Throughout this section $A$ denotes an $n \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

(1) Matrix $A$ is nonsingular if for every $v \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that

$$Ax = v \quad \text{or} \quad \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

if $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $v = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

Matrix $A$ is invertible if it has an inverse, i.e. there exists a matrix $B$ such that $AB = I = BA$, where $I = I_n$ is the identity matrix. If $AB = I$ and $AC = I$, then $B = C$. Thus, if $A$ has an inverse, it is unique, and denoted by $A^{-1}$. Furthermore,

$$A \text{ is nonsingular } \iff A \text{ is invertible } \iff \det A \neq 0$$

If $\det A \neq 0$, in the $n=2$ case $A^{-1}$ is given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det A = ad - bc$$

In general, there is a three-step procedure for computing $A^{-1}$. The last step of this procedure involves division by $\det A$. If $A$ and $B$ are square matrices,

$$\det (AB) = (\det A) \cdot (\det B) = \det (BA), \quad \text{but} \quad \det (A+B) \neq (\det A) + (\det B)$$

(2) The set of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, or in any vector space, is linearly independent if

$$c_1 v_1 + \ldots + c_k v_k = 0, \quad c_1, \ldots, c_k \in \mathbb{R} \text{ (or } \mathbb{C} \text{)} \implies c_1, \ldots, c_k = 0.$$

In other words, no nontrivial linear combination of the vectors $v_1, \ldots, v_k$ is the zero vector $0$. The set of vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$, or in any vector space $V$, is a basis for $\mathbb{R}^n$, or for $V$, if for every $v$ in $\mathbb{R}^n$, or in $V$, there exists a unique tuple $(c_1, \ldots, c_n)$ such that

$$v = c_1 v_1 + \ldots + c_n v_n.$$
Equivalently, the set of vectors \( v_1, \ldots, v_n \) is a \textit{basis} for \( V \) if the vectors \( v_1, \ldots, v_n \) are linearly independent and \textit{span} \( V \), i.e., for every \( v \) in \( V \), there exists a tuple \((c_1, \ldots, c_n)\) such that

\[
v = c_1v_1 + \ldots + c_nv_n.
\]

Can you show that these two definitions are equivalent? In the case of \( \mathbb{R}^n \):

(i) \( \{v_1, \ldots, v_n\} \) is a linearly independent set of vectors in \( \mathbb{R}^n \) if and only if

(ii) \( \{v_1, \ldots, v_n\} \) is a basis for \( \mathbb{R}^n \) if and only if

(iii) \( \det \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix} \neq 0. \)

(3) An \textit{eigenvector} \( v \) for \( A \) with \textit{eigenvalue} \( \lambda \in \mathbb{R} \) is a nonzero column \( n \)-vector such that

\[
Av = \lambda v
\]

or

\[
\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix} \text{ if } v = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
\]

If \( v \) is an eigenvector for \( A \) with eigenvalue \( \lambda \), so is \( cv \) for any number \( c \). If \( v_1 \) and \( v_2 \) are eigenvectors for \( A \) with the same eigenvalue \( \lambda \), so is \( v_1 + v_2 \). If \( v_1, \ldots, v_k \) are eigenvectors for \( A \) with \textit{distinct eigenvalues} \( \lambda_1, \ldots, \lambda_k \), i.e. \( \lambda_i \neq \lambda_j \) whenever \( i \neq j \), the vectors \( v_1, \ldots, v_k \) are linearly independent. If some of these eigenvalues are the same, the vectors \( v_1, \ldots, v_k \) may or may not be linearly independent.

(4) The \textit{eigenvalues} of \( A \) are the roots of the \textit{characteristic polynomial} for \( A \):

\[
\det(A - \lambda I) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n,n-1} & a_{nn} - \lambda \end{pmatrix}
\]

However, repeated roots of the characteristic polynomial may or may not correspond to different linearly independent eigenvectors. If the multiplicity of a root \( \lambda \) of the characteristic polynomial is \( q \), there exist \( q \) linearly independent \textit{generalized eigenvectors} \( v_1, \ldots, v_q \) for \( A \) with eigenvalue \( \lambda \), i.e.

\[
(A - \lambda)^r v_i = 0 \text{ for some } r
\]

In fact, \( r = q \) works in the given case. If \( v_i \) is an actual eigenvector, \( r = 1 \) suffices, by definition. Furthermore, \( v_1, \ldots, v_q \) can be chosen in such a way that

\[
Av_1 = \lambda v_1 \quad \text{and} \quad Av_{i+1} = v_i + \lambda v_{i+1} \text{ for } i = 1, 2, \ldots, q-1.
\]

Thus, it is always possible to find a basis \( \{v_1, \ldots, v_n\} \) of generalized eigenvectors for \( A \) such that

\[
Av_i = \lambda_i v_i + a_i v_{i-1}, \quad \text{where } a_i = 0 \text{ or } a_i = 1, \quad a_i = 0 \text{ if } i = 1 \text{ or } \lambda_{i-1} \neq \lambda_i,
\]
where $\lambda_i$ is the eigenvalue corresponding to the generalized eigenvector $v_i$. Then,

$$ A = B^{-1}DB, \quad \text{where} \quad D = \begin{pmatrix} \lambda_1 & a_2 & 0 & \ldots \\ 0 & \lambda_2 & a_3 & \ldots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}. \quad (1) $$

*Can you check this?* The above basis $v_1, \ldots, v_n$ and matrix $B$, however, may be complex. In such a case, $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis for $\mathbb{C}^n$, not an $\mathbb{R}$-basis for $\mathbb{R}^n$.

(5) If $A$ is an $n \times n$ matrix, the *exponential* of $A$ is the $n \times n$ matrix given by

$$ e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k $$

where $A^0 = I_n$, $A^2 = AA$, $A^3 = AAA$, \ldots

Note that this is the same power series as for $e^a$, if $a$ is a real or complex number. By definition, if $A$ is the zero matrix, $e^A = I_n$. Another property of the matrix exponential is

$$ \text{If } AB = BA, \text{ then } e^{A+B} = e^Ae^B = e^Be^A $$

(2)

Using this property, we can conclude that

(i) $e^A$ is an invertible matrix and $(e^A)^{-1} = e^{-A}$,

(ii) if $H(t) = e^{tA}$, then $H'(t) = AH(t) = H(t)A$.

If $A$ is a diagonal matrix, then $e^A$ is also a diagonal matrix, and the diagonal entries of $e^A$ are the exponentials of the corresponding diagonal entries of $A$. For example,

$$ A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \implies \quad e^A = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix} $$

However, if $A$ is not a diagonal matrix, the entries of $e^A$ are *not* usually the exponentials of the entries of $A$, and it may be very hard to determine them directly from the power series definition of the exponential. On the other hand, it may be possible to find a basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, or $\mathbb{C}^n$, such that $e^A v_i$ is easy to compute for each $i$. Since $\{v_1, \ldots, v_n\}$ is a basis, an arbitrary vector $v$ has the form

$$ v = C_1v_1 + \ldots + C_nv_n, \quad C_1, \ldots, C_n \in \mathbb{C} \quad \implies \quad e^Av = C_1e^Av_1 + \ldots + C_ne^Av_n. $$

This is usually sufficient for solving systems of linear ODEs with constant coefficients. The product $e^Av_i$ can be computed for generalized eigenvectors of $A$. For example,

$$ Av_1 = \lambda v_1, \quad Av_2 = a v_1 + \lambda v_2 \implies e^{Av_1} = e^\lambda v_1, \quad e^{Av_2} = ae^\lambda v_1 + e^\lambda v_2 $$

These two relations are sufficient for the $n=2$ case.

(6) In order to compute $e^A$ for an arbitrary square matrix, one makes use of the relation

$$ e^{B^{-1}DB} = B^{-1}e^DB $$

3
and (eq1). The exponential of the matrix $D$ as in (eq1) can be computed directly from the definition. This approach is analogous to the one described in Section 9.8: if \( \{v_1(t), \ldots, v_n(t)\} \) is a fundamental set of solutions for the ODE, then

\[
Y(t) = \begin{pmatrix}
v_1(t) \\
\vdots \\
v_n(t)
\end{pmatrix} \implies e^{tA} = Y(t)Y(0)^{-1}
\]

(4)

On the other hand, if $A$ has only one eigenvalue $\lambda$, \((A - \lambda I)^n\) is the zero matrix, and the power series for the exponential of $A - \lambda I$ quickly truncates. Since $\lambda I$ commutes with all matrices, one can compute $e^{A}$ by using (eq2) with $A = \lambda I$ and $B = A - \lambda I$.

**Systems of Linear ODEs with Constant Coefficients**

(1) A system of first-order linear ODEs with constant coefficients can be written as

\[
y' = Ay + f, \quad y = y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad f = f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.
\]

This system is called *homogeneous* if $f = 0$. A system of first-order linear ODEs with constant coefficients can be solved by the integrating factor method for first-order linear ODEs:

\[
y' = Ay + f \implies y(t) = e^{tA}v + e^{tA} \int_0^t e^{-sA}f(s) \, ds, \quad v \in \mathbb{R}^n
\]

(5)

Note that the function $y_h = y_h(t)$ defined by (eq5) with $f = 0$, i.e. the first term on the right-hand side, is the general solution of the corresponding homogeneous system of ODEs. Thus, the general solution to an inhomogeneous system of ODEs is given by

\[
y' = Ay + f \implies y = y_p + y_h
\]

(6)

where $y_p$ is a solution to the inhomogeneous system, e.g. the function $y$ corresponding to $v = 0$ to (eq5). The relation (eq6) is valid for any system of linear ODEs, with constant or non-constant coefficients.

(2) The main difficulty in solving a system of linear ODEs with constant coefficients is dealing with the terms in (eq5) involving $e^{tA}$. This is not difficult to do if there is a basis for $\mathbb{R}^n$, or $\mathbb{C}^n$, of eigenvectors for $A$:

\[
y' = Ay \implies y(t) = C_1 e^{\lambda_1 t}v_1 + \cdots + C_n e^{\lambda_n t}v_n, \quad C_1, \ldots, C_n \in \mathbb{R} \text{ (or } \mathbb{C})
\]

if \( \{v_1, \ldots, v_n\} \) is a basis for $\mathbb{R}$ (or $\mathbb{C}$) and $Av_i = \lambda_i v_i, \ldots, Av_n = \lambda_n v_n$

(7)

(3) If we are looking for real solutions, we will need to rewrite (eq7) in a different way if some of the eigenvalues $\lambda_i$ are complex, and not real. If $v_i$ is an eigenvector for $A$ with eigenvalue $\lambda_i$ and
\( \lambda_i \) is complex, \( \vec{v}_i \) is an eigenvector for \( A \) with eigenvalue \( \lambda_i \) and the vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent. Thus, if \( n = 2 \) and \( A \) has an eigenvector \( \vec{v}_1 \) with a complex eigenvalue \( \lambda_1 \), then the two eigenvalues of \( A \) are complex conjugates, \( \lambda_1, \lambda_2 = a \pm ib \), and \( \mathbb{C}^2 \) has a basis of conjugate eigenvectors \( \{ \vec{v}_1, \vec{v}_2 = \vec{w}_1 \pm i \vec{w}_2 \} \). The general solution in this case can be written as

\[
\begin{align*}
y'(t) &= Ay(t) \\
&= (A_1 \cos bt + A_2 \sin bt) e^{at} \vec{w}_1 + (A_2 \cos bt - A_1 \sin bt) e^{at} \vec{w}_2, \\
&= e^{at} (w_1 w_2) \begin{pmatrix} \cos bt & \sin bt \\ - \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad A_1, A_2 \in \mathbb{R} \text{ (or } \mathbb{C})
\end{align*}
\]

This expression is obtained by setting \( C_1, C_2 = (A_1 \mp i A_2)/2 \) in (eq7). Note that if \( A_1 \) and \( A_2 \) are arbitrary complex constants, so are \( C_1 \) and \( C_2 \). On the other hand, the solution corresponding to \( A_1 \) and \( A_2 \) is real if and only if \( A_1 \) and \( A_2 \) are real.

(3) Another potential problem with (eq7) is that \( \mathbb{R}^n \), or \( \mathbb{C}^n \), may not have a basis of eigenvectors for \( A \). If so, we can use a basis of generalized eigenvectors. If \( n = 2 \) and \( A \) has only one eigenvalue \( \lambda_i \) by (eq3),

\[
\begin{align*}
y'(t) &= Ay(t) \\
&= (C_1 e^{\lambda t} + C_2 ate^{\lambda t}) \vec{v}_1 + C_2 e^{\lambda t} \vec{v}_2, \quad C_1, C_2 \in \mathbb{R} \text{ (or } \mathbb{C})
\end{align*}
\]

if \( \vec{v}_1, \vec{v}_2 \) are lin. indep., \( Av_1 = \lambda v_1 \), and \( Av_2 = av_1 + \lambda v_2 \)

Once an eigenvector \( \vec{v}_1 \) for the eigenvalue \( \lambda \) is found, \( \vec{v}_2 \) can be taken to be any vector in \( \mathbb{R}^2 \) which is not a multiple of \( \vec{v}_1 \), and the number \( a \) is determined by computing \( Av_2 \).

(4) The general solution to an inhomogeneous system of linear first-order ODEs with constant-coefficients is given by (eq5), or more generally by

\[
\begin{align*}
y'(t) &= Ay + f \quad \implies \quad y(t) &= (e^{tA}B)v + (e^{tA}B) \int_0^t (e^{sA}B)^{-1}f(s)ds, \quad v \in \mathbb{R}^n
\end{align*}
\]

(5) A solution to an initial value problem can be obtained directly by

\[
\begin{align*}
y'(t) = Ay + f, \quad y(t_0) = y_0 \quad \implies \quad y(t) &= e^{tA}(e^{-t_0A}y_0 + \int_{t_0}^t e^{-sA}f(s)ds)
\end{align*}
\]

More generally, if \( Y = Y(t) \) is any fundamental matrix for \( y' = Ay \),

\[
\begin{align*}
y'(t) &= Ay + f, \quad y(t_0) = y_0 \quad \implies \quad y(t) &= Y(t)(Y(t_0)^{-1}y_0 + \int_{t_0}^t Y(s)^{-1}f(s)ds)
\end{align*}
\]
Qualitative Descriptions

(1) As is the case for linear ODEs, every initial-value problem
\[ y' = Ay + f, \quad y(t_0) = y_0, \quad A = A(t), \quad f = f(t), \]  
has a unique solution, provided the functions \( A \) and \( f \) are continuous near \( t_0 \). Furthermore, the interval of the existence of the solution to (eq9) is the largest interval on which \( A \) and \( f \) are defined. If \( A \) is a constant matrix, it follows that the phase-space solution curves for the system \( y' = Ay \) do not intersect. Can you explain why?

(2) Every homogeneous system of linear ODEs \( y' = Ay \) has an equilibrium solution, \( y(t) = 0 \). This solution can be asymptotically stable, stable, or unstable. If \( A \) is a constant matrix and the real part of every eigenvalue of \( A \) is negative, all solutions \( y = y(t) \) approach \( 0 \) at \( t \to \infty \), and thus \( 0 \) is an asymptotically stable equilibrium point of the system. If the real parts of some eigenvalues of \( A \) are negative and of some are zero, some solutions \( y = y(t) \) approach \( 0 \) at \( t \to \infty \), while others approach closed orbits. In this case, \( 0 \) is a stable equilibrium point of the system, as every solution starting near \( 0 \) stays near \( 0 \). Finally, if the real part of at least one eigenvalue of \( A \) is positive, some solutions \( y = y(t) \) move away from \( 0 \) and approach \( \infty \) at \( t \to 0 \), and thus \( 0 \) is an unstable equilibrium point of the system.

(3) If \( A \) is a constant matrix, the system \( y' = Ay \) is autonomous, i.e. it does not involve \( t \) explicitly. Thus, if \( y = y(t) \) is a solution to this system, so is \( \tilde{y}(t) = y(t-c) \). The latter solution traces the same curve \( y(t) \) in \( \mathbb{R}^n \), but is delayed by time \( c \). For this reason, the qualitative behavior of solutions of \( y' = Ay \) is well represented by the non-intersecting curves \( y(t) \) traced out in the phase space, i.e. \( \mathbb{R}^n \). For some sketches in the \( n=2 \) case, see Figures 2-4 of the solutions to PS4.

(4) While systems of first-order ODEs arise in applications by themselves, they can also be used to replace high-order ODEs. For example, the initial value problem
\[ y''' + y'y'' + ty = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y''(t_0) = y_2, \]
is equivalent to the initial value problem
\[
\begin{pmatrix}
y \\
u \\
v
\end{pmatrix}' =
\begin{pmatrix}
u \\
v \\
-uv - ty
\end{pmatrix}, \quad
\begin{pmatrix}
y(0) \\
u(0) \\
v(0)
\end{pmatrix} =
\begin{pmatrix}
y_0 \\
y_1 \\
y_2
\end{pmatrix}.
\]
Can you explain why? Such replacements are often useful, because many numerical methods and methods of qualitative analysis apply only to first-order ODEs and systems of first-order ODEs.