The Robustness of Extortion in Iterated Prisoner's Dilemma: Appendix

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In this appendix, we provide more details on the technical statements made in the *Main Result* section of the paper. The equation numbers refer to the equtions in the paper, unless stated otherwise.

Throughout this appendix, we assume that $\phi>0$. Thus, $p_1>0$, $p_1>p_2$, $p_3<1$, $p_1<1$ unless $\chi=1$,

$$\begin{gathered}\big(1-p_2-p_3\big)R=(1-p_1)(T+S), \qquad p_3(\chi-1)R=(1-p_1)(T-\chi S),\\ p_3(\chi T-S)-(p_1-p_2)(T-\chi S)=(\chi-1)R(T-\chi S)\phi.\end{gathered}$$

These observations are used below.

We first use equation 6 to show that $D(\mathbf{p},\mathbf{q},\mathbf{1})$ does not vanish on $(0,1)^4$ and determine where on the boundary it does vanish. Since this function is linear in each q_1,q_2,q_3,q_4 separately, it is sufficient to consider $D(\mathbf{p},\mathbf{q},\mathbf{1})$ for the extremal values of q_1,q_2,q_3,q_4. For $(q_3,q_4)=(0,0)$,

$$-\phi^{-1}D(\mathbf{p},\mathbf{q},\mathbf{1})=\big((1-q_2)(1-p_1q_1)+p_3q_2(1-q_1)\big)(\chi T-S)\ge 0;$$

the equality holds if and only if either $q_1,q_2=1$ or $\chi,q_1=1$. For $(q_3,q_4)=(1,0)$,

$$\begin{aligned}-\phi^{-1}D(\mathbf{p},\mathbf{q},\mathbf{1})&=(1-q_2)\big((1-p_1q_1)(\chi-1)(T+S)+p_2(\chi-1)R+(1-q_1)p_2(T-\chi S)\big)\\&\quad+q_2\big((1-q_1)(p_3(\chi T-S)-(p_1-p_2)(T-\chi S))+p_3(\chi-1)R-(1-p_1)(T-\chi S)\big)\ge 0;\end{aligned}$$

the equality holds if and only if either $q_1,q_2=1$ or $\chi=1$ along with one of $q_1=1$, $q_2=1$, or $p_2=0$. For $(q_3,q_4)=(0,1)$,

$$\begin{aligned}-\phi^{-1}D(\mathbf{p},\mathbf{q},\mathbf{1})&=(1-p_1q_1)(2-q_2)(\chi T-S)+\big((1-q_2)(1-p_1q_1)+p_1q_2(1-q_1)\big)(T-\chi S)\\&\quad+(2-q_1)p_3q_2(\chi T-S)+q_2\big((1-p_1)(T-\chi S)-p_3(\chi-1)R\big)\ge 0;\end{aligned}$$

the equality holds if and only if $\chi,q_1=1$ and $q_2=0$. For $(q_3,q_4)=(1,1)$,

$$\begin{aligned}-\phi^{-1}D(\mathbf{p},\mathbf{q},\mathbf{1})&=\big((1-p_1q_1+(2-q_1)p_3)q_2+2(1-q_2)(1-p_1q_1)\big)(\chi T-S)\\&\quad+p_2\big((2-q_2)(\chi-1)R+(2-q_1)(T-\chi S)\big)\ge 0;\end{aligned}$$

the equality holds if and only if $\chi,q_1=1$ and $p_2,q_2=0$.

We next use a similar approach to verify the inequalities 9 and determine when the equalities hold. By equations 1, 6, and 7,

$$D(\mathbf{p},\mathbf{q},\mathbf{1})^2\frac{\partial s_Y}{\partial q_4}=\begin{pmatrix}-1+p_1q_1 & R & R\\ p_2q_3 & S & T\\ p_3q_2 & T & S\end{pmatrix}\begin{pmatrix}-1+p_1q_1 & (1-\chi)R & 1-q_1\\ p_2q_3 & S-\chi T & -q_3\\ p_3q_2 & T-\chi S & 1-q_2\end{pmatrix}.$$

The first determinant above is

$$(1-p_1q_1)(T^2-S^2)+(p_2q_3+p_3q_2)R(T-S)\geq 0;$$

the equality holds if and only if $\chi=1$, $q_1=1$, $q_2=1$, and either $q_3=0$ or $p_2=0$. The second determinant above is linear in q_1, q_2, and q_3. Thus, it is sufficient to check that it is nonnegative at the eight extremal values $q_1,q_2,q_3=0,1$. For $q_3=0$, this determinant is

$$\big((1-q_2)(1-p_1q_1)+(1-q_1)p_3q_2\big)(\chi T-S)\geq 0;$$

the equality holds if and only if either $q_1,q_2=1$ or $\chi,q_1=1$. For $q_3=1$, we obtain

$$\begin{aligned}&q_2=0\colon (\chi-1)\big(p_2R+(1-p_1q_1)(T+S)\big)+(1-q_1)p_2(T-\chi S)\geq 0,\\&q_2=1\colon \phi(1-q_1)(\chi-1)R(T-\chi S)\geq 0.\end{aligned}$$

The equality holds if and only if either $q_1,q_2=1$, or $\chi,q_2=1$, $(\chi,q_1,q_2)=(1,1,0)$, or $(\chi,p_2,q_2)=(1,0,0)$.

By equations 1, 6, and 7, we find that

$$\begin{aligned}D(\mathbf{p},\mathbf{q},\mathbf{1})^2\frac{\partial s_Y}{\partial q_1}&=q_4(T-S)\big(p_3q_2(\chi T-S)+p_2q_3(T-\chi S)\big)\\&\quad\times\big(\big((1-q_2+2q_4)p_1-(p_1-p_2)q_3+p_3q_2\big)R+(1-p_1-p_1q_4)(T+S)\big).\end{aligned}$$

The factors on the first line are nonnegative and vanish if and only if $q_4=0$, or $(q_2,q_3)=0$, or $(p_2,q_2)=(0,0)$. The last factor above equals

$$p_1q_4(2R-(T+S))+\big((1-p_3)(1-q_2)+(1-p_1)q_2+(p_1-p_2)(1-q_3)\big)R>0.$$

Similarly,

$$\begin{aligned}D(\mathbf{p},\mathbf{q},\mathbf{1})^2\frac{\partial s_Y}{\partial q_2}&=q_4(T-S)\big(p_2q_3(\chi-1)R+(1-p_1q_1)(\chi T-S)\big)\\&\quad\times\big(p_3q_4(2R-(T+S))+\big((1-q_3)p_3+p_2q_3\big)R+\big((1-p_3)(1-q_1)+(1-p_1)q_1\big)(T+S)\big)\geq 0;\end{aligned}$$

the equality holds if and only if either $q_4=0$ or $(\chi,q_1)=(1,1)$. Finally,

$$\begin{aligned}D(\mathbf{p},\mathbf{q},\mathbf{1})^2\frac{\partial s_Y}{\partial q_3}&=q_4(T-S)\big(\big((1-q_2)p_3(\chi-1)-p_3(\chi-1)\big)R+\big((1-q_1)p_1+(1-p_1)\big)(T-\chi S)\big)\\&\quad\times\big(p_2q_4(2R-(T+S))+\big((1-q_2)p_2+p_3q_2\big)R+\big((1-p_2)(1-q_1)+(1-p_1)q_1\big)(T+S)\big)\geq 0;\end{aligned}$$

the equality holds if and only if either $q_4=0$, or $q_1,q_2=1$, or $\chi,q_1=1$.