Math53: Ordinary Differential Equations
Autumn 2004

Problem Set 4 Solutions

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 5.1:14, 5.4:18,36, and 5.7:22. In 5.1:14 and 5.7:22, complex numbers are used to simplify computations of integrals. In 5.4:36a, a particular solution is found via the complex approach of Section 4.5, while the constants are found from the general real solution. In 5.4:18 and 5.4:36b, "fast" partial fractions are used. In the latter case, they are used along with complex numbers.

Section 5.1: 14,26 (9pts)

5.1:14; 6pts: *Compute the Laplace Transform $F=F(s)$ of the function* $f=e^{at}\sin\omega t$.
We can compute $F(s)$ using two integrations by parts, but the integral becomes far easier if we use Euler's formula:

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{at}\sin\omega t\, e^{-st}dt = \int_0^\infty \left(\mathrm{Im}\, e^{i\omega t}\right) e^{(a-s)t}dt = \mathrm{Im}\int_0^\infty e^{i\omega t}e^{(a-s)t}dt \\
&= \mathrm{Im}\int_0^\infty e^{(a-s+i\omega)t}dt = \frac{1}{a-s+i\omega}e^{(a-s+i\omega)t}\Big|_{t=0}^{t=\infty} = \frac{1}{s-a-i\omega} = \frac{1}{s-a-i\omega}\cdot\frac{s-a+i\omega}{s-a+i\omega} \\
&= \mathrm{Im}\,\frac{s-a+i\omega}{(s-a)^2+\omega^2} = \boxed{\tfrac{\omega}{(s-a)^2+\omega^2} \qquad s>a}
\end{aligned}$$

5.1:26; 3pts: *Compute the Laplace Transform $F=F(s)$ of the function*

$$f(t) = \begin{cases} 0, & \text{if } 0\le t<4; \\ 5, & \text{if } t\ge 4. \end{cases}$$

By definition of the Laplace Transform,

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_4^\infty 5e^{-st}dt = -\frac{5}{s}e^{-st}\Big|_{t=4}^{t=\infty} = \frac{5}{s}e^{-4s} = \boxed{5\tfrac{e^{-4s}}{s}, \qquad s>0}$$

Section 5.2: 24,32,43 (20pts)

5.2:24; 4pts: *Find the Laplace transform $Y=\mathcal{L}(y)$ of the solution y to the initial value problem:*

$$y''+y'+2y=\cos 2t+\sin 3t \quad y(0)=-1, \quad y'(0)=1.$$

Take LT of both sides of this ODE and use the initial conditions:

$$y''+y'+2y = \cos 2t+\sin 3t \implies \big(s^2Y(s)-sy(0)-y'(0)\big)+\big(sY(s)-y(0)\big)+2Y(s) = \mathcal{L}(\cos 2t)+\mathcal{L}(\sin 3t)$$

$$\implies \big(s^2Y(s)+s-1\big) + \big(sY(s)+1\big) + 2Y(s) = \frac{s}{s^2+2^2} + \frac{3}{s^2+3^2}$$

$$\implies (s^2+s+2)Y(s) = -s + \frac{s}{s^2+4} + \frac{3}{s^2+9}$$

$$\implies \boxed{Y(s) = -\frac{s}{s^2+s+2} + \frac{s}{(s^2+4)(s^2+s+2)} + \frac{3}{(s^2+9)(s^2+s+2)}}$$

5.2:32; 4pts: *Find the Laplace transform $Y(t)$ of $y(t)=t^2\cos 2t$*

$$\{\mathcal{L}(t^2\cos t)\}(s) = -\{\mathcal{L}(t\cos t)\}'(s) = \{\mathcal{L}\cos t\}''(s) = \Big(\frac{s}{s^2+4}\Big)'' = \Big(\frac{s^2+4-s(2s)}{(s^2+4)^2}\Big)'$$

$$= \Big(\frac{4-s^2}{(s^2+4)^2}\Big)' = \frac{(-2s)(s^2+4)^2-(4-s^2)2(s^2+4)2s}{(s^2+4)^4} = \boxed{\frac{2s^3-24s}{(s^2+4)^3}}$$

5.2:43 *The gamma function is defined by:*

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt, \quad \alpha > 0.$$

(a; **2pts**) *Prove that $\Gamma(1)=1$.*

$$\Gamma(1) = \int_0^\infty e^{-t}t^{1-1}dt = \lim_{T\longrightarrow\infty}\int_0^T e^{-t}dt = \lim_{T\longrightarrow\infty} -e^{-t}\big|_0^T = -\lim_{T\longrightarrow\infty}(e^{-T}-1) = 1.$$

(b; **5pts**) *Prove that $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$. If n is a positive integer, show that $\Gamma(n+1)=n!$*

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t}t^{\alpha}dt = \lim_{T\longrightarrow\infty}\int_0^T e^{-t}t^\alpha dt = \lim_{T\longrightarrow\infty}\big[-e^{-t}t^\alpha\big|_0^T + \alpha\int_0^T e^{-t}t^{\alpha-1}dt\big]$$

$$= \lim_{T\longrightarrow\infty}\big[-e^{-T}T^\alpha + \alpha\int_0^T e^{-t}t^{\alpha-1}dt\big] = 0+\alpha\int_0^\infty e^{-t}t^{\alpha-1}dt = \alpha\Gamma(\alpha)$$

If n is integer, using the relation $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ n times and $\Gamma(1)=1$, we get:

$$\Gamma(n+1) = n\cdot\Gamma(n) = n(n-1)\cdot\Gamma(n-1) = \cdots = n(n-1)\dots(2)(1)\cdot\Gamma(1) = n!$$

(c; **5pts**) *Show that*

$$\mathcal{L}(t^\alpha)(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

If n is a positive integer, use this result to show that $\mathcal{L}(t^n)(s) = n!/s^{n+1}$.

Using the substitution $u=st$, we get:

$$\{\mathcal{L}t^\alpha\}(s) = \int_0^\infty t^\alpha e^{-st}dt = \int_0^\infty \Big(\frac{u}{s}\Big)^\alpha e^{-u}\frac{du}{s} = \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u}u^{(\alpha+1)-1}du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

If α is a positive integer n, we get:

$$\mathcal{L}(t^n)(s) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Section 5.3: 2,30 (10pts)

5.3:2; 3pts: *Find the inverse Laplace Transform of the function* $Y(s)=\frac{2}{3-5s}$

$$Y(s)=\frac{2}{3-5s}=-\frac{2}{5s-3}=-\frac{2}{5}\cdot\frac{1}{s-(3/5)}\implies y(t)=-\frac{2}{5}e^{(3/5)t}=\boxed{-\tfrac{2}{5}e^{3t/5}}$$

by the fifth row in Table 1 on p250.

5.3:30; 7pts: *Find the inverse Laplace Transform of the function*

$$Y(s)=\frac{7s^2+20s+53}{(s-1)(s^2+2s+5)}$$

Since the quadratic factor does not factor, we first need to find A, B, and C such that

$$\begin{aligned}\frac{7s^2+20s+53}{(s-1)(s^2+2s+5)}&=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}\\&=\frac{A(s^2+2s+5)+(Bs+C)(s-1)}{(s-1)(s^2+2s+5)}=\frac{(A+B)s^2+(2A-B+C)s+(5A-C)}{(s-1)(s^2+2s+5)}.\end{aligned}$$

Thus, we need to solve the system of equations

$$\begin{cases}A+B=7\\2A-B+C=20\\5A-C=53\end{cases}\implies\begin{cases}A+B=7\\7A-B=73\\5A-C=53\end{cases}\implies\begin{cases}A+B=7\\8A=80\\5A-C=53\end{cases}\implies\begin{cases}A=10\\B=-3\\C=-3\end{cases}$$

It follows that

$$\begin{aligned}Y(s)=\frac{7s^2+20s+53}{(s-1)(s^2+2s+5)}&=\frac{10}{s-1}+\frac{-3s-3}{s^2+2s+5}=10\frac{1}{s-1}-3\frac{s+1}{(s+1)^2+2^2}\\&\implies\quad\boxed{y(t)=10e^{t}-3e^{-t}\cos 2t}\end{aligned}$$

by the fifth and seventh rows of Table 1 on p250.

Section 5.4: 18,36 (28pts)

5.4:18; 12pts: *Use the Laplace transform to solve the second-order initial value problem*

$$y''-y'-2y=t^2e^{2t},\quad y(0)=0,\quad y'(0)=-1.$$

Let $\{\mathcal{L}y\}(s)=Y(s)$. Taking LT of both sides and using Table 1 on p250 we get:

$$\begin{aligned}y''-y'-2y=t^2e^{2t}&\implies\big(s^2Y(s)-sy(0)-y'(0)\big)-\big(sY(s)-y(0)\big)-2Y(s)=\frac{2}{(s-2)^3}\\&\implies(s^2-s-2)Y(s)=\frac{2}{(s-2)^3}-1\implies Y(s)=-\frac{1}{(s+1)(s-2)}+\frac{2}{(s+1)(s-2)^4}\end{aligned}$$

We need to find the partial fraction decompositions for the last two fractions. One way to do so is by using the method explained in class, which is far simpler than the standard method:

$$\frac{1}{(s+1)(s-2)} = \frac{1}{1-(-2)}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big) = \frac{1}{3}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big) \quad \Longrightarrow$$

$$\begin{aligned}\frac{2}{(s+1)(s-2)^4} &= \frac{2}{(s-2)^3}\cdot\frac{1}{3}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big) = \frac{2}{3}\cdot\frac{1}{(s-2)^4}-\frac{2}{3}\cdot\frac{1}{(s-2)^2}\cdot\frac{1}{3}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big)\\ &= \frac{2}{3}\cdot\frac{1}{(s-2)^4}-\frac{2}{9}\cdot\frac{1}{(s-2)^3}+\frac{2}{9}\cdot\frac{1}{(s-2)}\cdot\frac{1}{3}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big)\\ &= \frac{2}{3}\cdot\frac{1}{(s-2)^4}-\frac{2}{9}\cdot\frac{1}{(s-2)^3}+\frac{2}{27}\cdot\frac{1}{(s-2)^2}-\frac{2}{27}\cdot\frac{1}{3}\Big(\frac{1}{s-2}-\frac{1}{s+1}\Big).\end{aligned}$$

Combining the two decompositions, we obtain

$$\begin{aligned}Y(s) &= \frac{2}{3}\cdot\frac{1}{(s-2)^4}-\frac{2}{9}\cdot\frac{1}{(s-2)^3}+\frac{2}{27}\cdot\frac{1}{(s-2)^2}-\frac{29}{81}\cdot\frac{1}{s-2}+\frac{29}{81}\cdot\frac{1}{s+1}\\ &\Longrightarrow \quad \boxed{y(t)=\tfrac{29}{81}e^{-t}-\tfrac{29}{81}e^{2t}+\tfrac{2}{27}te^{2t}-\tfrac{1}{9}t^2e^{2t}+\tfrac{1}{9}t^3e^{2t}}\end{aligned}$$

For the standard approach, we would write

$$\begin{aligned}\frac{-1}{(s+1)(s-2)} &= \frac{A}{s+1}+\frac{B}{s-2} = \frac{(A+B)s+(A-2B)}{(s+1)(s-2)}\\ \Longrightarrow \quad A+B=0,\ A-2B=-1 \quad &\Longrightarrow \quad A=\frac{1}{3},\ B=-\frac{1}{3}\end{aligned}$$

For the second fraction, we have:

$$\begin{aligned}\frac{2}{(s+1)(s-2)^4} &= \frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}+\frac{E}{(s-2)^4}\\ \Longrightarrow \quad 2 &= A(s-2)^4+B(s+1)(s-2)^3+C(s+1)(s-2)^2+D(s+1)(s-2)+E(s+1).\end{aligned}$$

Multiplying these out and equating the coefficients of s^k, we get a system of five linear equations in five unknowns. Solve this system and proceed as in the final step of the first approach.

5.4:36 *Solve the initial value problem*

$$y''+y=-2\sin t, \quad y(0)=-1, \quad y'(0)=1.$$

in two ways: first by solving the associated homogeneous equation, and second by using the Laplace transform. Compare the two solutions.

(a; **7pts**) The characteristic polynomial is $r^2+1=0$; its roots are $\pm i$. Thus, the general solution to the associated homogeneous equation is

$$y_h = A_1e^{it}+A_2e^{-it} = C_1\cos t+C_2\sin t.$$

To find a particular solution $y_p(t)$ for $y''+y=-2\sin t$, find a particular solution $z_p(t)$ for $z''+z=-2e^{it}$ and take $y_p=\operatorname{Im} z_p$. Since e^{it} solves the homogeneous equation, we try $z_p(t)=Ate^{it}$:

$$\begin{aligned}z_p(t)=Ate^{it} \;\Longrightarrow\; z_p'=Aite^{it}+Ae^{it},\ z_p''=-Ate^{it}+2iAe^{it} \;&\Longrightarrow\; \big(-Ate^{it}+2iAe^{it}\big)+Ate^{it}=-2e^{it}\\ \Longrightarrow\; A=i \;\Longrightarrow\; z_p(t)=ite^{it}=it(\cos t+i\sin t)=-t\sin t+it\cos t.&\end{aligned}$$

Thus, $y_p(t) = Im(z_p(t)) = t\cos t$, and the general solution to $y''+y = -2\sin t$ is

$$y(t) = y_h(t) + y_p(t) = C_1\cos t + C_2\sin t + t\cos t.$$

Using the initial conditions, we obtain

$$y(0) = C_1 = -1,\ y'(t) = C_2 + 1 = 1 \implies C_1 = -1,\ C_2 = 0 \implies \boxed{y(t) = -\cos t + t\cos t}$$

(b; **9pts**) Let $\{\mathcal{L}y\}(s) = Y(s)$. Then,

$$\begin{aligned} y'' + y = -2\sin t &\implies \big(s^2Y(s) - sy(0) - y'(0)\big) + Y(s) = -\frac{2}{s^2+1} \\ &\implies Y(s) = \frac{-s+1}{s^2+1} - \frac{2}{(s^2+1)^2} = -\frac{s}{s^2+1} - \Big(\frac{s}{s^2+1}\Big)' \\ &\implies Y = -\mathcal{L}(\cos t) + \mathcal{L}(t\cos t) \implies \boxed{y(t) = -\cos t + t\cos t} \end{aligned}$$

The tricky part here is to see what to do with $2/(s^2+1)^2$. One approach is to use known LTs to produce more LTs that look like $1/(s^2+1)^2$. So, take the first derivative of $1/(s^2+1)$ and of $s/(s^2+1)$. These derivatives, multiplied by -1, are the LTs of $t\sin t$ and $t\cos t$. We find that

$$\Big(\frac{s}{s^2+1}\Big)' = \frac{1}{s^2+1} - \frac{s\cdot 2s}{(s^2+1)^2} = \frac{1}{s^2+1} - 2\frac{(s^2+1)-1}{(s^2+1)^2} = -\frac{1}{s^2+1} + \frac{2}{(s^2+1)^2}$$

Another approach is to use *complex* partial fractions, in either of the two ways described in 5.4:18:

$$\frac{1}{s^2+1} = \frac{1}{i-(-i)}\Big(\frac{1}{s-i} - \frac{1}{s+i}\Big) = \frac{1}{2i}\Big(\frac{1}{s-i} - \frac{1}{s+i}\Big) \implies$$

$$\frac{1}{(s^2+1)^2} = \Big(\frac{1}{2i}\Big)^2\Big(\frac{1}{s-i} - \frac{1}{s+i}\Big)^2 = -\frac{1}{4}\Big(\frac{1}{(s-i)^2} + \frac{1}{(s+i)^2}\Big) - \frac{1}{4}\cdot(-2)\cdot\frac{1}{2i}\Big(\frac{1}{s-i} - \frac{1}{s+i}\Big)$$

$$\begin{aligned} \implies \mathcal{L}^{-1}\Big(\frac{1}{(s^2+1)^2}\Big) &= -\frac{1}{4}\big(te^{it} + te^{-it}\big) + \frac{1}{4i}\big(e^{it} - e^{-it}\big) \\ &= -\frac{1}{4}\cdot t\cdot 2\cos t + \frac{1}{4i}\cdot 2i\,\sin t = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t \end{aligned}$$

Yet another way is to use the main relationship between the convolution and the Laplace Transform:

$$\mathcal{L}^{-1}\Big(\frac{1}{(s^2+1)^2}\Big) = \mathcal{L}^{-1}\Big(\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\Big) = \mathcal{L}^{-1}\Big(\frac{1}{s^2+1}\Big) * \mathcal{L}^{-1}\Big(\frac{1}{s^2+1}\Big) = (\sin t) * (\sin t).$$

Section 5.5: 20 (5pts)

Compute the inverse Laplace transform $f = f(t)$ of the function $F(s) = e^{-s}/(s^2+4)$.

$$\big\{\mathcal{L}\big(H(t-a)f(t-a)\big)\big\}(s) = e^{-as}\{\mathcal{L}f\}(s), \quad \mathcal{L}(\sin bt) = \frac{b}{s^2+b} \implies$$

$$f(t) = \Big\{\mathcal{L}^{-1}\Big(\frac{e^{-s}}{s^2+4}\Big)\Big\}(t) = \frac{1}{2}\cdot H(t-1)\sin 2(t-1) = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ \frac{1}{2}\sin 2(t-1), & \text{if } t \ge 1. \end{cases}$$

Section 5.6: 1,8 (20pts)

5.6:1 (a; **4pts**) *Compute the Laplace transform* $\mathcal{L}(\delta_p^\epsilon) = F_p^\epsilon$ *of the function*

$$\delta_p^\epsilon(t) = \epsilon^{-1}\big(H_p(t) - H_{p+\epsilon}(t)\big).$$

By definition of the Laplace Transform,

$$F_p^\epsilon(s) = \int_0^\infty \delta_p^\epsilon(t)e^{-st}\,dt = \int_p^{p+\epsilon} \epsilon^{-1}\cdot e^{-st}\,dt = \frac{1}{-\epsilon s}e^{-st}\Big|_p^{p+\epsilon} = \frac{e^{-ps} - e^{-(p+\epsilon)s}}{\epsilon s} = \boxed{\frac{1-e^{-\epsilon s}}{\epsilon s}e^{-ps}}$$

(b; **4pts**) *Compute* $\lim_{\epsilon\longrightarrow 0} F_p^\epsilon(s)$.
Using l'Hôpital's rule, i.e. differentiating the top and bottom of the above fraction with respect to ϵ, we obtain

$$\lim_{\epsilon\longrightarrow 0} F_p^\epsilon(s) = \lim_{\epsilon\longrightarrow 0} \frac{1-e^{-\epsilon s}}{\epsilon s}e^{-ps} = \lim_{\epsilon\longrightarrow 0} \frac{s\cdot e^{-\epsilon s}}{s}e^{-ps} = \boxed{e^{-ps}} = \mathcal{L}(\delta_p)$$

5.6:8 (a; **5pts**) *Use the Laplace Transform to find the solution* $y_0 = y_0(t)$ *to the initial value problem*

$$y'' = 2\delta, \qquad y(0) = y'(0) = 0.$$

Sketch the solution curve for this IVP.
Taking the Laplace Transform of both sides and then the Inverse Laplace Transform, we obtain

$$s^2Y - sy(0) - y'(0) = 2 \implies s^2Y = 2 \implies Y(s) = 2\cdot\frac{1}{s^2} \implies y_0(t) = \begin{cases} 2t, & \text{if } t\geq 0; \\ 0, & \text{if } t<0, \end{cases}$$

by Table 1 on p250. Note that the Inverse Laplace Transform of any function is zero for negative values of t. We conclude that $\boxed{y_0(t) = 2t\cdot H(t)}$ The corresponding solution curve is shown below.

(b,c; **7pts**) *Use the Laplace Transform to find the solution* y_0^ϵ *to the initial value problem*

$$y'' = 2\delta_0^\epsilon, \qquad y(0) = y'(0) = 0.$$

Sketch the solution curve for this IVP.
Taking the Laplace Transform of both sides, with the help of 5.6:1a, and then the Inverse Laplace

Transform, we obtain

$$s^2Y_0^\epsilon - sy(0) - y'(0) = 2\frac{1-e^{-\epsilon s}}{\epsilon s} \quad\Longrightarrow\quad Y_0^\epsilon = \frac{2}{\epsilon}\cdot\frac{1}{s^3} - \frac{2}{\epsilon}\cdot\frac{e^{-\epsilon s}}{s^3}$$
$$\Longrightarrow\quad y_0^\epsilon(t) = \frac{1}{\epsilon}t^2 - \frac{1}{\epsilon}H(t-\epsilon)\cdot(t-\epsilon)^2 = \begin{cases} 2t-\epsilon, & \text{if } t\geq\epsilon; \\ \epsilon^{-1}t^2, & \text{if } 0\leq t\leq\epsilon; \\ 0, & \text{if } t<0, \end{cases}$$

by Table 1 on p250 and Proposition 5.6 on p. 250. Thus, $y_0^\epsilon(t)\longrightarrow y_0(t)$ as $\epsilon\longrightarrow 0$ for all t, i.e. $y_0^\epsilon\longrightarrow y_0$ as $\epsilon\longrightarrow 0$ *pointwise.*

Section 5.7: 14,22,28 (30pts)

5.7:14; 12pts: *Let $f(t)=e^{3t}$ and $g(t)=t^2$. Compute $f*g$ using the definition of convolution. Then find Laplace transforms $F=\mathcal{L}f$, $G=\mathcal{L}g$, and $\mathcal{L}(f*g)$, and check that $\mathcal{L}(f*g)=F\cdot G$ holds.*

$$\begin{aligned} f*g(t) &= \int_0^t f(t)g(t-u)\,du = \int_0^t e^{3u}(t-u)^2du = \frac{1}{3}\Big(e^{3u}(t-u)^2\big|_0^t - \int_0^t e^{3u}(-2(t-u))du\Big) \\ &= -\frac{1}{3}t^2 + \frac{2}{3}\int_0^t e^{3u}(t-u)du = -\frac{1}{3}t^2 + \frac{2}{9}\Big(e^{3u}(t-u)\big|_0^t - \int_0^t e^{3u}(-1)du\Big) \\ &= -\frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27} + \frac{2}{27}e^{3t} \qquad\Longrightarrow \end{aligned}$$
$$\{\mathcal{L}(f*g)\}(s) = -\frac{2}{3s^3} - \frac{2}{9s^2} - \frac{2}{27s} + \frac{2}{27(s-3)} = \frac{2}{(s-3)s^3} = \frac{1}{s-3}\cdot\frac{2}{s^3} = F(s)G(s),$$

by Table 1 on p250.

5.7:22; 10pts: *Use the formula for LT of convolution to find the inverse Laplace transform of the function*

$$Y(s)=\frac{1}{(s+1)(s^2+4)}$$

Using the third and fifth rows of Table 1 on p.250, we get that:

$$Y(s) = \frac{1}{2}\cdot\frac{1}{s+1}\cdot\frac{2}{s^2+4} = \frac{1}{2}\mathcal{L}(e^{-t})\cdot\mathcal{L}(\sin 2t) \qquad\Longrightarrow$$
$$\begin{aligned} \mathcal{L}^{-1}(Y) &= \frac{1}{2}(e^{-t})*(\sin 2t) = \frac{1}{2}\int_0^t e^{-(t-u)}\sin 2u\,du = \frac{1}{2}e^{-t}\,\mathrm{Im}\int_0^t e^{(1+2i)u}du \\ &= \frac{1}{2}e^{-t}\,\mathrm{Im}\Big(\frac{1}{1+2i}e^{(1+2i)u}\big|_0^t\Big) = \frac{1}{2}e^{-t}\,\mathrm{Im}\Big(\frac{1-2i}{5}(e^{(1+2i)t}-1)\Big) \\ &= \frac{e^{-t}}{10}\big(e^t\sin 2t - 2e^t\cos 2t - 2\big) = \boxed{\tfrac{1}{5}e^{-t} - \tfrac{1}{5}\cos 2t + \tfrac{1}{10}\sin 2t} \end{aligned}$$

5.7:28; 8pts: *Find the solution to the initial value problem*

$$y'' + 5y' + 4y = g(t), \quad y(0)=1, \quad y'(0)=0$$

where g is a piecewise continuous function.
First, compute the impulse response function $e(t)$ satisfying:

$$e'' + 5e' + 4e = \delta(t), \quad e(0) = 0, \quad e'(0) = 0.$$

Its Laplace transform $E(s)$ is given by:

$$\begin{aligned} E(s) &= \frac{1}{P(s)} = \frac{1}{s^2+5s+4} = \frac{1}{(s+1)(s+4)} = \frac{1}{3}\Big(\frac{1}{s+1} - \frac{1}{s+4}\Big) \\ \Longrightarrow \quad & e(t) = \frac{1}{3}\big(e^{-t} - e^{-4t}\big) \quad \Longrightarrow \quad e'(t) = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}. \end{aligned}$$

The solution $y(t)$ to the initial value problem is then:

$$\begin{aligned} y(t) &= \{e*g\}(t) + y(0)e'(t) + \big(y'(0)+5y(0)\big)e(t) = \int_0^t e(u)g(t-u)\,du + e'(t) + 5e(t) \\ &= \frac{1}{3}\int_0^t \big(e^{-u} - e^{-4u}\big)g(t-u)\,du + \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}. \end{aligned}$$

Problem C; 10pts

(a; **4pts**) *If g is a piecewise continuous function on the real line, show that the operators given by*

$$T_g f = \int_{-\infty}^{\infty} g(t)f(t)\,dt, \quad f \in C_c^\infty(\mathbb{R}), \qquad \text{and} \qquad T_g' f = -\int_{-\infty}^{\infty} g(t)f'(t)\,dt \quad f \in C_c^\infty(\mathbb{R}),$$

are well-defined distributions.
We first need to check that for every compactly supported function f both integrals are finite real numbers. Since f is compactly supported, there exist a and b such that $f(t)=0$ for all $t \le a$ and for all $t \ge b$. Thus,

$$T_g f = \int_{-\infty}^{\infty} g(t)f(t)\,dt = \int_a^b g(t)f(t)\,dt \quad \text{and} \quad T_g' f = -\int_{-\infty}^{\infty} g(t)f'(t)\,dt = -\int_a^b g(t)f'(t)\,dt.$$

Since the functions $g(t)f(t)$ and $g(t)f'(t)$ are piecewise continuous on $[a,b]$, the two integrals are finite.
We also need to check the linearity property:

$$\begin{aligned} T_g(\alpha f_1 + \beta f_2) &= \int_{-\infty}^{\infty} g(t)\big(\alpha f_1(t) + \beta f_2(t)\big)\,dt \\ &= \alpha\int_{-\infty}^{\infty} g(t)f_1(t)\,dt + \beta\int_{-\infty}^{\infty} g(t)f_2(t)\,dt = \alpha T_g(f_1) + \beta T_g(f_2); \\ T_g'(\alpha f_1 + \beta f_2) &= -\int_{-\infty}^{\infty} g(t)\big(\alpha f_1(t) + \beta f_2(t)\big)'\,dt \\ &= -\alpha\int_{-\infty}^{\infty} g(t)f_1'(t)\,dt - \beta\int_{-\infty}^{\infty} g(t)f_2'(t)\,dt = \alpha T_g'(f_1) + \beta T_g'(f_2). \end{aligned}$$

(b; **6pts**) *Use integration by parts to show that if g is a continuous function with a piecewise continuous first derivative and H is the Heaviside function, then*

$$T_g' f = T_{g'} f \quad \text{and} \quad T_H' f = T_\delta f \qquad \text{for all} \quad f \in C_c^\infty(\mathbb{R}).$$

Since f is compactly supported, there exist a and b such that $f(t)=0$ for all $t \le a$ and for all $t \ge b$. Thus,

$$\begin{aligned} T_g' f &= -\int_{-\infty}^{\infty} g(t) f'(t)\, dt = -\int_a^b g(t) f'(t)\, dt = -\Big(f(t)g(t)\Big|_{t=a}^{t=b} - \int_a^b g'(t) f(t)\, dt \Big) \\ &= -\big(0\cdot g(b) - 0\cdot g(a)\big) + \int_a^b g'(t) f(t)\, dt = \int_a^b g'(t) f(t)\, dt = T_{g'} f, \end{aligned}$$

as claimed. In order to prove the second identity, we assume that $b>0$. Since $H(t)=0$ for $t<0$ and $H(t)=1$ for $t \ge 1$,

$$T_H' f = -\int_{-\infty}^{\infty} H(t) f'(t)\, dt = -\int_0^b f'(t)\, dt = -\big(f(b) - f(0)\big) = f(0) = T_\delta f.$$

Problem D (18pts)

(a; **14pts**) *Use the integrating-factor approach to second-order linear ODEs to find a solution $y_p = y_p(t)$ to*

$$y'' + py' + qy = f, \qquad f = f(t), \tag{1}$$

in the form $y_p = G * f$ for some function $G = G(t)$.
If λ_1 and λ_2 are the two roots of the quadratic equation $\lambda^2 + p\lambda + q = 0$, the ODE (1) is equivalent to

$$\big(e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t} y)'\big)' = e^{-\lambda_2 t}(y'' + py' + qy) = e^{-\lambda_2 t} f.$$

Integrating both sides of this identity from 0, we obtain

$$e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t} y)' = \int_0^t e^{-\lambda_2 u} f(u)\, du \quad \Longrightarrow \quad (e^{-\lambda_1 t} y)' = e^{(\lambda_2-\lambda_1)t} \int_0^t e^{-\lambda_2 u} f(u)\, du.$$

Integrating from 0 once more gives

$$e^{-\lambda_1 t} y(t) = \int_0^t e^{(\lambda_2-\lambda_1)v} \int_0^v e^{-\lambda_2 u} f(u)\, du\, dv \quad \Longrightarrow \quad y_p(t) = e^{\lambda_1 t} \int_0^t \int_0^v e^{(\lambda_2-\lambda_1)v} e^{-\lambda_2 u} f(u)\, du\, dv.$$

The last double integral is taken over all (u, v) such that $0 \le u \le v \le t$. Thus, interchanging the order of integration, we obtain

$$\begin{aligned} y_p(t) &= e^{\lambda_1 t} \int_0^t \int_0^v e^{(\lambda_2-\lambda_1)v} e^{-\lambda_2 u} f(u)\, du\, dv \\ &= e^{\lambda_1 t} \int_0^t \int_u^t e^{(\lambda_2-\lambda_1)v} e^{-\lambda_2 u} f(u)\, dv\, du = e^{\lambda_1 t} \int_0^t e^{-\lambda_2 u} f(u) \Big(\int_u^t e^{(\lambda_2-\lambda_1)v}\, dv \Big)\, du. \end{aligned} \tag{2}$$

Our next step is to evaluate the inner integral in (2), but there are two cases. First, if $\lambda_1 = \lambda_2$,

$$y_p(t) = e^{\lambda_1 t} \int_0^t e^{-\lambda_2 u} f(u) \Big(\int_u^t 1\, dv \Big)\, du = \int_0^t e^{\lambda_1(t-u)} (t-u) f(u)\, du \qquad \text{if} \quad \lambda_1 = \lambda_2. \tag{3}$$

If $\lambda_1 \neq \lambda_2$, (2) gives

$$\begin{aligned} y_p(t) &= e^{\lambda_1 t} \int_0^t e^{-\lambda_2 u} f(u) \frac{e^{(\lambda_2-\lambda_1)t} - e^{(\lambda_2-\lambda_1)u}}{\lambda_2 - \lambda_1}\, du \\ &= \int_0^t \frac{e^{\lambda_2(t-u)} - e^{\lambda_1(t-u)}}{\lambda_2 - \lambda_1} f(u)\, du, \qquad\qquad \text{if} \quad \lambda_1 \neq \lambda_2. \end{aligned} \tag{4}$$

If $\lambda_1, \lambda_2 = a+ib$ are complex, the fraction above involves complex numbers, but

$$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = \frac{e^{(a+ib)t} - e^{(a-ib)t}}{(a+ib) - (a-ib)} = e^{at} \frac{e^{ibt} - e^{-ibt}}{2ib} = \frac{e^{at} \sin bt}{b}. \tag{5}$$

Combining (2)-(5), we conclude that a particular solution y_p to the ODE (1) is given by

$$\boxed{y_p = G * f, \qquad \text{where} \qquad G(t) = \begin{cases} te^{\lambda_1 t}, & \text{if } \lambda_1 = \lambda_2 \text{ or } p^2 = 4q; \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, & \text{if } \lambda_1 \neq \lambda_2 \text{ are real, or } p^2 > 4q; \\ \frac{e^{at} \sin bt}{b}, & \text{if } \lambda_1, \lambda_2 = a \pm ib \text{ are complex, or } p^2 < 4q. \end{cases}}$$

(b; **4pts**) *Compare your expression for y_p with that for y_s in Theorem 7.16, on p293.*
By Theorems 6.10 and 7.16, a particular solution to the ODE (1) is given by

$$y_s = e * f, \qquad \text{where} \qquad e = \mathcal{L}^{-1}(E), \quad E(s) = \frac{1}{s^2 + ps + q}.$$

Thus, we need to determine the inverse Laplace transform of E. If $\lambda_1 = \lambda_2$,

$$E(s) = \frac{1}{s^2 + ps + q} = \frac{1}{(s - \lambda_1)^2} \quad \Longrightarrow \quad e = \mathcal{L}^{-1}(E) = te^{\lambda_1 t}, \qquad \text{if} \quad \lambda_1 = \lambda_2, \tag{6}$$

by the last row of Table 1, on p250. On the other hand, if $\lambda_1 \neq \lambda_2$,

$$\begin{aligned} E(s) &= \frac{1}{s^2 + ps + q} = \frac{1}{\lambda_1 - \lambda_2} \Big(\frac{1}{s - \lambda_1} - \frac{1}{s - \lambda_2} \Big) \\ \Longrightarrow \quad e(s) &= \frac{1}{\lambda_1 - \lambda_2} \big(e^{\lambda_1 t} - e^{\lambda_2 t} \big) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \qquad\qquad \text{if} \quad \lambda_1 \neq \lambda_2. \end{aligned} \tag{7}$$

Comparing (6) and (7) with (3) and (4), we conclude that the expression for y_p obtained in part (a) is exactly the same as the expression for y_s in Theorem 7.16:

$$\boxed{y_p = y_s = e * f, \qquad \text{where} \qquad e(t) = G(t)}$$

Remark: In part (a), instead of changing the order of integration, one could observe that the double integral is

$$(f * e^{\lambda_2 t}) * e^{\lambda_1 t} = f * (e^{\lambda_2 t} * e^{\lambda_1 t}).$$

Thus, $G = e^{\lambda_2 t} * e^{\lambda_1 t}$. The above equality uses the fact that $(f*g)*h = f*(g*h)$; its proof involves changing the order of integration in a double integral.