Math53: Ordinary Differential Equations
Autumn 2004

Problem Set 6 Solutions

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 9.2:38,40 and 9.8:6. In the first two problems, phase-plane portraits are discussed in detail. In 9.8:6, complex numbers are used to greatly simplify the computation.

Section 9.2: 38,40,44 (25pts)

9.2:38; 10pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
The characteristic polynomial for this system is

$$\lambda^2 - (-3-1)\lambda + \big((-3)\cdot(-1) - 1\cdot(-1)\big) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2.$$

Thus, there is only one eigenvalue, $\lambda = -2$. We next find an eigenvector $\mathbf{v}_1$ for $\lambda = -2$:

$$\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{cases} \iff c_1 = c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now pick a simple vector $\mathbf{v}_2$, express $A\mathbf{v}_2 - \lambda\mathbf{v}_2$ in terms of $\mathbf{v}_1$, and then compute $e^{tA}\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = (-1)\cdot\mathbf{v}_1$$

$$\implies tA\mathbf{v}_2 = (-t)\mathbf{v}_1 + (-2t)\mathbf{v}_2 \implies e^{tA}\mathbf{v}_2 = -te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2.$$

The general solution to the ODE is thus given by

$$\mathbf{y}(t) = C_1 e^{-2t}\mathbf{v}_1 + C_2\big(-te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2\big) = \boxed{e^{-2t}\begin{pmatrix} C_1 + C_2 - C_2 t \\ C_1 - C_2 t \end{pmatrix}}$$

A phase-plane sketch is the first plot in Figure 1. The origin is a *degenerate nodal sink*. Each solution curve descends to the origin as $t \longrightarrow \infty$, and its slope approaches 1 as $t \longrightarrow \pm\infty$. In order to see which way the solution curves move on the two sides of the line $\mathbb{R}\mathbf{v}_1$, we need to determine whether $C_2 > 0$ or $C_2 < 0$ on each of the two sides of this line. The line itself corresponds to $C_2 = 0$. We know that if $C_2 > 0$, the point $\mathbf{y}(t)$ corresponding to C_1 and t will lie either to the left or to the right of the line, with left or right being the same for all C_1 and t. Thus, we can test this using $C_1 = 0$ and $t = 0$. In this case, $\mathbf{y}(t) = (1,0)$ lies to the right of the line. Thus, C_2 is positive to the right of the line. By looking at $\mathbf{y}(t)$, we see that if $C_2 > 0$, the x- and y-coordinates of $\mathbf{y}(t)$ become

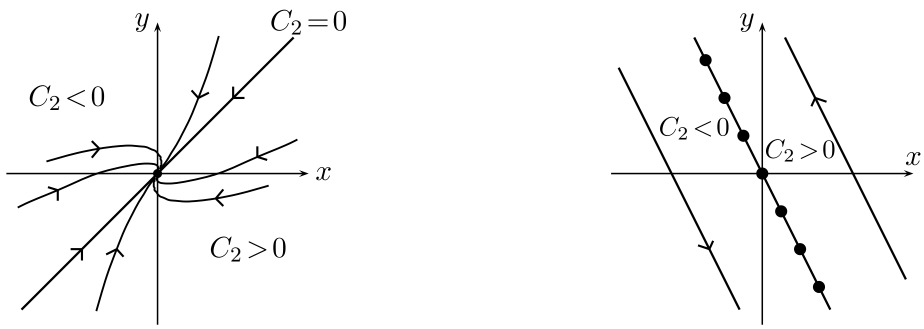

Figure 1: Phase-Plane Plots for Problems 9.2:38 and 9.2:40

very large and positive as $t \longrightarrow -\infty$, and become negative as $t \longrightarrow \infty$. Thus, the solution curves on the right of the line $\mathbb{R}\mathbf{v}_1$ rise up in the direction of $+\mathbf{v}_1$ as $t \longrightarrow -\infty$ and approach the origin from below left as $t \longrightarrow \infty$. The picture on the left side of the line $\mathbb{R}\mathbf{v}_1$ is just a reflection about the origin.

9.2:40; 10pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
The characteristic polynomial for this system is

$$\lambda^2 - (-2+2)\lambda + \big((-2)\cdot 2 - (-1)\cdot 4\big) = \lambda^2.$$

Thus, there is only one eigenvalue, $\lambda\!=\!0$. We next find an eigenvector $\mathbf{v}_1$ for $\lambda\!=\!0$:

$$\begin{pmatrix} -2-\lambda & -1 \\ 4 & 2-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -2c_1 - c_2 = 0 \\ 4c_1 + 2c_2 = 0 \end{cases} \iff c_2 = -2c_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We now pick a simple vector $\mathbf{v}_2$, express $A\mathbf{v}_2 - \lambda\mathbf{v}_2$ in terms of $\mathbf{v}_1$, and then compute $e^{tA}\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (-2)\cdot\mathbf{v}_1$$
$$\implies tA\mathbf{v}_2 = (-2t)\mathbf{v}_1 + (0t)\mathbf{v}_2 \implies e^{tA}\mathbf{v}_2 = (-2t)e^{-0t}\mathbf{v}_1 + e^{-0t}\mathbf{v}_2.$$

The general solution to the ODE is thus given by

$$\mathbf{y}(t) = C_1 e^{-0t}\mathbf{v}_1 + C_2\big(-2te^{-0t}\mathbf{v}_1 + e^{-0t}\mathbf{v}_2\big) = \boxed{\begin{pmatrix} C_1 + C_2 - 2C_2 t \\ -2C_1 + 4C_2 t \end{pmatrix}}$$

A phase-plane sketch is the second plot in Figure 1. Note that if $C_2\!=\!0$, the corresponding solution $\mathbf{y}(t) = (C_1 \;\; -2C_1)^t$ is a constant function, i.e. *every* point on the line $y = -2x$ is an equilibrium point. If $C_2 \neq 0$, the solution $\mathbf{y}(t)$ traces the line of slope -2 through the point $(C_2 \;\, 0)^t$. In order to tell whether it moves up or down along the line, we need to determine whether $C_2\!>\!0$ or $C_2\!<\!0$

on each of the two sides of the line $y=-2x$. The line itself corresponds to $C_2=0$, with the values of C_1 corresponding to the points on the line. We know that if $C_2>0$, the point $\mathbf{y}(t)$ corresponding to C_1 and t will lie either to the left or to the right of the line $y=-2x$, with left or right being the same for all C_1 and t. Thus, we can test this using $C_1=0$ and $t=0$. In this case, $\mathbf{y}(t)=(1,0)$ lies to the right of the line. Thus, C_2 is positive to the right of the line. Since the y-coordinate increases with t for $C_2>0$, solutions to the right of the line $y=-2x$ move up. Similarly, solutions to the left of this line move down. The origin is an unstable equilibrium, and so is every point on the line.

9.2:44; 5pts: *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

By 9.2:38, it remains to find C_1 and C_2 such that

$$\mathbf{y}(0) = \begin{pmatrix} C_1+C_2 \\ C_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \iff \begin{cases} C_1 + C_2 = 0 \\ C_1 = -3 \end{cases} \iff \begin{cases} C_1 = -3 \\ C_2 = 3 \end{cases}$$

Thus, the solution to the IVP is $\boxed{\mathbf{y}(t) = -3e^{-2t} \begin{pmatrix} t \\ 1+t \end{pmatrix}}$

Section 9.4: 14 (12pts)

Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The characteristic polynomial $p(\lambda)$ for the matrix is:

$$\begin{aligned} \det(A-\lambda I) = \det \begin{pmatrix} -3-\lambda & 0 & -1 \\ 3 & 2-\lambda & 3 \\ 2 & 0 & -\lambda \end{pmatrix} &= (2-\lambda)\det \begin{pmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} \\ &= -(\lambda-2)(\lambda^2+3\lambda+2) = -(\lambda-2)(\lambda+1)(\lambda+2). \end{aligned}$$

The eigenvalues are $\lambda_1=-2$, $\lambda_2=-1$, $\lambda_3=2$. For each of these, we find an eigenvector:

$$\lambda_1 = -2: \quad \begin{pmatrix} -3-\lambda_1 & 0 & -1 \\ 3 & 2-\lambda_1 & 3 \\ 2 & 0 & -\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 - c_3 = 0 \\ 3c_1 + 4c_2 + 3c_3 = 0 \\ 2c_1 + 2c_3 = 0 \end{cases}$$

$$\iff \begin{cases} c_3 = -c_1 \\ c_2 = 0 \end{cases} \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_2 = -1: \quad \begin{pmatrix} -3-\lambda_2 & 0 & -1 \\ 3 & 2-\lambda_2 & 3 \\ 2 & 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -2c_1 - c_3 = 0 \\ 3c_1 + 3c_2 + 3c_3 = 0 \\ 2c_1 + c_3 = 0 \end{cases}$$

$$\iff \begin{cases} c_3 = -2c_1 \\ c_2 = c_1 \end{cases} \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_3 = 2: \quad \begin{pmatrix} -3-\lambda_3 & 0 & -1 \\ 3 & 2-\lambda_3 & 3 \\ 2 & 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -5c_1 - c_3 = 0 \\ 3c_1 + 3c_3 = 0 \\ 2c_1 - 2c_3 = 0 \end{cases}$$

$$\iff \begin{cases} c_1 = 0 \\ c_3 = 0 \end{cases} \implies \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, the general solution is:

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 = C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

From the initial condition, we obtain

$$\mathbf{y}(0) = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \iff \begin{cases} C_1 + C_2 = 1 \\ C_2 + C_3 = -1 \\ -C_1 - 2C_2 = 2 \end{cases}$$

$$\iff \begin{cases} C_1 = 1 - C_2 \\ C_3 = -1 - C_2 \\ -1 - C_2 = 2 \end{cases} \iff \begin{cases} C_2 = -3 \\ C_1 = 4 \\ C_3 = 2 \end{cases}$$

Plugging these constants into the general solution, we get

$$\boxed{\mathbf{y}(t) = 4e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 3e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ -3e^{-t} + 2e^{2t} \\ -4e^{-2t} + 6e^{-t} \end{pmatrix}}$$

Section 9.6: 7,9; 10pts

9.6:7; 4pts *Determine whether the origin is an unstable, stable, or asymptotically stable equilibrium for the system*

$$\mathbf{y}' = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
The characteristic polynomial for this system is

$$\lambda^2 - (1 + (-3))\lambda + \big(1 \cdot (-3) - (-4) \cdot 1\big) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

Thus, the matrix has only one eigenvalue $\lambda = \lambda_1 = -1$. Since this eigenvalue is negative and it is the only eigenvalue, the origin is an $\boxed{\text{asymptotically stable}}$ point. It is a degenerate sink. The phase-plane portrait is similar to that in the first sketch of Figure 1, except the half-lines have slope .5, instead of 1.

9.6:9; 6pts *Determine whether the origin is an unstable, stable, or asymptotically stable equilibrium for the system*

$$\mathbf{y}' = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

The characteristic polynomial for this system is

$$\det \begin{pmatrix} -3-\lambda & -4 & 2 \\ -2 & -7-\lambda & 4 \\ -3 & -8 & 4-\lambda \end{pmatrix} = -\big(\lambda^3 + 6\lambda^2 + 11\lambda + 6\big) = -(\lambda+1)(\lambda+2)(\lambda+3).$$

All three eigenvalues $\lambda_1, \lambda_2, \lambda_3 = -1, -2, -3$ are negative. Thus, the origin is a nodal sink and an $\boxed{\text{asymptotically stable}}$ equilibrium point.

Section 9.7: 17 (4pts)

Find the general solution of the equation $y^{(4)} + 36y = 13y''$.
The characteristic polynomial for $y^{(4)} - 13y'' + 36y = 0$ is:

$$\lambda^4 - 13\lambda^2 + 36 = (\lambda^2 - 4)(\lambda^2 - 9) = (\lambda+2)(\lambda-2)(\lambda+3)(\lambda-3).$$

It has four distinct roots: ± 2, ± 3. Thus, the general solution is:

$$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{-2t} + C_3 e^{2t} + C_4 e^{3t}}$$

Section 9.8: 6,18,29 (29pts)

9.8:6; 15pts: *Find the general solution of the system* $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$, *where*

$$A = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{f} = \begin{pmatrix} t \\ e^{3t} \end{pmatrix}.$$

The characteristic polynomial for A is

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 10.$$

The eigenvalues of A are the roots of this polynomial: $\lambda_1, \lambda_2 = 3 \pm i$. We next find an eigenvector for λ_1:

$$\begin{pmatrix} 4-\lambda_1 & 2 \\ -1 & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} (1-i)c_1 + 2c_2 = 0 \\ -c_1 - (1+i)c_2 = 0 \end{cases}$$

$$\iff \quad c_1 = -(1+i)c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}.$$

The complex conjugate of $\mathbf{v}_1$, i.e. $\mathbf{v}_2 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$, is an eigenvector for $\lambda_2 = \bar{\lambda}_1$. Thus, the general solution to the homogeneous system $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y}_h(t) &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^{(3+i)t} \begin{pmatrix} 1+i \\ -1 \end{pmatrix} + C_2 e^{(3-i)t} \begin{pmatrix} 1-i \\ -1 \end{pmatrix} \\ &= (A_1 \cos t + A_2 \sin t) e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (A_2 \cos t - A_1 \sin t) e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{1}$$

The next step is to find a particular solution $\mathbf{y}_p$ to the inhomogeneous system, using

$$\mathbf{y}_p(t) = Y(t) \int_0^t Y(s)^{-1}\mathbf{f}(s)\, ds,$$

where $Y(t) = (\mathbf{y}_1(t)\ \mathbf{y}_2(t))$ is a fundamental matrix and $\{\mathbf{y}_1(t), \mathbf{y}_2(t)\}$ is a fundamental set of solutions for the homogeneous system. We can use either complex or real solutions:

$$Y(t) = e^{3t} \begin{pmatrix} (1+i)e^{it} & (1-i)e^{-it} \\ -e^{it} & -e^{-it} \end{pmatrix} \qquad \text{or} \qquad Y(t) = e^{3t} \begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ -\cos t & -\sin t \end{pmatrix}. \tag{2}$$

In the first case, the fundamental solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of the homogeneous system correspond to the $(C_1 = 1, C_2 = 0)$ and $(C_1 = 0, C_2 = 1)$ cases of (eq1). In the second case, they correspond to the $(A_1 = 1, A_2 = 0)$ and $(A_1 = 0, A_2 = 1)$ cases of (eq1). As (eq2) might suggest, it is easier to use the complex solutions. In the complex case:

$$Y(t)^{-1} = e^{-3t} \cdot \frac{1}{-2i} \begin{pmatrix} -e^{-it} & -(1-i)e^{-it} \\ e^{it} & (1+i)e^{it} \end{pmatrix} \implies Y^{-1}(s)\mathbf{f}(s) = \frac{i}{2} \begin{pmatrix} -se^{-(3+i)s} - (1-i)e^{-is} \\ se^{-(3-i)s} + (1+i)e^{is} \end{pmatrix}.$$

We next compute

$$\begin{aligned} &\int_0^t (1+i)e^{is} ds = \frac{1+i}{i} e^{is}\Big|_{s=0}^{s=t} = (1-i)\big(e^{it} - 1\big) \implies \int_0^t (1-i)e^{-is} ds = (1+i)\big(e^{-it} - 1\big); \\ &\int s e^{-(3+i)s} ds = \frac{1}{-(3+i)} \Big(s e^{-(3+i)s} - \int e^{-(3+i)s} ds \Big) = -\frac{3-i}{10} s e^{-(3+i)s} - \frac{4-3i}{50} e^{-(3+i)s} \\ &\qquad \implies \int_0^t s e^{-(3+i)s} ds = -\frac{3-i}{10} t e^{-(3+i)t} - \frac{4-3i}{50} \big(e^{-(3+i)t} - 1\big) \\ &\qquad \implies \int_0^t s e^{-(3-i)s} ds = -\frac{3+i}{10} t e^{-(3-i)t} + \frac{4+3i}{50} \big(e^{-(3-i)t} - 1\big). \end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
\mathbf{y}_p(t) &= Y(t)\int_0^t Y(s)^{-1}\mathbf{f}(s)\,ds \\
&= e^{3t}\begin{pmatrix} (1+i)e^{it} & (1-i)e^{-it} \\ -e^{it} & -e^{-it}\end{pmatrix}\cdot\frac{i}{2}\begin{pmatrix} \frac{3-i}{10}te^{-(3+i)t}+\frac{4-3i}{50}e^{-(3+i)t}-(1+i)e^{-it} \\ -\frac{3+i}{10}te^{-(3-i)t}-\frac{4+3i}{50}e^{-(3-i)t}+(1-i)e^{it}\end{pmatrix}+Y(t)\mathbf{v} \\
&= \frac{ie^{3t}}{2}\begin{pmatrix} \frac{2i}{5}te^{-3t}+\frac{i}{25}e^{-3t}-4i \\ \frac{i}{5}te^{-3t}+\frac{3i}{25}e^{-3t}+2i\end{pmatrix}+Y(t)\mathbf{v} = -\frac{1}{50}\begin{pmatrix} 10t+1-100e^{3t} \\ 5t+3+50e^{3t}\end{pmatrix}+Y(t)\mathbf{v},
\end{aligned}$$

for some $\mathbf{v}\in\mathbb{C}$. Since $Y(t)\mathbf{v}$ is a solution of the homogeneous system, the last expression is still a solution of the inhomogeneous system even if we drop the last term. Thus, the general solution of the inhomogeneous system is

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h(t)+\mathbf{y}_p(t) \\
&= \boxed{(A_1\cos t+A_2\sin t)e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}+(A_2\cos t-A_1\sin t)e^{3t}\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{50}\begin{pmatrix}10t+1-100e^{3t}\\5t+3+50e^{3t}\end{pmatrix}}
\end{aligned}$$

Another way of finding $\mathbf{y}_p$ is to use the method of undetermined coefficients. In this case, this would be mean finding a_1, b_1, c_1 and a_2, b_2, c_2, such that

$$\mathbf{y}_p' = A\mathbf{y}_p+\mathbf{f} \qquad \text{for} \qquad \mathbf{y}_p(t)=\begin{pmatrix} a_1e^{3t}+b_1t+c_1 \\ a_2e^{3t}+b_2t+c_2\end{pmatrix}.$$

9.8:18; 10pts: *Solve the initial value problem*

$$\mathbf{y}'=\begin{pmatrix}-7 & -3 \\ 6 & 2\end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0)=\begin{pmatrix}1\\0\end{pmatrix}.$$

The characteristic polynomial for A is

$$\lambda^2+(\operatorname{tr}A)\lambda+\det A=\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4).$$

The eigenvalues of A are the roots of this polynomial: $\lambda_1,\lambda_2=-1,-4$. We next find the corresponding eigenvectors:

$$\lambda_1=-1: \qquad \begin{pmatrix}-7-\lambda_1 & -3 \\ 6 & 2-\lambda_1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}-6c_1-3c_2=0\\ 6c_1+3c_2=0\end{cases}$$

$$\iff \quad c_2=-2c_1 \implies \mathbf{v}_1=\begin{pmatrix}1\\-2\end{pmatrix}.$$

$$\lambda_2=-4: \qquad \begin{pmatrix}-7-\lambda_2 & -3 \\ 6 & 2-\lambda_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}-3c_1-3c_2=0\\ 6c_1+6c_2=0\end{cases}$$

$$\iff \quad c_2=-c_1 \implies \mathbf{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}.$$

Thus, a fundamental matrix for this system is

$$Y(t) = \big(e^{\lambda_1 t}\mathbf{v}_1\ e^{\lambda_2 t}\mathbf{v}_2\big) = \begin{pmatrix} e^{-t} & e^{-4t} \\ -2e^{-t} & -e^{-4t} \end{pmatrix} \Longrightarrow Y(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \Longrightarrow Y(0)^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{-t} & e^{-4t} \\ -2e^{-t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2e^{-4t}-e^{-t} & e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} & 2e^{-t}-e^{-4t} \end{pmatrix}}$$

Finally,

$$\mathbf{y}(t) = e^{tA}\mathbf{y}(0) = \begin{pmatrix} 2e^{-4t}-e^{-t} & e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} & 2e^{-t}-e^{-4t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} \end{pmatrix}}$$

9.8:29; 4pts: *Show that if A is an $n\times n$ matrix, the function*

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0 + \int_0^t e^{(t-s)A}\mathbf{f}(s)\,ds$$

solves the initial value problem $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{y}_0$.
We first check that the initial condition is satisfied:

$$\mathbf{y}(0) = e^{0A}\left(\mathbf{y}_0 + \int_0^0 e^{-s}\mathbf{f}(s)ds\right) = I\mathbf{y}_0 = \mathbf{y}_0,$$

as required. We next use the product rule to check that the ODE is satisfied

$$\mathbf{y}(t) = e^{tA}\left(\mathbf{y}_0 + \int_0^t e^{-sA}\mathbf{f}(s)ds\right)$$

$$\Longrightarrow \quad \mathbf{y}'(t) = Ae^{tA}\left(\mathbf{y}_0 + \int_0^t e^{-sA}\mathbf{f}(s)ds\right) + e^{tA}\big(e^{-tA}\mathbf{f}(t)\big) = A\mathbf{y}(t) + \mathbf{f}(t).$$

Problem E (20pts)

(a; **2pts**) *Find simple conditions on smooth functions $P = P(t)$, $Q = Q(t)$, and $a = a(t)$ that are equivalent to*

$$\big(Q(y'+ay)\big)' = P(y'' + py' + qy), \qquad p = p(t),\ q = q(t), \tag{3}$$

for every smooth function $y = y(t)$.
Expand LHS and compare with RHS:

$$\big(Q(y'+ay)\big)' = Qy'' + (Q'+Qa)y' + (Qa)'y = Py'' + Ppy' + Pqy \qquad \Longrightarrow$$

$$P = Q,\ \ Q' + Qa = pP,\ \ (Qa)' = qP \quad \Longleftrightarrow \quad \boxed{P = Q,\ \ P' + Pa = pP,\ \ (Pa)' = qP}$$

(b; **8pts**) *Find an integrating factor for the second-order ODE (eq3) with constant p and q. Use it to find $R_1 = R_1(t)$ and $R_2 = R_2(t)$ such that*

$$\big(R_2(R_1 y)'\big)' = P(y'' + py' + qy), \qquad p, q = const.$$

By (a), we need to find a nonzero solution to the system

$$\begin{pmatrix} P \\ (Pa) \end{pmatrix}' = \begin{pmatrix} p & -1 \\ q & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \end{pmatrix} \qquad P=P(t),\ a=a(t). \tag{4}$$

The characteristic polynomial for the constant-coefficient matrix in (eq4) is $\lambda^2-p\lambda+q=0$. Let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be the two roots of this quadratic equation. Note that $\lambda_1=-\tilde{\lambda}_1$ and $\lambda_2=-\tilde{\lambda}_2$ must then be the roots of $\lambda^2+p\lambda+q=0$, i.e. the characteristic polynomial for the second-order ODE. The reason is that

$$\lambda_1+\lambda_2 = -(\tilde{\lambda}_1+\tilde{\lambda}_2) = -p \quad \text{and} \quad \lambda_1\cdot\lambda_2 = (-\tilde{\lambda}_1)\cdot(-\tilde{\lambda}_2) = \tilde{\lambda}_1\cdot\tilde{\lambda}_2 = q.$$

We next find an eigenvector for the eigenvalue $\tilde{\lambda}_2$:

$$\begin{pmatrix} p-\tilde{\lambda}_2 & -1 \\ q & -\tilde{\lambda}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} \tilde{\lambda}_1 c_1 - c_2 = 0 \\ q c_1 - \tilde{\lambda}_2 c_2 = 0 \end{cases} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \tilde{\lambda}_1 \end{pmatrix}$$

$$\implies \begin{pmatrix} P \\ (Pa) \end{pmatrix} = e^{\tilde{\lambda}_2 t} \begin{pmatrix} 1 \\ \tilde{\lambda}_1 \end{pmatrix} = \begin{pmatrix} e^{-\lambda_2 t} \\ -\lambda_1 e^{-\lambda_2 t} \end{pmatrix}$$

Thus, we can take $\boxed{P(t) = e^{-\lambda_2 t}, \quad a(t) = -\lambda_1}$ By the above,

$$e^{-\lambda_2 t}(y''+py'+qy) = \big(e^{-\lambda_2 t}(y'-\lambda_1 y)\big)' = \big(e^{-\lambda_2 t}e^{\lambda_1 t}(e^{-\lambda_1 t}y)'\big)' = \big(e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t}y)'\big)', \tag{5}$$

where λ_1 and λ_2 are the roots of the characteristic polynomial associated to the ODE (eq3). The middle equality above is obtained from our knowledge of an integrating factor for a first-order ODE, especially one with a constant coefficient. The equality of the first and last terms in (eq5) recovers the formula used in the integrating-factor approach to solving any linear second-order ODE with constant coefficients.

(c; **10pts**) *If $p,q,r=const$, find functions $P=P(t)\neq 0$, $R_1=R_1(t)$, $R_2=R_2(t)$, and $R_3=R_3(t)$, such that*

$$\big(R_3\big(R_2(R_1 y)'\big)'\big)' = P(y'''+py''+qy'+ry)$$

for all smooth function $y=y(t)$.

We first find functions $P=P(t)$, $Q=Q(t)$, $a=a(t)$, and $b=b(t)$, such that

$$\begin{aligned} P(y'''+py''+qy'+ry) &= \big(Q(y''+ay'+by)\big)' = Qy''' + (Q'+Qa)y'' + \big((Qa)'+Qb\big)y' + (Qb)'y \\ \iff \qquad & P=Q, \quad P'+Pa=pP, \quad (Pa)'+(Pb)=qP, \quad (Pb)'=rP. \end{aligned}$$

Thus, we need a nonzero solution to the ODE

$$\begin{pmatrix} P \\ (Pa) \\ (Pb) \end{pmatrix}' = \begin{pmatrix} p & -1 & 0 \\ q & 0 & -1 \\ r & 0 & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \\ (Pb) \end{pmatrix} \qquad P=P(t),\ a=a(t),\ b=b(t). \tag{6}$$

The characteristic polynomial for this equation is

$$\begin{aligned} \det\left(\begin{pmatrix} p & -1 & 0 \\ q & 0 & -1 \\ r & 0 & 0 \end{pmatrix} - \lambda I\right) &= \det\begin{pmatrix} p-\lambda & -1 & 0 \\ q & -\lambda & -1 \\ r & 0 & -\lambda \end{pmatrix} \\ &= (p-\lambda)(-\lambda)(-\lambda) + r(-1)(-1) - (-\lambda)q(-1) = -(\lambda^3 - p\lambda^2 + q\lambda - r). \end{aligned}$$

Let $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, and $\tilde{\lambda}_3$ be the roots of this cubic polynomial. Note that $\lambda_1=-\tilde{\lambda}_1$, $\lambda_2=-\tilde{\lambda}_2$, and $\lambda_3=-\tilde{\lambda}_3$ must then be the roots of

$$\lambda^3+p\lambda^2+q\lambda+r=0,$$

i.e. the characteristic polynomial for the third-order ODE $y'''+py''+qy'+ry=f$, since

$$\begin{aligned} &\lambda_1+\lambda_2+\lambda_3=-(\tilde{\lambda}_1+\tilde{\lambda}_2+\tilde{\lambda}_3)=-p, \quad \lambda_1\lambda_2\lambda_3=(-\tilde{\lambda}_1)(-\tilde{\lambda}_2)(-\tilde{\lambda}_3)=-\tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3=-r,\\ \text{and} \qquad &\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=(-\tilde{\lambda}_1)(-\tilde{\lambda}_2)+(-\tilde{\lambda}_1)(-\tilde{\lambda}_3)+(-\tilde{\lambda}_2)(-\tilde{\lambda}_3)=q. \end{aligned}$$

We next find an eigenvector for the eigenvalue $\tilde{\lambda}_3$ of the matrix in (eq6):

$$\begin{pmatrix} p-\tilde{\lambda}_3 & -1 & 0 \\ q & -\tilde{\lambda}_3 & -1 \\ r & 0 & -\tilde{\lambda}_3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} (\tilde{\lambda}_1+\tilde{\lambda}_2)c_1-c_2=0 \\ qc_1-\tilde{\lambda}_3c_2-c_3=0 \\ rc_1-\tilde{\lambda}_3c_3=0 \end{cases} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}=\begin{pmatrix} 1 \\ \tilde{\lambda}_1+\tilde{\lambda}_2 \\ \tilde{\lambda}_1\tilde{\lambda}_2 \end{pmatrix}$$

$$\implies \begin{pmatrix} P \\ (Pa) \\ (Pb) \end{pmatrix}=e^{\tilde{\lambda}_3 t}\begin{pmatrix} 1 \\ \tilde{\lambda}_1+\tilde{\lambda}_2 \\ \tilde{\lambda}_1\tilde{\lambda}_2 \end{pmatrix}=\begin{pmatrix} e^{-\lambda_3 t} \\ -(\lambda_1+\lambda_2)e^{-\lambda_3 t} \\ \lambda_1\lambda_2 e^{-\lambda_3 t} \end{pmatrix}$$

Thus, we can take $P(t)=e^{-\lambda_3 t}$, $a(t)=-(\lambda_1+\lambda_2)$, and $b(t)=\lambda_1\lambda_2$. By the above,

$$\begin{aligned} e^{-\lambda_3 t}(y'''+py''+qy'+ry) &= \big(e^{-\lambda_3 t}(y''-(\lambda_1+\lambda_2)y'+\lambda_1\lambda_2 y)\big)' \\ &= \big(e^{-\lambda_3 t}e^{\lambda_2 t}\big(e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t}y)'\big)'\big)' \\ &= \big(e^{(\lambda_2-\lambda_3)t}\big(e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t}y)'\big)'\big)', \end{aligned} \tag{7}$$

where λ_1, λ_2, and λ_3 are the roots of the characteristic polynomial associated to the ODE

$$y'''+py''+qy'+ry=f.$$

The middle equality in (eq7) is obtained from (eq5). The equality of the first and last terms in (eq7) can be used to solve any linear third-order ODE with constant coefficients.

Can you guess and prove the analogue of (eq7) for linear ODEs with constant coefficients of any order?