum_{d=1}^{\infty}N_dQ^d=-\frac{5}{2}\left[\!\!\left[e^{-J(q)w}\frac{F(q,w)}{I_0(q)}\right]\!\!\right]_{w;3}=-\frac{5}{2}\left[\!\!\left[\ln\left(\frac{F(q,w)}{I_0(q)}\right)\right]\!\!\right]_{w;3}, \tag{17.6}$$

where q and Q are related by the mirror transformations (17.5).

By Exercise 17.2, the right-hand side of (17.6) can be written as a power series in Q; thus, (17.6) specifies all N_d. These are rational numbers. Table 4.1 lists the first 20 numbers n_d defined from N_d by

$$N_d=\sum_{r|d}\frac{n_{d/r}}{r^3} \qquad \forall\ d=1,2,\ldots; \tag{17.7}$$

note that these 20 numbers are *integers*!

Exercise 17.4. Show that the identities (17.7) describe the numbers n_d as functions of the numbers N_d, with various d.

Since X_5 is a Calabi-Yau 3-fold, the *expected* dimension of the space of curves in X_5 is zero. Thus, ideally (i.e. if everything is as expected), the space of genus 0 curves consists of isolated elements and every such curve is smooth. The normal bundle to a smooth rational curve in X_5 is holomorphic, of rank 2, and of degree -2. Any such holomorphic bundle splits as

$$\mathcal{O}(-a)\oplus\mathcal{O}(-b)=\gamma^{\otimes a}\oplus\gamma^{\otimes b}\longrightarrow\mathbb{P}^1,$$

with $a+b=2$. Ideally, $a=b$.

d	n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
11	1017913203569692432490203659468875
12	1512323901934139334751675234074638000
13	2299488568136266648325160104772265542625
14	3565959228158001564810294084668822024070250
15	5624656824668483274179483938371579753751395250
16	9004003639871055462831535610291411200360685606000
17	14602074714589033874568888115959699651605558686799250
18	23954445228532694121482634657728114956109652255216482000
19	39701666985451876233836105884497728824100003703180307231625
20	66408603312404471392397268104340892583652834904833089314920000

Table 4.1: BPS states n_d for a quintic threefold

Exercise 17.5. Use Gromov's Compactness Theorem to show that if every genus 0 curve in X is smooth and its normal bundle splits as $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, then there are finitely many genus 0 curves in each homology class.

If X_5 is ideal as above, the contribution of every genus 0 degree d curve to the Gromov-Witten invariant N_{rd} is $1/r^3$ by the Aspinwall-Morrison formula. Thus, if n_d is the number of (reduced) genus 0 curves of degree d in an ideal quintic 3-fold X_5, then the number N_d is given by (17.7) for all d. In such a case, the numbers n_d defined from N_d by (17.7) are counts of curves in X_5 and are thus nonnegative integers. However, it is known that even a generic quintic 3-fold (zero set in $\mathbb{P}^4$ of a generic degree 5 homogeneous polynomial in 5 variables) is not ideal. Nevertheless, it is still conjectured that the numbers n_d are integers.

Conjecture 17.6. *The rational numbers n_d defined from the Gromov-Witten invariants N_d of a quintic* 3*-fold are integers.*

This conjecture is generalized to all compact Calabi-Yau 3-folds X by viewing d and d/r as elements of $H_2(X;\mathbb{Z})$. Using (17.6) and a simple computer program, Conjecture 17.6 can be confirmed to a very high degree.

In a generic quintic 3-fold, rational curves of degree $d\leq 4$ are smooth and their normal bundles split as $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Thus, the numbers n_d defined by (17.7) with $d\leq 4$ are indeed counts of curves.

The first number in Table 4.1, the number of lines on a general quintic 3-fold, can be confirmed via a straightforward computation on the Grassmannian $\mathbb{G}(2,5)$; see [19, Chapter 7]. A more elaborate Schubert calculus computation [18, Section 3] confirms n_2. A still classical, but even more involved, approach of [8] verifies n_3.

18 Generalization to other hypersurfaces

For the rest of this section we fix positive integers n and a and denote by X a smooth degree a hypersurface in $\mathbb{P}^{n-1}$. If $a>n$ and $d(a-n)>n-5$, the expected dimension of $\overline{\mathfrak{M}}_{0,0}(X,d)$ is negative. Thus, if $a>n$ only finitely many genus 0 GW-invariants of X are nonzero. Our attention will be on the $a\le n$ cases. If $a=n$, X is a Calabi-Yau $(n-2)$-fold; the formula we will obtain for its GW-invariants will look very different from the $a<n$ cases.

Similarly to (17.2), (17.3), and (17.4), let

$$F(w,q)=\sum_{d=0}^{\infty}q^d\frac{\prod_{r=1}^{r=ad}(aw+r)}{\prod_{r=1}^{r=d}(w+r)^n}\in\mathbb{Q}(w)\big[\big[q\big]\big] \tag{18.1}$$

For each $d\in\mathbb{Z}^+$, let

$$\pi\colon\mathcal{L}\equiv\mathcal{O}(a)=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^{n-1}\qquad\text{and}\qquad \begin{aligned}&\tilde{\pi}\colon\mathcal{V}_d\equiv\overline{\mathfrak{M}}_{0,k}(\mathcal{L},d)\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d),\\ &\tilde{\pi}\big([\xi\colon\Sigma\longrightarrow\mathcal{L}]\big)=\big[\pi\circ\xi\colon\Sigma\longrightarrow\mathbb{P}^{n-1}\big].\end{aligned}$$

Exercise 18.1. Show that $\mathcal{V}_d\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ is a vector orbi-bundle of rank $da+1$ and the vector bundle homomorphism

$$\widetilde{\mathrm{ev}}_1\colon\mathcal{V}_d\longrightarrow\mathrm{ev}_1^*\mathcal{L},\qquad \big[\xi\colon\Sigma\longrightarrow\mathcal{L}\big]\longrightarrow\xi(x_1(\Sigma)), \tag{18.2}$$

where $x_1(\Sigma)\in\Sigma$ is the first marked point, is surjective.

It follows that $\mathcal{V}_d'\equiv\ker\widetilde{\mathrm{ev}}_1$ is a vector orbi-bundle and there is a short exact sequence of vector orbi-bundles

$$0\longrightarrow\mathcal{V}_d'\longrightarrow\mathcal{V}_d\longrightarrow\mathrm{ev}_1^*\mathcal{L}\longrightarrow0 \tag{18.3}$$

over $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ for $k\ge1$. With ψ_1 and ev_1 denoting the first Chern class of the cotangent line bundle at the first marked point and the evaluation map on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$, respectively, let

$$Z(H,Q)=1+\sum_{d=1}^{\infty}Q^d\mathrm{ev}_{1*}\bigg(\frac{e(\mathcal{V}_d')}{1-\psi_1}\bigg)\in H^*(\mathbb{P}^{n-1})\big[\big[Q\big]\big]. \tag{18.4}$$

Theorem 18.2. *If n and $a\le n$ are positive integers, X is a smooth degree a hypersurface in $\mathbb{P}^{n-1}$, and Y and Z are given by (18.1) and (18.4), then*

$$\begin{aligned}&Z(H,Q)=\begin{cases}F(H,Q), & \text{if } a<n-1;\\ e^{-a!Q}F(H,Q), & \text{if } a=n-1;\end{cases}\\ &Z(H,Q)=e^{-J(q)H}F(H,q)\big/I_0(q)\quad\text{if}\quad a=n,\end{aligned} \tag{18.5}$$

with Q and q related by the mirror transformation (17.5) in the last case.

Exercise 18.3. Using (7.5), show that

$$Z(H,Q) = 1 + \sum_{d=1}^{\infty} Q^d \bigg(\sum_{m=0}^{n-1} H^m \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} e(\mathcal{V}'_d)\psi_1^{(n-a)d+m-1} \mathrm{ev}_1^* H^{n-1-m} \bigg),$$

where $H = c_1(\gamma^*) \in H^2(\mathbb{P}^{n-1})$ is the Poincare dual to the hyperplane class and $\psi_1^p \equiv 0$ if $p<0$.

Exercise 18.3 implies that $Z(H,Q)$ encodes many Gromov-Witten invariants of X, since

$$\int_{[\overline{\mathfrak{M}}_{0,k}(X,d)]^{vir}} \eta|_{\overline{\mathfrak{M}}_{0,k}(X,d)} = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_d)\eta \qquad \forall\ \eta \in H^*\big(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)\big) \tag{18.6}$$

by the Hyperplane Theorem. Furthermore, if

$$f_1 : \overline{\mathfrak{M}}_{0,1}(X,d) \longrightarrow \overline{\mathfrak{M}}_{0,0}(X,d) \qquad \text{and} \qquad f_2 : \overline{\mathfrak{M}}_{0,2}(X,d) \longrightarrow \overline{\mathfrak{M}}_{0,1}(X,d)$$

are the forgetful morphisms dropping the (first) marked point in the first case and the second marked point in the second case (and contracting any resulting unstable components), then

$$\begin{aligned}
\int_{[\overline{\mathfrak{M}}_{0,1}(X,d)]^{vir}} \psi_1^m f_1^*\eta_0 &= \begin{cases} 0, & \text{if } m=0; \\ -2\int_{[\overline{\mathfrak{M}}_{0,0}(X,d)]^{vir}} \eta_0, & \text{if } m=1; \end{cases} \quad \forall\, \eta_0 \in H^*\big(\overline{\mathfrak{M}}_{0,0}(X,d)\big);\\
\int_{[\overline{\mathfrak{M}}_{0,2}(X,d)]^{vir}} \psi_1^m f_2^*\eta_1 &= \begin{cases} 0, & \text{if } m=0; \\ \int_{[\overline{\mathfrak{M}}_{0,1}(X,d)]^{vir}} \psi_1^{m-1}\eta_1, & \text{if } m\ge 1; \end{cases} \quad \forall\, \eta_1 \in H^*\big(\overline{\mathfrak{M}}_{0,1}(X,d)\big).
\end{aligned} \tag{18.7}$$

Exercise 18.4. Verify (18.7).
`Remark:` For our purposes it is sufficient to verify (18.7) only for the classes

$$\eta_k = \eta'_k|_{\overline{\mathfrak{M}}_{0,k}(X,d)}, \qquad \text{with} \quad \eta'_k \in H^*\big(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)\big).$$

For such classes, (18.7) follows from the analogous identities for $\mathbb{P}^{n-1}$ (in place of X) and (18.6).

Exercise 18.5. Deduce Theorem 17.3 from Theorem 18.2.

19 Equivariant version

Theorem 18.2 will be proved by applying the Atiyah-Bott Localization Theorem; see Theorem 8.4. In fact, we will prove a stronger, equivariant, version of Theorem 18.2.

With $\mathbb{T}$ denoting the n-torus, let

$$\mathcal{H}_{\mathbb{T}}^* \approx \mathbb{C}(\alpha_1,\ldots,\alpha_n) \equiv \mathbb{C}_\alpha \tag{19.1}$$

be the field of fractions of the ring $H_{\mathbb{T}}^*$ as in Section 8; $\alpha_1,\ldots,\alpha_k$ are the generators of $H_{\mathbb{T}}^*$ as in (3.3). With respect to the standard action of $\mathbb{T}$,

$$\mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \equiv H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^* \approx \mathbb{C}_\alpha[x]\big/(x-\alpha_1)\ldots(x-\alpha_n); \tag{19.2}$$

see Proposition 5.3. Via the standard lift to the line bundle $\gamma \longrightarrow \mathbb{P}^{n-1}$, the $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ lifts to a $\mathbb{T}$-action on the line bundle

$$\mathcal{L} \equiv \mathcal{O}(a) \equiv \gamma^{*\otimes a} \longrightarrow \mathbb{P}^{n-1}.$$

Via composition of maps, the $\mathbb{T}$-actions on $\mathbb{P}^{n-1}$ and $\mathcal{L}$ induce actions on $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ and

$$\mathcal{V}_d \equiv \overline{\mathfrak{M}}_{0,2}(\mathcal{L},d) \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d).$$

Since the evaluation map $\widetilde{\mathrm{ev}}_1$ in (18.2) is $\mathbb{T}$-equivariant, the vector orbi-bundle

$$\mathcal{V}_d' \subset \mathcal{V}_d \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$$

is also $\mathbb{T}$-equivariant and thus has a well-defined equivariant Euler class

$$\mathbf{e}(\mathcal{V}_d') \in \mathcal{H}_{\mathbb{T}}^*\big(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\big).$$

Since the forgetful map

$$\overline{\mathfrak{M}}_{0,3}(\mathbb{P}^{n-1},d) \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$$

is $\mathbb{T}$-equivariant, the cotangent line bundle at the first marked point also has an equivariant Euler class

$$\psi_1 \in \mathcal{H}_{\mathbb{T}}^*\big(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\big).$$

Since the morphism $\mathrm{ev}_1\colon \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d) \longrightarrow \mathbb{P}^{n-1}$ is $\mathbb{T}$-equivariant, the power series

$$\mathcal{Z}(x,\hbar,Q) = 1 + \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\bigg(\frac{\mathbf{e}(\mathcal{V}_d')}{\hbar-\psi_1}\bigg) \in \mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1})\big[\big[Q\big]\big] \tag{19.3}$$

is well-defined. Let

$$\mathcal{Y}(x,\hbar,q) = \sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=ad}(ax+r\hbar)}{\prod_{r=1}^{r=d}\prod_{k=1}^{k=n}(x-\alpha_k+r\hbar)}\,. \tag{19.4}$$

This is a power series in q with coefficients in $\mathbb{C}_\alpha[x][[\hbar^{-1}]]$ if $a \le n$. We will also denote by $\mathcal{Y}(x,\hbar,q)$ the image of $\mathcal{Y}(x,\hbar,q)$ in

$$\big\{\mathbb{C}_\alpha[x]/(x-\alpha_1)\dots(x-\alpha_n)\big\}(\hbar)\big[\big[q\big]\big] = \big\{\mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1})\big\}(\hbar)\big[\big[q\big]\big].$$

Let

$$\sigma_1 = \sum_{k=1}^{k=n}\alpha_k\,, \qquad C_1(q) = \bigg(\sum_{d=1}^{\infty} q^d\frac{(ad)!}{(d!)^n}\sum_{r=1}^{r=d}\frac{1}{r}\bigg)\Big/ I_0(q) \quad \text{if} \quad a=n\,.$$

Theorem 19.1. *If n and $a \le n$ are positive integers, X is a smooth degree a hypersurface in $\mathbb{P}^{n-1}$, and $\mathcal{Z}$ and $\mathcal{Y}$ are given by (19.3) and (19.4), then*

$$\begin{aligned} \mathcal{Z}(x,\hbar,Q) &= \begin{cases} \mathcal{Y}(x,\hbar,Q), & \text{if } a < n-1; \\ e^{-a!Q/\hbar}\mathcal{Y}(x,\hbar,Q), & \text{if } a = n-1; \end{cases} \\ \mathcal{Z}(x,\hbar,Q) &= e^{-C_1(q)\sigma_1/\hbar}e^{-J(q)x/\hbar}\mathcal{Y}(x,\hbar,q)\big/I_0(q) \quad \text{if} \quad a=n, \end{aligned} \tag{19.5}$$

with Q and q related by the mirror transformation (17.5) in the last case.

Theorem 18.2 follows immediately from Theorem 19.1 by setting $\hbar=1$ and $\alpha_i=0$.

By (5.8), the power series

$$\mathcal{Z}(x,\hbar,Q), \mathcal{Y}(x,\hbar,Q) \in \big\{\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big\}\big[\big[\hbar^{-1},Q\big]\big]$$

are determined by their values at n points:

$$\mathcal{Z}(x=\alpha_i,\hbar,Q), \mathcal{Y}(x=\alpha_i,\hbar,Q) \in \mathbb{C}_\alpha\big[\big[\hbar^{-1},Q\big]\big] \qquad i=1,2,\ldots,n.$$

Exercise 19.2. Show that for all $i=1,2,\ldots,n$

$$\mathcal{Z}(\alpha_i,\hbar,Q) = 1 + \sum_{d=1}^{\infty} Q^d \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)} \frac{\mathbf{e}(\mathcal{V}'_d)}{\hbar-\psi_1} \mathrm{ev}_1^*\phi_i\,,$$

where ϕ_i is the equivariant Poincare dual of the fixed point $P_i \in \mathbb{P}^{n-1}$ given by (8.7).

The proof of (19.5) consists of showing that

Step 1: $\mathcal{Y}(x,\hbar,Q)$ and $\mathcal{Z}(x,\hbar,Q)$ satisfy the recursion (20.12) on the Q-degree;

Step 2: $\mathcal{Y}(x,\hbar,Q)$ and $\mathcal{Z}(x,\hbar,Q)$ satisfy the self-polynomiality condition (SPC) of Lemma 20.2;

Step 3: the two sides of (19.5), viewed as a powers series in $\hbar^{-1}$, agree mod $\hbar^{-2}$;

Step 4: if $\mathcal{F}(x,\hbar,Q)$ satisfies the recursion and the SPC, so do certain transforms of $\mathcal{F}(x,\hbar,Q)$;

Step 5: if $\mathcal{F}(x,\hbar,Q)$ satisfies the recursion and the SPC and $\mathcal{F}(\alpha_i,\hbar,0) \in \mathbb{C}^*_\alpha$ for every $i=1,2,\ldots,n$, then $\mathcal{F}$ is determined by its "mod $\hbar^{-2}$ part".

For the purposes of these statements, in particular in Steps 3 and 5,

$$\mathcal{F}(x,\hbar,Q), \mathcal{Y}(x,\hbar,Q), \mathcal{Z}(x,\hbar,Q) \in \mathbb{C}_\alpha[x]\big[\big[\hbar^{-1},Q\big]\big] \Big/ \prod_{k=1}^{k=n}(x-\alpha_k).$$

For example, the statement in Step 5 means

$$\begin{aligned} &\mathcal{F}(\alpha_i,\hbar,Q) = \bar{\mathcal{F}}(\alpha_i,\hbar,Q) \quad (\text{mod } \hbar^{-2}) \quad \forall\, i=1,2,\ldots,n \\ &\qquad\qquad \Longrightarrow \qquad \mathcal{F}(\alpha_i,\hbar,Q) = \bar{\mathcal{F}}(\alpha_i,\hbar,Q) \quad \forall\, i=1,2,\ldots,n. \end{aligned}$$

None of the many steps involved in the proof of Theorem 19.1 is very difficult. The proof that $\mathcal{Y}(x,\hbar,Q)$ satisfies the recursion (20.12) and the SPC is a straightforward algebraic computation involving the Residue Theorem on S^2 stated in Exercise 8.12 below; see Lemma 21.2. Proofs of Steps 4 and 5 also consist of fairly simple algebraic computations; see Lemma 21.1 and Proposition 20.5, respectively. The Atiyah-Bott Localization Theorem is used to show that $\mathcal{Z}(x,\hbar,Q)$ satisfies the recursion (20.12) and the SPC. However, overall the proof of Theorem 17.3 involves several key insights, originating in [12] and clarified in [29] (and later in more detail in [15, Part IV]):

- integrating $e(\mathcal{V}_d)$ over $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$, with $k>0$, rather than over $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^{n-1},d)$ as one might expect. This makes it possible to pushforward cohomology classes to $\mathbb{P}^{n-1}$, thus re-formulating the mirror symmetry statement of Theorem 17.3 as a direct relation with the power series (17.2), and to set up a degree recursion for Gromov-Witten invariants;
- augmenting the integrand by the denominator $\hbar-\psi_1$ to beautifully capture the essential nature of the recursion;
- discovering the uniqueness property of Step 5;
- constructing a morphism from $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ to a space with a larger torus action in order to verify that $\mathcal{Z}(x,\hbar,Q)$ satisfies the SPC; see Section 23.

Exercise 19.3. Show that both sides of (19.5) equal 1 modulo $\hbar^{-2}$, in the sense described above.
`Hint:` Use (18.7) and Exercise 19.2.

20 On rigidity of certain polynomial conditions

This section describes the extent of rigidity of double power series that satisfy a certain recursion and a polynomiality condition.

Denote by $\mathbb{Z}^{\geq 0}$ the set of nonnegative integers and by $[n]$, whenever $n\in\mathbb{Z}^{\geq 0}$, the set of positive numbers not exceeding n:

$$\mathbb{Z}^{\geq 0}=\{0,1,2,\ldots,\},\qquad [n]=\{1,2,\ldots,n\}.$$

Whenever f is a function of w (and possibly of other variables) which is holomorphic at $w=0$ (for a dense subspace of the other variables) and $s\in\mathbb{Z}^{\geq 0}$, let $[\![f]\!]_{w;s}$ denote the coefficient of w^s in the power series expansion of f around $w=0$; this is a function of the other variables if there are any. For any ring R, let

$$R[\![\hbar]\!]\equiv R[[\hbar^{-1}]]+R[\hbar]$$

denote the R-algebra of Laurent series in $\hbar^{-1}$. If

$$\mathcal{F}(\hbar,Q)=\sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{\infty}\mathcal{F}_d^{(r)}\hbar^{-r}\bigg)Q^d\in R[\![\hbar]\!]\big[\big[Q\big]\big]$$

for some $\mathcal{F}_d^{(r)}\in R$, we define

$$\mathcal{F}(\hbar,Q)\cong\sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{p-1}\mathcal{F}_d^{(r)}\hbar^{-r}\bigg)Q^d\quad(\text{mod }\hbar^{-p}),$$

i.e. we drop $\hbar^{-p}$ and higher powers of $\hbar^{-1}$, instead of higher powers of $\hbar$.

For any element

$$\mathcal{F}\equiv\mathcal{F}(x,\hbar,Q)\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big],$$

we define

$$\Phi_{\mathcal{F}} \equiv \Phi_{\mathcal{F}}(\hbar, z, Q) \in \mathbb{C}_\alpha\llbracket\hbar\rrbracket\big[\big[z, Q\big]\big] \qquad \text{by}$$

$$\Phi_{\mathcal{F}}(\hbar, z, Q) = \sum_{i=1}^{i=n} \frac{a\alpha_i\, e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i - \alpha_k)} \mathcal{F}\big(\alpha_i, \hbar, Qe^{\hbar z}\big)\mathcal{F}(\alpha_i, -\hbar, Q). \tag{20.1}$$

Lemma 20.1. *For every* $\mathcal{F} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\llbracket\hbar\rrbracket\big[\big[Q\big]\big]$ *such that*

$$\mathcal{F}(x=\alpha_i, \hbar, Q) \in \mathbb{C}_\alpha(\hbar)\big[\big[Q\big]\big] \subset \mathbb{C}_\alpha\llbracket\hbar\rrbracket\big[\big[Q\big]\big] \qquad \forall\, i \in [n],$$

there exists a unique collection

$$\big(E_{\mathcal{F};d} \equiv E_{\mathcal{F};d}(\hbar, \Omega)\big)_{d\in\mathbb{Z}^{\geq 0}} \subset \mathbb{C}_\alpha(\hbar)[\Omega]$$

such that the Ω*-degree of* $E_{\mathcal{F};d}$ *is at most* $(d+1)n-1$ *for every* $d \in \mathbb{Z}^{\geq 0}$ *and*

$$\Phi_{\mathcal{F}}(\hbar, z, Q) = \sum_{d=0}^{\infty} Q^d \bigg(\frac{1}{2\pi\mathfrak{i}} \oint e^{\Omega z} \frac{E_{\mathcal{F};d}(\hbar, \Omega)}{\prod\limits_{r=0}^{r=d}\prod\limits_{k=1}^{k=n}(\Omega - \alpha_k - r\hbar)} \mathrm{d}\Omega\bigg), \tag{20.2}$$

where each path integral is taken over a simple closed loop in $\mathbb{C}$ *enclosing all points* $\Omega = \alpha_k + r\hbar$ *with* $k = 1, \ldots, n$ *and* $r = 0, 1, \ldots, d$*. The equality holds for a dense collection of complex parameters* $\hbar$*.*

Proof. It can be assumed that

$$\alpha_k + r\hbar \neq \alpha_{k'} + r'\hbar \qquad \forall\ k, k' \in [n],\ r, r' \in \mathbb{Z}^{\geq 0},\ (r, k) \neq (r', k').$$

Note that for every $i = 1, \ldots, n$ and $d' = 0, 1, \ldots, d$,

$$\begin{aligned}
&\prod_{r=0}^{r=d'-1}(\alpha_i + d'\hbar - \alpha_i - r\hbar) \prod_{r=d'+1}^{r=d}(\alpha_i + d'\hbar - \alpha_i - r\hbar) \prod_{r=0}^{r=d}\prod_{k\neq i}(\alpha_i + d'\hbar - \alpha_k - r\hbar)\\
&\qquad = d'!\hbar^{d'}(d-d')!(-\hbar)^{d-d'}\bigg(\prod_{r=1}^{r=d'}\prod_{k\neq i}(\alpha_i - \alpha_k + r\hbar)\bigg)\bigg(\prod_{k\neq i}(\alpha_i - \alpha_k)\bigg)\bigg(\prod_{r=1}^{r=d-d'}\prod_{k\neq i}(\alpha_i - \alpha_k - r\hbar)\bigg)\\
&\qquad = \bigg(\prod_{k\neq i}(\alpha_i - \alpha_k)\bigg)\Delta_{d'}(\alpha_i, \hbar)\,\Delta_{d-d'}(\alpha_i, -\hbar),
\end{aligned}$$

where

$$\Delta_d(x, \hbar) \equiv \prod_{r=1}^{r=d}\prod_{k=1}^{k=n}(x - \alpha_k + r\hbar) \qquad \forall\ d \in \mathbb{Z}^{\geq 0}. \tag{20.3}$$

By Cauchy's Formula,

$$\begin{aligned}
&\frac{1}{2\pi\mathfrak{i}} \oint e^{\Omega z} \frac{E_{\mathcal{F};d}(\hbar, \Omega)}{\prod_{r=0}^{r=d}\prod_{k=1}^{k=n}(\Omega - \alpha_k - r\hbar)} \mathrm{d}\Omega\\
&\qquad = \sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n} e^{(\alpha_i + d'\hbar)z} \frac{E_{\mathcal{F};d}(\hbar, \alpha_i + d'\hbar)}{\big(\prod\limits_{k\neq i}(\alpha_i - \alpha_k)\big)\Delta_{d'}(\alpha_i, \hbar)\,\Delta_{d-d'}(\alpha_i, -\hbar)} \qquad\qquad (20.4)\\
&\qquad = \sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n}\bigg(\frac{e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i - \alpha_k)}\bigg)\bigg(\frac{(e^{\hbar z})^{d'}}{\Delta_{d'}(\alpha_i, \hbar)\Delta_{d-d'}(\alpha_i, -\hbar)}\bigg) E_{\mathcal{F};d}(\hbar, \alpha_i + d'\hbar).
\end{aligned}$$

On the other hand, by the assumptions on $\mathcal{F}$,

$$\mathcal{F}(x,\hbar,Q)=\sum_{d=0}^{\infty}\frac{N_{\mathcal{F};d}(x,\hbar)}{\Delta_d(x,\hbar)}Q^d \tag{20.5}$$

for a unique $N_{\mathcal{F};d}\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]$ such that $N_{\mathcal{F};d}(x=\alpha_i,\hbar)\in\mathbb{C}_\alpha(\hbar)$ for every $i\in[n]$. By (20.1) and (20.5),

$$\begin{aligned}\Phi_{\mathcal{F}}(\hbar,z,Q)&=\sum_{d=0}^{\infty}\sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n}\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\bigg(\frac{N_{\mathcal{F};d'}(\alpha_i,\hbar)}{\Delta_{d'}(\alpha_i,\hbar)}\bigg)(Qe^{\hbar z})^{d'}\bigg(\frac{N_{\mathcal{F};d-d'}(\alpha_i,-\hbar)}{\Delta_{d-d'}(\alpha_i,-\hbar)}\bigg)Q^{d-d'}\\&=\sum_{d=0}^{\infty}Q^d\Bigg(\sum_{d'=0}^{d'=d}\sum_{i=1}^{i=n}\frac{e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\bigg(\frac{(e^{\hbar z})^{d'}}{\Delta_{d'}(\alpha_i,\hbar)\Delta_{d-d'}(\alpha_i,-\hbar)}\bigg)\\&\qquad\qquad\times a\alpha_i\,N_{\mathcal{F};d'}(\alpha_i,\hbar)N_{\mathcal{F};d-d'}(\alpha_i,-\hbar)\Bigg).\end{aligned} \tag{20.6}$$

By (20.4) and (20.6), (20.2) is satisfied if and only if

$$E_{\mathcal{F};d}(\hbar,\alpha_i+d'\hbar)=a\alpha_i\,N_{\mathcal{F};d'}(\alpha_i,\hbar)\cdot N_{\mathcal{F};d-d'}(\alpha_i,-\hbar)\quad\forall\, i\in[n],\ d'=0,\ldots,d. \tag{20.7}$$

For a dense collection of complex parameters $\hbar$, there exists a unique polynomial

$$E_{\mathcal{F};d}(\hbar,\Omega)\in\mathbb{C}_\alpha(\hbar)[\Omega]$$

of Ω-degree at most $(d+1)n-1$ that satisfies (20.7). □

Lemma 20.2. *If $\mathcal{F}\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ and $(E_{\mathcal{F};d})_{d\in\mathbb{Z}^{\ge0}}\subset\mathbb{C}_\alpha(\hbar)[\Omega]$ are as in Lemma 20.1, then*

$$\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big]\qquad\Longleftrightarrow\qquad E_{\mathcal{F};d}\in\mathbb{C}_\alpha[\hbar,\Omega]\ \ \forall\, d\in\mathbb{Z}^{\ge0}. \tag{20.8}$$

Proof. By Exercise 8.12,

$$\frac{1}{2\pi\mathfrak{i}}\oint\frac{\Omega^s\mathrm{d}\Omega}{\prod\limits_{r=0}^{r=d}\prod\limits_{k=1}^{k=n}(\Omega-\alpha_k-r\hbar)}=\begin{cases}0, & \text{if } s<(d+1)n-1;\\ 1, & \text{if } s=(d+1)n-1;\\ R^d_{s-(d+1)n+1}(\hbar), & \text{if } s>(d+1)n-1,\end{cases} \tag{20.9}$$

where $R^d_s\in\mathbb{C}_\alpha[\hbar]$ is given by

$$R^d_s(\hbar)=\Bigg[\!\!\Bigg[\prod_{r=0}^{r=d}\prod_{k=1}^{k=n}(1-(\alpha_k+r\hbar)w)^{-1}\Bigg]\!\!\Bigg]_{w;s}\qquad\forall\ s\in\mathbb{Z}^{\ge0}.$$

The path integral in (20.9) is again taken over a simple closed loop enclosing all points $\Omega=\alpha_k+r\hbar$ with $r\le d$. Write

$$\Phi_{\mathcal{F}}(\hbar,z,Q)=\sum_{d=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}F_{d,m}(\hbar)z^mQ^d\,,\qquad E_{\mathcal{F};d}(\hbar,\Omega)=\sum_{s=0}^{(d+1)n-1}f_{d,s}(\hbar)\Omega^s. \tag{20.10}$$

By (20.2), (20.9), and (20.10),

$$
\begin{aligned}
F_{d,m}(\hbar) &= \sum_{s=0}^{(d+1)n-1} \frac{1}{2\pi\mathfrak{i}} \oint \frac{f_{d,s}(\hbar)\,\Omega^{m+s}\mathrm{d}\Omega}{\prod\limits_{r=0}^{r=d}\prod\limits_{k=1}^{k=n}(\Omega-\alpha_k-r\hbar)} \\
&= \sum_{s=\max(0,(d+1)n-1-m)}^{(d+1)n-1} R^d_{m+s-(d+1)n+1}(\hbar)\, f_{d,s}(\hbar).
\end{aligned}
\tag{20.11}
$$

Since $R^d_s \in \mathbb{C}_\alpha[\hbar]$, it follows that $F_{d,m} \in \mathbb{C}_\alpha[\hbar]$ if $f_{d,s} \in \mathbb{C}_\alpha[\hbar]$ for all s. Conversely, since $R^d_0(\hbar)=1$,

$$
F_{d,0},\ldots,F_{d,(d+1)n-1} \in \mathbb{C}_\alpha[\hbar] \qquad \Longrightarrow \qquad f_{d,(d+1)n-1},\ldots,f_{d,0} \in \mathbb{C}_\alpha[\hbar].
$$

These observations imply Lemma 20.2. □

Definition 20.3. *Let $C \equiv (C^j_i(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{C}_\alpha$. A Laurent series $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ is C-recursive if*

$$
\mathcal{F}\big(\alpha_i,(\alpha_i-\alpha_j)/d,Q\big) \in \mathbb{C}_\alpha[[Q]]
$$

is well-defined for all $d\in\mathbb{Z}^+$ and $i,j\in[n]$ and

$$
\mathcal{F}(\alpha_i,\hbar,Q) = \sum_{d=0}^{\infty}\bigg(\sum_{r=-N_d}^{r=N_d}\mathcal{F}^r_{i;d}\hbar^{-r}\bigg)Q^d + \sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C^j_i(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\,\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big) \tag{20.12}
$$

for every $i\in[n]$ and for some $N_d\in\mathbb{Z}$ and $\mathcal{F}^r_{i;d}\in\mathbb{C}_\alpha$.

Exercise 20.4. Suppose $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ satisfies (20.12). Show that

(a) if $\mathcal{F}(\alpha_i,\hbar,0)\in\mathbb{C}_\alpha[\hbar,\hbar^{-1}]$ for all $i\in[n]$, then $\mathcal{F}(\alpha_i,\hbar,Q)\in\mathbb{C}_\alpha(\hbar)\big[\big[Q\big]\big]$ for all $i\in[n]$;

(b) $\mathcal{F}$ is determined by $\mathcal{F}(x,\hbar,0)$ and the collections $\{C^j_i(d)\}$ and $\{\mathcal{F}^r_{i;d}\}$.

Proposition 20.5. *Let $C \equiv (C^j_i(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{C}_\alpha$. If Laurent series $\mathcal{F},\bar{\mathcal{F}}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$ are C-recursive, $\Phi_{\mathcal{F}},\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, and*

$$
\mathcal{F}(x=\alpha_i,\hbar,Q=0) = \bar{\mathcal{F}}(x=\alpha_i,\hbar,Q=0) \in \mathbb{C}^*_\alpha \subset \mathbb{C}_\alpha[\![\hbar]\!] \qquad \forall\, i\in[n],
$$

then $\mathcal{F}\cong\bar{\mathcal{F}}$ (mod $\hbar^{-2}$) if and only $\mathcal{F}=\bar{\mathcal{F}}$.

Proof. Let $\mathcal{F}^r_{i;d},\bar{\mathcal{F}}^r_{i;d}\in\mathbb{C}_\alpha$ be the coefficients in (20.12) corresponding to $\mathcal{F}$ and $\bar{\mathcal{F}}$, respectively,

$$
\delta\mathcal{F}^r_{i;d} = \mathcal{F}^r_{i;d} - \bar{\mathcal{F}}^r_{i;d}, \quad \text{and} \quad \delta\mathcal{F}(x,\hbar,Q) = \mathcal{F}(x,\hbar,Q) - \bar{\mathcal{F}}(x,\hbar,Q).
$$

We show by induction on d that $\delta\mathcal{F}^r_{i;d}=0$ for all i and r. For each $i\in[n]$, let

$$
f_i = \mathcal{F}(\alpha_i,\hbar,Q=0) = \bar{\mathcal{F}}(\alpha_i,\hbar,Q=0) \in \mathbb{C}^*_\alpha.
$$

Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ are C-recursive and

$$\mathcal{F}(x,\alpha_i,0), \bar{\mathcal{F}}(x,\alpha_i,0) \in \mathbb{C}_\alpha(\hbar) \qquad \forall\, i\in[n],$$

$\mathcal{F}$ and $\bar{\mathcal{F}}$ satisfy the assumptions of Lemmas 20.1 and 20.2. Let

$$N_{\mathcal{F};d}, N_{\bar{\mathcal{F}};d} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]$$

be as in the proof of Lemma 20.1 and

$$\delta N_d = N_{\mathcal{F};d} - N_{\bar{\mathcal{F}};d}.$$

Since $\mathcal{F}(\alpha_i,\hbar,0)=\bar{\mathcal{F}}(\alpha_i,\hbar,0)$, $\mathcal{F}^r_{i;0}=\bar{\mathcal{F}}^r_{i;0}$ for all i and r. Suppose $d>0$ and we have shown that

$$\mathcal{F}^r_{i;d'} = \bar{\mathcal{F}}^r_{i;d'} \qquad \forall\, d'=0,1,\ldots,d-1,\ i\in[n],\ r. \tag{20.13}$$

Then, by (20.12),

$$\mathcal{F}(\alpha_i,\hbar,Q) \equiv \bar{\mathcal{F}}(\alpha_i,\hbar,Q) \pmod{Q^d}\ \ \forall\, i\in[n], \quad N_{\mathcal{F};d'} = N_{\bar{\mathcal{F}};d'}\ \ \forall\, d'=0,1,\ldots,d-1. \tag{20.14}$$

Since $\mathcal{F}$ and $\bar{\mathcal{F}}$ agree modulo $\hbar^{-2}$, by (20.12) and the first equation in (20.14)

$$\delta\mathcal{F}(\alpha_i,\hbar,Q) \equiv Q^d \sum_{r=2}^{r=N} \delta\mathcal{F}^r_{i;d}\hbar^{-r} \pmod{Q^{d+1}} \qquad \forall\, i\in[n] \tag{20.15}$$

for some $N\in\mathbb{Z}^+$ and $\delta\mathcal{F}^r_{i;d}\in\mathbb{C}_\alpha$. Then,

$$N_{\mathcal{F};0}(\alpha_i,\hbar) = N_{\bar{\mathcal{F}};0}(\alpha_i,\hbar) = f_i\,, \quad \delta N_{d'}(\alpha_i,\hbar) = \begin{cases} 0, & \text{if } d'<d;\\ \Delta_d(\alpha_i,\hbar)\sum\limits_{r=2}^{r=N}\delta\mathcal{F}^r_{i;d}\hbar^{-r}, & \text{if } d'=d, \end{cases} \tag{20.16}$$

by (20.5), (20.14), and (20.15). Let $\delta E_d = E_{\mathcal{F};d} - E_{\bar{\mathcal{F}};d}$. Since

$$\delta E_d(\hbar,\alpha_i+d'\hbar) = 0 \ \ \forall\, d'=1,\ldots,d-1,\ i\in[n]$$

by (20.7) and (20.16) and $\delta E_d\in\mathbb{C}_\alpha[\hbar,\Omega]$ by Lemma 20.2,

$$\delta E_d(\hbar,\Omega) = \bigg(\prod_{d'=1}^{d-1}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-d'\hbar\big)\bigg)\cdot R_d(\hbar,\Omega)$$

for some $R_d\in\mathbb{C}_\alpha[\hbar,\Omega]$. Thus,

$$\delta E_d(\hbar,\alpha_i+d\hbar) = \bigg(\prod_{d'=1}^{d-1}\prod_{k=1}^{k=n}\big(\alpha_i+d\hbar-\alpha_k-d'\hbar\big)\bigg)\cdot R_d(\hbar,\alpha_i+d\hbar) = \hbar^{d-1}\tilde{R}_d(\hbar) \tag{20.17}$$

for some $\tilde{R}_d\in\mathbb{C}_\alpha[\hbar]$. On the other hand, by (20.7) and (20.16)

$$\begin{aligned} \delta E_d(\hbar,\alpha_i+d\hbar) &= a\alpha_i\,\delta N_d(\alpha_i,\hbar)\cdot f_i = a\alpha_i f_i\cdot\bigg(d!\hbar^d\prod_{r=1}^{r=d}\prod_{k\neq i}(\alpha_i-\alpha_k+r\hbar)\bigg)\sum_{r=2}^{r=N}\delta\mathcal{F}^r_{i;d}\hbar^{-r}\\ &= a\alpha_i f_i\cdot\bigg(d!\prod_{r=1}^{r=d}\prod_{k\neq i}(\alpha_i-\alpha_k+r\hbar)\bigg)\sum_{r=2}^{r=N}\delta\mathcal{F}^r_{i;d}\hbar^{d-r}. \end{aligned} \tag{20.18}$$

By (20.17) and (20.18),

$$\delta\mathcal{F}^r_{i;d} = 0 \quad \forall\, r=2,\ldots,N,\ i\in[n].$$

Along with (20.15), this implies that (20.13) holds with d replaced by $d+1$. $\square$

21 Other algebraic observations

In this section we show that the recursion and polynomiality conditions of Section 20 are preserved under certain transformations and are satisfied by the function $\mathcal{Y}$ defined in (19.4). For $i,j\in[n]$ with $i\neq j$ and $d\in\mathbb{Z}^+$, let

$$\mathfrak{C}^j_i(d) = \frac{\prod_{r=1}^{ad}\big(a\alpha_i + r(\alpha_j-\alpha_i)/d\big)}{d\prod\limits_{\substack{r=1\\(r,k)\neq(d,j)}}^{r=d}\prod\limits_{k=1}^{k=n}\big(\alpha_i-\alpha_k + r(\alpha_j-\alpha_i)/d\big)} \in \mathbb{C}_\alpha. \tag{21.1}$$

Lemma 21.1. *Suppose $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][[Q]]$ is C-recursive and $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$. Then,*

(i) if $f\in\mathbb{C}_\alpha[[Q]]$, then $f\mathcal{Z}$ is C-recursive and $\Phi_{f\mathcal{Z}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$;

(ii) if $f\in Q\cdot\mathbb{C}_\alpha[[Q]]$, then $\bar{\mathcal{F}}\equiv e^{f/\hbar}\mathcal{F}$ is C-recursive and $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$;

(iii) if $g\in Q\cdot\mathbb{C}[[Q]]$ and

$$\bar{\mathcal{F}}(x,\hbar,Q) = e^{g(Q)x/\hbar}\mathcal{F}\big(x,\hbar,Qe^{g(Q)}\big),$$

then $\bar{\mathcal{F}}$ is C-recursive and $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[Q,z]]$.

Proof. (i) Since $\mathcal{F}$ is C-recursive and the multiplication by f preserves the structure of each of the terms in (20.12), $f\mathcal{F}$ is also C-recursive. Since

$$\Phi_{f\mathcal{F}}(\hbar,z,Q) = f\big(Qe^{\hbar z}\big)f(Q)\Phi_{\mathcal{F}}(\hbar,z,Q)$$

and $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, $\Phi_{f\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$.

(ii) Since the coefficient of Q^0 in f is 0, the multiplication by $e^{f(Q)/\hbar}$ preserves the structure of the first term on the right-hand side of (20.12). The (d,j)-summand in the last term becomes

$$e^{f(Q)/\hbar}\frac{C^j_i(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big) = \frac{C^j_i(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\bar{\mathcal{F}}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big)$$
$$+\Big(e^{f(Q)/\hbar}-e^{f(Q)/((\alpha_j-\alpha_i)/d)}\Big)\frac{C^j_i(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big).$$

Since $\mathcal{F}$ is C-recursive and

$$\frac{e^{f(Q)/\hbar}-e^{f(Q)/((\alpha_j-\alpha_i)/d)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}} \in \mathbb{C}_\alpha[\hbar,\hbar^{-1}]\big[[Q]\big],$$

it follows that $\bar{\mathcal{F}}$ is C-recursive as well. On the other hand,

$$\Phi_{\bar{\mathcal{F}}}(\hbar,z,Q) = e^{(f(Qe^{\hbar z})-f(Q))/\hbar}\Phi_{\mathcal{F}}(\hbar,z,Q). \tag{21.2}$$

Since $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ and

$$\big(f(Qe^{\hbar z})-f(Q)\big)/\hbar\in\mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big],$$

(21.2) implies that $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ as well.

(iii) Since the coefficient of Q^0 in g is 0, the operation of replacing Q with $Qe^{g(Q)}$ followed by multiplication by $e^{\alpha_i g(Q)/\hbar}$ preserves the structure of the first term on the right-hand side of (20.12). The (d,j)-summand in the last term becomes

$$\begin{split} e^{\alpha_i g(Q)/\hbar}\frac{C_i^j(d)Q^d e^{dg(Q)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Qe^{g(Q)}\big) &= \frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\bar{\mathcal{F}}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Q\big)\\ &+\Big(e^{(\alpha_i/\hbar+d)g(Q)}-e^{(\alpha_j/((\alpha_j-\alpha_i)/d))g(Q)}\Big)\frac{C_i^j(d)Q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{F}\big(\alpha_j,(\alpha_j-\alpha_i)/d,Qe^{g(Q)}\big). \end{split}$$

Since $\mathcal{F}$ is C-recursive and

$$\frac{e^{(\alpha_i/\hbar+d)g(Q)}-e^{(\alpha_j/((\alpha_j-\alpha_i)/d))g(Q)}}{\hbar-\frac{\alpha_j-\alpha_i}{d}}=\frac{d}{(\alpha_i+d\hbar)-\alpha_j}e^{g(Q)\frac{dz}{z-\alpha_i}}\Big|_{z=\alpha_j}^{z=\alpha_i+d\hbar}\in\mathbb{C}_\alpha[\hbar,\hbar^{-1}]\big[\big[Q\big]\big],$$

it follows that $\bar{\mathcal{F}}$ is C-recursive as well. On the other hand,

$$\Phi_{\bar{\mathcal{F}}}(\hbar,z,Q)=\Phi_{\mathcal{F}}\big(\hbar,\tilde{z},Qe^{g(Q)}\big),\qquad\text{where}\quad \tilde{z}=z+\frac{g(Qe^{\hbar z})-g(Q)}{\hbar}\in\mathbb{C}[\hbar]\big[\big[z,Q\big]\big]. \tag{21.3}$$

Since $\Phi_{\mathcal{F}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$, (21.3) implies that $\Phi_{\bar{\mathcal{F}}}\in\mathbb{C}_\alpha[\hbar][[z,Q]]$ as well. □

Lemma 21.2. *The function $\mathcal{Y}$ defined by (19.4) satisfies the C-recursivity condition of Definition 20.3 with the collection of coefficients given by (21.1) and $\Phi_{\mathcal{Y}}\in\mathbb{C}_\alpha[\hbar][[z,q]]$.*

Proof. (1) In this argument, we view $\mathcal{Y}$ as an element of $\mathbb{C}_\alpha(x,\hbar)[[q]]$. By (19.4) and (21.1),

$$\frac{C_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{Y}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big)=\mathop{\mathfrak{R}}_{z=\frac{\alpha_j-\alpha_i}{d}}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}.$$

Thus, by the Residue Theorem on S^2,

$$\begin{split}\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{Y}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big)&=-\mathop{\mathfrak{R}}_{z=\hbar,0,\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}\\ &=\mathcal{Y}(\alpha_i,\hbar,q)-\mathop{\mathfrak{R}}_{z=0,\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}.\end{split}\tag{21.4}$$

On the other hand,

$$\mathop{\mathfrak{R}}_{z=\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}(\alpha_i,z,q)\Big\}=\begin{cases}1, & \text{if } a<n;\\ \sum\limits_{d=0}^{\infty}q^d\frac{(nd)!}{(d!)^n}, & \text{if } a=n;\end{cases}$$

$$\mathop{\mathfrak{R}}_{z=0}\Big\{\frac{1}{\hbar-z}[\![\mathcal{Y}_d(\alpha_i,z,q)]\!]_{q;d}\Big\}=\left[\!\!\left[\frac{1}{\hbar-z}\frac{\prod\limits_{r=1}^{r=ad}(a\alpha_i+rz)}{d!\prod\limits_{r=1}^{r=d}\prod\limits_{k\neq i}(\alpha_i-\alpha_k+rz)}\right]\!\!\right]_{z;d-1}\in\mathbb{Q}_\alpha[\hbar^{-1}].$$

Thus, (21.4) implies that $\mathcal{Y}$ satisfies the recursion (20.12).

(2) In this argument, we view $\mathcal{Y}$ as an element of $\mathbb{C}_\alpha[x][[\hbar^{-1},q]]$; in particular,

$$\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,qe^{\hbar z}\big)\mathcal{Y}(x,-\hbar,q)$$

viewed as a function of x has residues only at $x=\alpha_i$ with $i\in[n]$ and $x=\infty$. By (19.4),

$$\frac{a\alpha_i\,e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Y}\big(\hbar,\alpha_i,qe^{\hbar z}\big)\mathcal{Y}(-\hbar,\alpha_i,q)=\mathop{\mathfrak{R}}_{x=\alpha_i}\left\{\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(\hbar,x,qe^{\hbar z}\big)\mathcal{Y}(-\hbar,x,q)\right\}.$$

Thus, by Exercise 8.12,

$$\begin{aligned}\Phi_{\mathcal{Y}}(\hbar,z,q)&=-\mathop{\mathfrak{R}}_{x=\infty}\left\{\frac{ax\,e^{xz}}{\prod\limits_{k=1}^{k=n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,qe^{\hbar z}\big)\mathcal{Y}(x,-\hbar,q)\right\}\\&=\sum_{d_1,d_2=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^{n-2+p+(n-a)(d_1+d_2)}}{(n-2+p+(n-a)(d_1+d_2))!}q^{d_1+d_2}e^{d_1\hbar z}\\&\qquad\left[\!\!\left[\frac{1}{\prod\limits_{k\neq i}(1-\alpha_k w)}\frac{\prod\limits_{r=1}^{r=ad_1}(a+r\hbar w)}{\prod\limits_{r=1}^{r=d_1}\prod\limits_{k=1}^{k=n}(1-(\alpha_k-r\hbar)w)}\cdot\frac{\prod\limits_{r=1}^{r=ad_2}(a-r\hbar w)}{\prod\limits_{r=1}^{r=d_2}\prod\limits_{k=1}^{k=n}(1-(\alpha_k+r\hbar)w)}\right]\!\!\right]_{w;p}.\end{aligned}$$

The (d_1,d_2,p)-th summand above is $q^{d_1+d_2}$ times an element of $\mathbb{C}_\alpha[\hbar]\big[\big[z\big]\big]$. Thus, $\Phi_{\mathcal{Y}}\in\mathbb{C}_\alpha[\hbar][[z,q]]$. □

22 Proof of the recursion property for GW-invariants

As described in detail in [15, Section 27.3], the fixed loci of the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$ are indexed by decorated graphs that have no loops. A graph consists of a set Ver of vertices and a collection Edg of edges, i.e. of two-element subsets of Ver. A loop in a graph (Ver, Edg) is a subset of Edg of the form

$$\big\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_N,v_1\}\big\},\qquad v_1,\ldots,v_N\in\mathrm{Ver},\ N\ge 1.$$

Neither of the three graphs in Figure 22.1 has a loop. A decorated graph with two marked points is a tuple

$$\Gamma=\big(\mathrm{Ver},\mathrm{Edg};\mu,\mathfrak{d},\eta\big),\tag{22.1}$$

where (Ver, Edg) is a graph and

$$\mu\colon\mathrm{Ver}\longrightarrow[n],\qquad\mathfrak{d}\colon\mathrm{Edg}\longrightarrow\mathbb{Z}^+,\quad\text{and}\quad\eta\colon\{1,2\}\longrightarrow\mathrm{Ver}$$

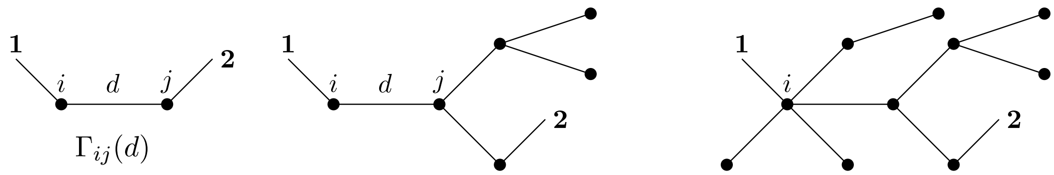


Figure 22.1: Two graphs of type $A_i(j;d)\subset A_i$ and a graph of type B_i

are maps such that

$$\mu(v_1) \neq \mu(v_2) \qquad \text{if} \quad \{v_1, v_2\} \in \text{Edg}. \tag{22.2}$$

In Figure 22.1, the values of the map μ on some of the vertices are indicated by letters next to those vertices. Similarly, the value of the map $\mathfrak{d}$ on one of the edges is indicated by a letter next to the edge. The two elements of the set $\{1, 2\}$ are shown in bold face. They are linked by line segments to their images under η. By (22.2), no two consecutive vertex labels are the same and thus $j \neq i$.

The fixed locus $\mathcal{Z}_\Gamma$ of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d)$ corresponding to a decorated graph Γ consists of the stable maps f from a genus-zero nodal curve $\mathcal{C}_f$ with 2 marked points into $\mathbb{P}^{n-1}$ that satisfy the following conditions. The components of $\mathcal{C}_f$ on which the map f is not constant are rational and correspond to the edges of Γ. Furthermore, if $e = \{v_1, v_2\}$ is an edge, the restriction of f to the component $\mathcal{C}_{f,e}$ corresponding to e is a degree-$\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}^1_{\mu(v_1),\mu(v_2)} \subset \mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. The map $f|_{\mathcal{C}_{f,e}}$ is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, $f|_{\mathcal{C}_{f,e}}$ is unique up to isomorphism. The remaining, contracted, components of $\mathcal{C}_f$ are indexed by the vertices $v \in \text{Ver}$ such that

$$\text{val}(v) \equiv \big|\{v' \in \text{Ver}\colon \{v, v'\} \in \text{Edg}\}\big| + \big|\{i \in \{1, 2\}\colon \eta(i) = v\}\big| \geq 3.$$

The map f takes such a component $\mathcal{C}_{f,v}$ to the fixed point $\mu(v)$. Thus,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \prod_{v \in \text{Ver}} \overline{\mathcal{M}}_{0,\text{val}(v)}, \tag{22.3}$$

where $\overline{\mathcal{M}}_{0,l}$ denotes the moduli space of stable genus-zero curves with l marked points. For the purposes of this definition, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are one-point spaces. For example, in the case of the last diagram in Figure 22.1,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{M}}_{0,3}^2 \times \overline{\mathcal{M}}_{0,2}^2 \times \overline{\mathcal{M}}_{0,1}^5 \approx \overline{\mathcal{M}}_{0,5}$$

is a fixed locus[1] in $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d)$ for some $d \geq 9$.

[1] after dividing by an appropriate automorphism group $\mathbb{A}_\Gamma$ as in [15, Section 27.3]; in what follows $\int_{\mathcal{Z}_\Gamma}$ will denote integration over the orbifold $\mathcal{Z}_\Gamma/\mathbb{A}_\Gamma$

We will show that the function $\mathcal{Z}(x,\hbar,Q)$ defined in (19.3) is C-recursive in the sense of Definition 20.3 with the collection of coefficients given by (21.1) by determining the contribution to

$$\mathcal{Z}(\alpha_i,\hbar,Q)=1+\sum_{d=1}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{e(\mathcal{V}_d')}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\,, \tag{22.4}$$

from the $\mathbb{T}$-fixed loci of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)$, with $d\geq 1$. Suppose Γ is a decorated graph as in (22.1) that contributes to (22.4), in the sense of the localization formula (8.2). By (8.7) and (5.7),

$$\mathrm{ev}_1^*\phi_i\big|_{\mathcal{Z}_\Gamma}=\prod_{k\neq i}\big(\alpha_{\mu(\eta(1))}-\alpha_k\big)=\delta_{i,\mu(\eta(1))}\prod_{k\neq i}(\alpha_i-\alpha_k),$$

where $\delta_{i,\mu(\eta(1))}$ is the Kronecker delta function. Thus, by (8.2), Γ does not contribute to $\mathcal{Z}(\hbar,\alpha_i,Q)$ unless $\mu(\eta(1))=i$, i.e. the marked point 1 of the map is taken to the point $P_i\in\mathbb{P}^{n-1}$. There are two types of graphs that do (or may) contribute to (22.4); they will be called A_i and B_i-types. In a graph of the A_i-type, the marked point 1 is attached to a vertex $v_0\in\mathrm{Ver}$ of valence two which is labeled i. In a graph of the B_i-type, the marked point 1 is attached to a vertex v_0 of valence at least 3, which is still labeled i. Examples of the two types are depicted in Figure 22.1.

Suppose Γ is a graph of type B_i and

$$\mathcal{Z}_\Gamma\subset\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d),$$

so that Γ contributes to the coefficient of Q^d in (22.4). In this case, the restriction of ψ_1 to $\mathcal{Z}_\Gamma$ is the pull-back of a ψ-class from the component $\overline{\mathcal{M}}_{0,\mathrm{val}(v_0)}$ in the decomposition (22.3). Since the $\mathbb{T}$-action on the corresponding tautological line bundle is trivial,

$$\psi_1^k\big|_{\mathcal{Z}_\Gamma}=0\qquad\forall\ k\geq d>\mathrm{val}(v_0)-3.$$

Thus, Γ contributes a polynomial in $\hbar^{-1}$, of degree at most d, to the coefficient of Q^d in (22.4). Therefore, the contributions of the loci of type B_i to (22.4) are accounted for by the middle term in (20.12).

A graph Γ as in (22.1) of type A_i has a unique vertex v joined to v_0. Denote by $A_i(j;d_0)$ the set of all graphs Γ of type A_i such that $\mu(v)=j$ and $\mathfrak{d}(\{v_0,v\})=d_0$, i.e. the unique vertex v of Γ joined to v_0 is mapped to $P_j\in\mathbb{P}^{n-1}$ and the edge $\{v_0,v\}$ corresponds to the d_0-fold cover of $\mathbb{P}^1_{ij}$ branched only over P_i and P_j. By (22.2),

$$A_i=\bigcup_{d_0=1}^{\infty}\bigcup_{j\neq i}A_i(j;d_0). \tag{22.5}$$

Suppose $\Gamma\in A_i(j;d_0)$ and v is the unique vertex joined to v_0 by an edge. We break Γ at v into two graphs:

(i) Γ_0 consisting of the vertices v_0 and v, the edge $\{v_0,v\}$, and marked points 1 and 2 attached to v_0 and v, respectively;

(ii) Γ_c consisting of all vertices and edges of Γ, other than the vertex v_0 and the edge $\{v_0,v\}$, with a new marked point attached to v;

,

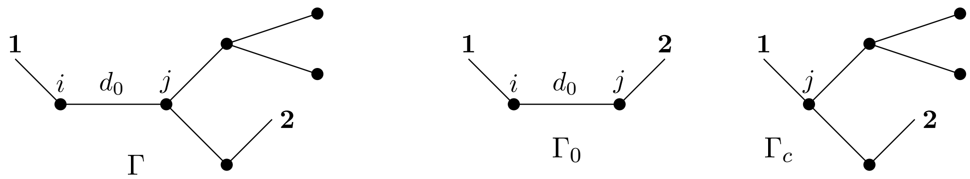


Figure 22.2: A graph of type $A_i^*(j;d_0)$ and its two subgraphs

see Figure 22.2. Let d_c denote the degree of Γ_c, i.e. the sum of all edge labels. By (22.3),

$$\mathcal{Z}_\Gamma \approx \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_c}. \tag{22.6}$$

Denote by π_0 and π_c the two component projection maps.

By [15, Section 27.4],

$$\begin{aligned} \mathcal{V}'_{d_0+d_c}\big|_{\mathcal{Z}_\Gamma} &= \pi_0^*\mathcal{V}'_{d_0} \oplus \pi_c^*\mathcal{V}'_{d_c}, \\ \frac{\mathcal{N}\mathcal{Z}_\Gamma}{T_{P_i}\mathbb{P}^{n-1}} &= \pi_0^*\biggl(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_0}}{T_{P_i}\mathbb{P}^{n-1}}\biggr) \oplus \pi_c^*\biggl(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_c}}{T_{P_j}\mathbb{P}^{n-1}}\biggr) \oplus \pi_0^*L_2 \otimes \pi_c^*L_1, \end{aligned} \tag{22.7}$$

where $L_2 \longrightarrow \mathcal{Z}_{\Gamma_0}$ and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_c}$ are the tautological tangent line bundles. Thus, by (5.10),

$$\begin{aligned} \frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma} &= \pi_0^*\biggl(\frac{\mathbf{e}(\mathcal{V}'_{d_0})}{\hbar-\psi_1}\biggr) \cdot \pi_c^*\Bigl(\mathbf{e}(\mathcal{V}'_{d_c})\Bigr), \\ \frac{\mathrm{ev}_1^*\phi_i|_{\mathcal{Z}_\Gamma}}{\mathbf{e}(N\mathcal{Z}_\Gamma)} &= \pi_0^*\biggl(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_0})}\biggr) \cdot \pi_c^*\biggl(\frac{\mathrm{ev}_1^*\phi_j}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_c})}\biggr) \cdot \frac{1}{\pi_0^*c_1(L_2) - \pi_c^*\psi_1}. \end{aligned} \tag{22.8}$$

By [15, Sections 27.1, 27.2],

$$\begin{aligned} \mathbf{e}(\mathcal{V}'_{d_0})\big|_{\mathcal{Z}_{\Gamma_0}} &= \prod_{r=1}^{ad_0}\Bigl(a\alpha_i + r\frac{\alpha_j-\alpha_i}{d_0}\Bigr), \qquad \psi_1\big|_{\mathcal{Z}_{\Gamma_0}} = c_1(L_2) = \frac{\alpha_j-\alpha_i}{d_0}, \\ \mathbf{e}(N\mathcal{Z}_{\Gamma_0}) &= (-1)^{d_0}\prod_{r=1}^{r=d_0}\Bigl(r\frac{\alpha_j-\alpha_i}{d_0}\Bigr)^2 \prod_{r=0}^{r=d_0}\prod_{k\neq i,j}\Bigl(\alpha_i-\alpha_k+r\frac{\alpha_j-\alpha_i}{d_0}\Bigr). \end{aligned} \tag{22.9}$$

Thus, using (5.10) and taking into the account the automorphism group $\mathbb{A}_{\Gamma_0} = \mathbb{Z}_{d_0}$ of $\mathcal{Z}_{\Gamma_0}$, we obtain

$$\int_{\mathcal{Z}_{\Gamma_0}} \frac{\mathbf{e}(\mathcal{V}'_{d_0})\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(N\mathcal{Z}_{\Gamma_0})} = \frac{\mathfrak{C}_i^j(d_0)}{\hbar - \frac{\alpha_j-\alpha_i}{d_0}}. \tag{22.10}$$

By (22.6), (22.9), and (22.10), the contribution of Γ to (22.4) is

$$\begin{aligned} &Q^{d_0+d_c}\int_{\mathcal{Z}_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma} \frac{1}{\mathbf{e}(N\mathcal{Z}_\Gamma)} \\ &\qquad = \frac{\mathfrak{C}_i^j(d_0)Q^{d_0}}{\hbar - \frac{\alpha_j-\alpha_i}{d_0}} \cdot \Biggl(\biggl\{Q^{d_c}\int_{\mathcal{Z}_{\Gamma_c}} \frac{\mathbf{e}(\mathcal{V}'_{d_c})\mathrm{ev}_1^*\phi_j}{\hbar-\psi_1}\frac{1}{\mathbf{e}(N\mathcal{Z}_{\Gamma_c})}\biggr\}\Big|_{\hbar=\frac{\alpha_j-\alpha_i}{d_0}}\Biggr). \end{aligned} \tag{22.11}$$

We next sum (22.11) over $\Gamma\in A_i(j;d_0)$. This is the same as summing the expression in the curly brackets over all 2-pointed graphs with the marked point 1 attached to a vertex v labeled j, i.e. all graphs of types A_j and B_j. By the localization formula (8.2), the sum of the terms in the curly brackets over all such graphs Γ_c is $\mathcal{Z}(Q,\alpha_j,\hbar)$. Thus,

$$\sum_{\Gamma\in A_i(j;d_0)} Q^{d_0+d_c}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}'_{d_0+d_c})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_\Gamma}\frac{1}{\mathbf{e}(N\mathcal{Z}_\Gamma)}=\frac{\mathfrak{C}_i^j(d_0)Q^{d_0}}{\hbar-\frac{\alpha_j-\alpha_i}{d_0}}\cdot\mathcal{Z}\big(\alpha_j,(\alpha_j-\alpha_i)/d_0,Q\big). \tag{22.12}$$

We conclude that $\mathcal{Z}(x,\hbar,Q)$ is $\mathfrak{C}$-recursive in the sense of Definition 20.3:

- the middle term in (20.12) consists of the contributions from the graphs of type B_i;
- the (d_0,j)-summand in (20.12) consists of the contributions from the graphs of type $A_i(j;d_0)$.

23 Proof of the polynomiality property for GW-invariants

In this section we use Lemma 23.1, which is proved in the next section, to show that the function $\mathcal{Z}(x,\hbar,Q)$ defined in (19.3) satisfies the polynomiality property of Lemma 20.2. The argument presented here is a more detailed version of Section 30.2 of [15] that describes what is perhaps the most unexpected idea in [12].

We will denote the weight of the standard action of the one-torus $\mathbb{T}^1$ on $\mathbb{C}$ by $\hbar$. Let $\widetilde{\mathbb{T}}=\mathbb{T}^1\times\mathbb{T}$. By (3.3),

$$H^*_{\mathbb{T}^1}\approx\mathbb{C}[\hbar],\quad H^*_{\widetilde{\mathbb{T}}}\approx\mathbb{C}[\hbar,\alpha_1,\ldots,\alpha_n]\qquad\Longrightarrow\qquad\mathcal{H}^*_{\widetilde{\mathbb{T}}}\approx\mathbb{C}_\alpha(\hbar).$$

Throughout this section, $V=\mathbb{C}\oplus\mathbb{C}$ will denote the representation of $\mathbb{T}^1$ with the weights 0 and $-\hbar$. The induced action on $\mathbb{P}V$ has two fixed points:

$$q_1\equiv[1,0],\qquad q_2\equiv[0,1].$$

Let $\gamma_1\longrightarrow\mathbb{P}V$ be the tautological line bundle. Then,

$$\mathbf{e}(\gamma_1^*)\big|_{q_1}=0,\quad\mathbf{e}(\gamma_1^*)\big|_{q_2}=-\hbar,\quad\mathbf{e}(T_{q_1}\mathbb{P}V)=\hbar,\quad\mathbf{e}(T_{q_2}\mathbb{P}V)=-\hbar. \tag{23.1}$$

For each $d\in\mathbb{Z}^{\ge0}$, the action of $\widetilde{\mathbb{T}}$ on $\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ induces an action on

$$\overline{\mathfrak{X}}_d\equiv\mathbb{P}\big(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*\big).$$

It has $(d+1)n$ fixed points:

$$P_i(r)\equiv\big[\tilde{P}_i\otimes u^{d-r}v^r\big],\qquad i\in[n],\ r\in\{0\}\cup[d],$$

if (u,v) are the standard coordinates on V and $\tilde{P}_i\in\mathbb{C}^n$ is the i-th coordinate vector (so that $[\tilde{P}_i]=P_i\in\mathbb{P}^{n-1}$). Let

$$\Omega\equiv\mathbf{e}(\gamma^*)\in H^*_{\widetilde{\mathbb{T}}}\big(\overline{\mathfrak{X}}_d\big)$$

denote the equivariant hyperplane class.

For all $i\in[n]$ and $r\in\{0\}\cup[d]$,

$$\Omega|_{P_i(r)}=\alpha_i+r\hbar, \qquad \mathbf{e}(T_{P_i(r)}\overline{\mathfrak{X}}_d)=\left\{\prod_{\substack{s=0\\(s,k)\neq(r,i)}}^{s=d}\prod_{k=1}^{k=n}(\Omega-\alpha_k-s\hbar)\right\}\Big|_{\Omega=\alpha_i+r\hbar}.^{2} \tag{23.2}$$

Since

$$B_{\widetilde{\mathbb{T}}}\overline{\mathfrak{X}}_d=\mathbb{P}\big(B_{\widetilde{\mathbb{T}}}(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*)\big)\longrightarrow B\widetilde{\mathbb{T}} \qquad \text{and}$$
$$c\big(B_{\widetilde{\mathbb{T}}}(\mathbb{C}^n\otimes\mathrm{Sym}^dV^*)\big)=\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(1-(\alpha_k+s\hbar)\big)\in H^*\big(B\widetilde{\mathbb{T}}\big),^{3}$$

the $\widetilde{\mathbb{T}}$-equivariant cohomology of $\overline{\mathfrak{X}}_d$ is given by

$$\begin{aligned}H^*_{\widetilde{\mathbb{T}}}\big(\overline{\mathfrak{X}}_d\big)\equiv H^*\big(B_{\widetilde{\mathbb{T}}}\overline{\mathfrak{X}}_d\big)&=H^*\big(B\widetilde{\mathbb{T}}\big)[\Omega]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-(\alpha_k+s\hbar)\big)\\&\approx\mathbb{Q}\big[\Omega,\hbar,\alpha_1,\ldots,\alpha\big]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-s\hbar\big)\\&\subset\mathbb{Q}_\alpha[\hbar,\Omega]\Big/\prod_{s=0}^{s=d}\prod_{k=1}^{k=n}\big(\Omega-\alpha_k-s\hbar\big).\end{aligned}\tag{23.3}$$

There is a natural $\widetilde{\mathbb{T}}$-equivariant morphism

$$\Theta\colon\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)\longrightarrow\overline{\mathfrak{X}}_d.$$

A general element b of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$ determines a map

$$(f,g)\colon\mathbb{P}^1\longrightarrow(\mathbb{P}V,\mathbb{P}^{n-1}),$$

up to an automorphism of the domain $\mathbb{P}^1$. Thus, the map

$$g\circ f^{-1}\colon\mathbb{P}V\longrightarrow\mathbb{P}^{n-1}$$

is well-defined and determines an element $\Theta(b)\in\overline{\mathfrak{X}}_d$. The map Θ extends continuously over the boundary of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$; see Exercise 23.2.[4] We denote the restriction of Θ to the smooth substack

$$\mathfrak{X}_d\equiv\big\{b\in\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)\colon\mathrm{ev}_1(b)\in q_1\times\mathbb{P}^{n-1},\ \mathrm{ev}_2(b)\in q_2\times\mathbb{P}^{n-1}\big\}\tag{23.4}$$

[2]The weight (i.e. negative first Chern class) of the $\widetilde{\mathbb{T}}$-action on the line $P_i(r)\subset\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ is $\alpha_i+r\hbar$. The tangent bundle of $\overline{\mathfrak{X}}_d$ at $P_i(r)$ is the direct sum of the lines $P_i(r)^*\otimes P_k(s)$ with $(k,s)\neq(i,r)$.

[3]The vector space $\mathbb{C}^n\otimes\mathrm{Sym}^dV^*$ is the direct sum of the one-dimensional representations $P_k(s)$ of $\widetilde{\mathbb{T}}$.

[4]For a complete algebraic proof, see [21, Lemma 2.6].

of $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$ by θ_d, or simply by θ whenever there is no ambiguity.

Let

$$\pi\colon \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)\longrightarrow \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}^{n-1},d\big)$$

be the natural projection map. In light of (23.3), the following lemma implies that

$$\Phi_{\mathcal{Z}}(\hbar,z,Q)\in\mathbb{C}_\alpha[\hbar]\big[\big[z,Q\big]\big].$$

Lemma 23.1. *With $\mathcal{Z}(\hbar,x,Q)$ as in (19.3) and Φ as in (20.1),*

$$\Phi_{\mathcal{Z}}(\hbar,z,Q)=\sum_{d=0}^{\infty}Q^d\int_{\mathfrak{X}_d}e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_d). \tag{23.5}$$

Exercise 23.2. Suppose $[f,g_s]$ is a sequence of elements in $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ so that the domain of each map is $\mathbb{P}^1$ and the sequence converges to some element

$$[\Sigma,\tilde{f},\tilde{g}]\in\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)).$$

There is a unique irreducible component $\Sigma_0\approx\mathbb{P}^1$ of Σ so that $\tilde{f}_0\equiv\tilde{f}|_{\Sigma_0}$ is not constant and can thus be chosen to be f; let $\tilde{g}_0=\tilde{g}|_{\Sigma_0}$. The maps

$$g_s\circ f^{-1},\ \tilde{g}_0\circ f^{-1}\colon\mathbb{P}V\longrightarrow\mathbb{P}^{n-1}$$

correspond to n-tuples of homogeneous polynomials

$$[\mathbf{R}_s]=\big[R_{s;1},\ldots,R_{s;n}\big]\in\overline{\mathfrak{X}}_d\,,\qquad [\tilde{\mathbf{S}}]=\big[\tilde{S}_1,\ldots,\tilde{S}_n\big]\in\overline{\mathfrak{X}}_{\tilde{d}_0},$$

with no common factors. Let

$$[\mathbf{R}]\equiv\big[(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_1,\ldots,(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_n\big]\in\overline{\mathfrak{X}}_d\,,$$

be a limit point of $\{[\mathbf{R}_s]\}$, i.e. the limit of a subsequence of $\{[\mathbf{R}_s]\}$, where $d_l\in\mathbb{Z}^+$,

$$[u_1,v_1],\ldots,[u_m,v_m]\in\mathbb{P}V$$

are distinct points, and the tuple of homogeneous polynomials

$$[\mathbf{S}]\equiv[S_1,\ldots,S_n]\in\overline{\mathfrak{X}}_{d_0}$$

has no common factor. In particular, $d_0+\ldots+d_m=d$. Show that

(a) $\tilde{d}_0=d_0$, $[\tilde{\mathbf{S}}]=[\mathbf{S}]$, Σ consists of Σ_0 along with connected rational curves $\Sigma_1,\ldots,\Sigma_m$ attached to Σ_0 at $f^{-1}([u_1,v_1]),\ldots,f^{-1}([u_m,v_m])$, and the degree of $\tilde{g}|_{\Sigma_l}$ is d_l;

(b) show that the map Θ extends continuously over $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ as claimed above, and the extension is $\tilde{\mathbb{T}}$-equivariant.

`Remark:` The first part is the hard one, as it requires a hands-on understanding of the topology of $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d))$ either from the algebro-geometric or symplectic point of view. First try the case when $m=1$, $d_1=1$, and $v_1=0$.

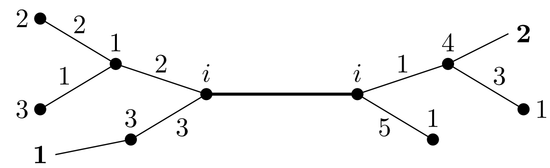


Figure 24.1: A graph representing a fixed locus in $\mathfrak{X}_d$; $i\neq 1,3,4$

24 Proof of Lemma 23.1

In this section we use the localization formula (8.2) to prove Lemma 23.1. We show that each fixed locus of the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}_d$ contributing to the right-hand side of (23.5) corresponds to a pair (Γ_1,Γ_2) of a graphs, with Γ_1 and Γ_2 contributing to $\mathcal{Z}(\alpha_i,\hbar,Qe^{\hbar z})$ and $\mathcal{Z}(\alpha_i,-\hbar,Q)$, respectively, for some $i\in[n]$.

Similarly to Section 22, the fixed loci of the $\widetilde{\mathbb{T}}$-action on $\overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(d',d)\big)$ correspond to decorated graphs Γ with 2 marked points and no loops. Each edge should be labeled by a pair of integers, indicating the degrees of the corresponding maps in $\mathbb{P}V$ and in $\mathbb{P}^{n-1}$. Each vertex should be labeled either $(1,j)$ or $(2,j)$ for some $j\in[n]$, indicating the fixed point, (q_1,P_j) or (q_2,P_j), to which the vertex is mapped. No two consecutive vertex labels are the same, but if two consecutive vertex labels differ in the k-th component, with $k=1,2$, the k-th component of the label for edge connecting them must be nonzero.

The situation for the $\widetilde{\mathbb{T}}$-action on

$$\mathfrak{X}_d \subset \overline{\mathfrak{M}}_{0,2}\big(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)\big)$$

is simpler, however. There is a unique edge of positive $\mathbb{P}V$-degree. We draw it as a thick horizontal line. The first component of all other edge labels must be 0; so we drop it. The first components of the vertex labels change only when the thick edge is crossed. Thus, we drop the first components of the vertex labels as well, with the convention that these components are 1 on the left side of the thick edge and 2 on the right. In particular, the vertices to the left of the thick edge (including the left endpoint) lie in $q_1\times\mathbb{P}^{n-1}$ and the vertices to its right lie in $q_2\times\mathbb{P}^{n-1}$. Thus, by (23.4), the marked point 1 is attached to a vertex to the left of the thick edge and the marked point 2 is attached to a vertex to the right. Finally, both vertices of the thick edge have the same (remaining, second) label $i\in[n]$. Let $\mathcal{A}_i$ denote the set of graphs as above so that the two endpoints of the thick edge are labeled i; see Figure 24.1.

If $\Gamma\in\mathcal{A}_i$, we break it into three sub-graphs:

(i) Γ_1 consisting of all vertices and edges of Γ to the left of the thick edge, including its left vertex v_1, and a new marked point attached to v_1; we label the new marked point 1, while re-labeling the old marked point 1 by 2;

(ii) Γ_0 consisting of the thick edge, its two vertices v_1 and v_2, and new marked points 1 and 2 attached to v_1 and v_2, respectively;

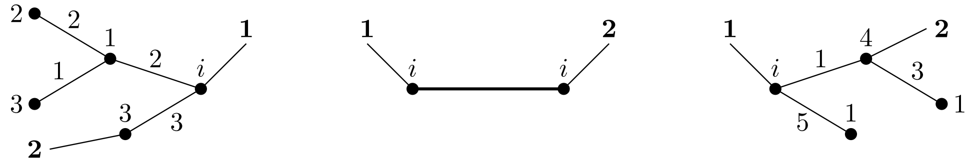


Figure 24.2: The three sub-graphs of the graph in Figure 24.1

(iii) Γ_2 consisting of all vertices and edges of Γ to the right of the thick edge, including its right vertex v_2, and a new marked point 1 attached to v_2;

see Figure 24.2. The fixed locus in $\mathfrak{X}_d$ corresponding to Γ is then

$$\mathcal{Z}_\Gamma \approx \mathcal{Z}_{\Gamma_1} \times \mathcal{Z}_{\Gamma_0} \times \mathcal{Z}_{\Gamma_2}. \tag{24.1}$$

The middle term is a single point. Let π_1, π_0, and π_2 denote the three component projection maps. Denote by d_1 and d_2 the degrees of Γ_1 and Γ_2, i.e.

$$\mathcal{Z}_{\Gamma_1} \subset \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d_1), \qquad \mathcal{Z}_{\Gamma_2} \subset \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d_2).$$

The exceptional case for the first statement is $d_1=0$, in which case the corresponding moduli space does not exist.

Suppose $\Gamma\in\mathcal{A}_i$, d_1 and d_2 are as above, and $d_1>0$. Similarly to (22.7),

$$\begin{aligned} \pi^*\mathcal{V}_{d_1+d_2}\big|_{\mathcal{Z}_\Gamma} &= \mathcal{L}_{P_i} \oplus \pi_1^*\mathcal{V}'_{d_1} \oplus \pi_2^*\mathcal{V}''_{d_2}, \\ \frac{\mathcal{N}\mathcal{Z}_\Gamma}{T_{P_i}\mathbb{P}^{n-1}} &= \pi_1^*\bigg(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_1}}{T_{P_i}\mathbb{P}^{n-1}}\bigg) \oplus \pi_2^*\bigg(\frac{\mathcal{N}\mathcal{Z}_{\Gamma_2}}{T_{P_i}\mathbb{P}^{n-1}}\bigg) \oplus \pi_1^*L_1\otimes\pi_0^*L_1 \oplus \pi_0^*L_2\otimes\pi_2^*L_1, \end{aligned} \tag{24.2}$$

where $\mathcal{N}\mathcal{Z}_\Gamma \longrightarrow \mathcal{Z}_\Gamma$ is the normal bundle of $\mathcal{Z}_\Gamma$ in $\mathfrak{X}_d$ and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_1}$, $L_1, L_2 \longrightarrow \mathcal{Z}_{\Gamma_0}$, and $L_1 \longrightarrow \mathcal{Z}_{\Gamma_2}$ are the tautological tangent line bundles. We note that

$$L_1 = T_{q_1}\mathbb{P}^1 \quad \text{and} \quad L_2 = T_{q_2}\mathbb{P}^1 \qquad \text{on} \quad \mathcal{Z}_{\Gamma_0}.$$

Thus, by (24.2), (5.10), and (23.1),

$$\begin{aligned} \pi^*\big(\mathbf{e}(\mathcal{V}_{d_1+d_2})\big)\Big|_{\mathcal{Z}_\Gamma} &= a\alpha_i\cdot\pi_1^*\mathbf{e}(\mathcal{V}'_{d_1})\cdot\pi_2^*\big(\mathbf{e}(\mathcal{V}'_{d_2})\big), \\ \frac{\prod_{k\neq i}(\alpha_i-\alpha_k)}{\mathbf{e}(N\mathcal{Z}_\Gamma)} &= \pi_1^*\bigg(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\bigg)\cdot\pi_2^*\bigg(\frac{\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\bigg)\cdot\frac{1}{\hbar-\pi_1^*\psi_1}\cdot\frac{1}{(-\hbar)-\pi_2^*\psi_1}. \end{aligned} \tag{24.3}$$

The map θ takes the locus $\mathcal{Z}_\Gamma$ to a fixed point $P_k(r)\in\overline{\mathfrak{X}}_d$. It is immediate that $k=i$. By Exercise 23.2, $r=d_1$. Thus, by the first identity in (23.2),

$$\theta^*\Omega\big|_{\mathcal{Z}_\Gamma} = \alpha_i+d_1\hbar.$$

Combining (24.1) and (24.3) with this observation, we obtain

$$\begin{aligned}\int_{\mathcal{Z}_\Gamma}\frac{e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_{d_1+d_2})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)} &= \frac{a\alpha_i\, e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)} \\ &\times\left\{e^{d_1\hbar z}\int_{\mathcal{Z}_{\Gamma_1}}\frac{\mathbf{e}(\mathcal{V}'_{d_1})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\right\}\left\{\int_{\mathcal{Z}_{\Gamma_2}}\frac{\mathbf{e}(\mathcal{V}'_{d_2})}{(-\hbar)-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\right\}.\end{aligned} \tag{24.4}$$

We note that this identity remains valid for $d_1=0$ if we set the term in the first curly brackets to 1 for $d_1=0$.

We now sum up (24.4), multiplied by $Q^{d_1+d_2}$, over all $\Gamma\in\mathcal{A}_i$. This is the same as summing over all pairs (Γ_1,Γ_2) of graphs such that

(1) Γ_1 is a 2-pointed graph of a degree $d_1\geq 0$ such that the marked point 1 is attached to the vertex labeled i;

(2) Γ_2 is an 2-pointed graph of a degree $d_2\geq 0$ such that the marked point 1 is attached to the vertex labeled i.

By the localization formula (8.2),

$$\begin{aligned}\sum_{\Gamma_1}Q^{d_1}\left\{e^{d_1\hbar z}\int_{\mathcal{Z}_{\Gamma_1}}\frac{\mathbf{e}(\mathcal{V}'_{d_1})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_1})}\right\} &= 1+\sum_{d=1}^{\infty}(Qe^{\hbar z})^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}'_d)}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i \\ &= \mathcal{Z}\big(\alpha_i,\hbar,Qe^{\hbar z}\big); \\ \sum_{\Gamma_2}Q^{d_2}\left\{\int_{\mathcal{Z}_{\Gamma_2}}\frac{\mathbf{e}(\mathcal{V}'_{d_2})\mathrm{ev}_1^*\phi_i}{(-\hbar)-\psi_1}\Big|_{\mathcal{Z}_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_2})}\right\} &= \sum_{d=0}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}'_d)}{(-\hbar)-\psi_1}\mathrm{ev}_1^*\phi_i \\ &= \mathcal{Z}\big(\alpha_i,-\hbar,Q\big).\end{aligned} \tag{24.5}$$

Finally, by (8.2), (24.4), and (24.5),

$$\sum_{d=0}^{\infty}Q^d\int_{\mathfrak{X}_d}e^{(\theta^*\Omega)z}\pi^*\mathbf{e}(\mathcal{V}_d)=\sum_{i=1}^{i=n}\frac{a\alpha_i\, e^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Z}\big(\alpha_i,\hbar,Qe^{\hbar z}\big)\mathcal{Z}\big(\alpha_i,-\hbar,Q\big)=\Phi_{\mathcal{Z}}(\hbar,z,Q),$$

as claimed in (23.5).

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