Math53: Ordinary Differential Equations
Autumn 2004

Solutions to Problem Set 3

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 4.5:26 and 4.6:13. In the first problem, complex numbers are used to simplify computations. In the second problem, the variation of parameters method itself is applied, instead of the final formulas given in the book.

Section 4.1, Problems 12,14 (15pts)

4.1:12; 5pts: *Show that $y_1(t)=e^{-t}\cos 2t$ and $y_2(t)=e^{-t}\sin 2t$ form a fundamental set of solutions for*

$$y''+2y'+5y=0.$$

Find a solution satisfying $y(0)=-1$ and $y'(0)=0$.
The functions $y_1(t)$ and $y_2(t)$ are linearly independent, since $\tan 2t=y_2(t)/y_1(t)$ is not a constant function. Thus, in order to prove the first statement, we only need to check that $y_1(t)$ and $y_2(t)$ solve the ODE:

$$\begin{aligned}
y_1'(t) &= e^{-t}\big(-2\sin 2t-\cos 2t\big) &\Longrightarrow\quad y_1''(t) &= e^{-t}\big(-4\cos 2t+2\sin 2t+2\sin 2t+\cos 2t\big)\\
&&&= e^{-t}\big(4\sin 2t-3\cos 2t\big);\\
y_2'(t) &= e^{-t}\big(2\cos 2t-\sin 2t\big) &\Longrightarrow\quad y_2''(t) &= e^{-t}\big(-4\sin 2t-2\cos 2t-2\cos 2t+\sin 2t\big)\\
&&&= -e^{-t}\big(4\cos 2t+3\sin 2t\big).
\end{aligned}$$

Plugging these expressions into the ODE, we obtain

$$\begin{aligned}
y_1''+2y_1'+5y_1 &= e^{-t}\big(4\sin 2t-3\cos 2t-4\sin 2t-2\cos 2t+5\cos 2t\big)=0;\\
y_2''+2y_2'+5y_2 &= e^{-t}\big(-4\cos 2t-3\sin 2t+4\cos 2t-2\sin 2t+5\sin 2t\big)=0,
\end{aligned}$$

as needed. Thus, $y=C_1y_1+C_2y_2$ is the general solution of the ODE. For the initial-value problem, we need to find C_1 and C_2 such that $y(0)=-1$ and $y'(0)=0$. Using the above expressions for y_1' and y_2', we find that

$$y(0)=C_1=-1 \qquad \text{and} \qquad y'(0)=-C_1+2C_2=0.$$

Thus, $C_2=-1/2$, and the solution to the initial value problem is $\boxed{y(t)=-e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t}$

4.1:14 (a; **2pts**) *Show that $y_1(t)=t^2$ is a solution of*

$$t^2y''+ty'-4y=0. \tag{1}$$

We need to plug in y_1 into (1). Since $y_1'=2t$ and $y_1''=2$,

$$t^2y_1''+ty_1'-4y_1=t^2\cdot 2+t\cdot 2t-4\cdot t^2=0,$$

as needed.
(b; **8pts**) *Let $y_2(t)=v(t)y_1(t)=v(t)t^2$. Show that y_2 is a solution of (1) if and only if v satisfies*

$$5v' + tv'' = 0. \tag{2}$$

Solve this equation for v and describe the general solution of (1).
We need to plug in y_2 into (1):

$$y_2'(t) = t^2v'(t) + 2tv(t) \implies y_2''(t) = t^2v''(t) + 2tv'(t) + 2tv'(t) + 2v(t) = t^2v'' + 4tv' + 2v$$
$$\implies \quad 0 = t^2y_2'' + ty_2' - 4y_2 = \big(t^4v'' + 4t^3v' + 2t^2v\big) + \big(t^3v' + 2t^2v\big) - 4t^2v = t^4v'' + 5t^3v'.$$

Dividing the last expression by t^3, we obtain (2). In order to solve (2), we first divide this equation by t and then multiply by the integrating factor $e^{\int(5/t)dt}=|t|^5$, or just by t^5:

$$v'' + 5t^{-1}v' = 0 \implies t^5v'' + 5t^4v' = 0 \implies (t^5v')' = 0 \implies t^5v'(t) = C_1$$
$$\implies v'(t) = C_1t^{-5} \implies v(t) = -\frac{C_1}{4}t^{-4} + C_2.$$

Since we need to find a single non-constant solution of (2), we can take

$$v(t) = t^{-4} \qquad \text{and} \qquad y_2(t) = v(t)y_1(t) = t^{-4}t^2 = t^{-2}.$$

The general solution of (1) is thus given by $\boxed{y(t) = C_1t^2 + C_2t^{-2}}$

Section 4.2, Problems 4 (3pts)

Use the substitution $v=y'$ to write the second-order ODE

$$y'' + 2y' + 2y = \sin 2\pi t$$

as a system of two first-order equations.
Since $v = y'$,

$$v' = y'' = -2y' - 2y + \sin 2\pi t = -2v - 2y + \sin 2\pi t.$$

Thus, the above second-order ODE is equivalent to the system

$$\begin{cases} y' = v \\ v' = -2v - 2y + \sin 2\pi t. \end{cases}$$

Section 4.5, Problems 2,6,16,18,26,30,32,42 (70pts)

4.5:2; 4pts: *Using an exponential forcing term, find a particular solution of the equation*

$$y'' + 6y' + 8y = -3e^{-t}.$$

We look for a solution of the form $y_p(t) = Ae^{-t}$. After plugging in

$$y_p(t) = Ae^{-t}, \qquad y_p'(t) = -Ae^{-t}, \qquad y_p''(t) = Ae^{-t},$$

into the equation, we obtain

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t} \implies 3Ae^{-t} = -3e^{-t} \implies A = -1.$$

Thus, a solution of the ODE is $\boxed{y(t) = -e^{-t}}$

4.5:6; 6pts: *Use the form $y = a\cos\omega t + b\sin\omega t$ to find a particular solution of the equation*

$$y'' + 9y = \sin 2t.$$

Let $y_p(t) = a\cos 2t + b\sin 2t$. After plugging in

$$y_p(t) = a\cos 2t + b\sin 2t, \qquad y_p'(t) = -2a\sin 2t + 2b\cos 2t, \qquad y_p''(t) = -4a\cos 2t - 4b\sin 2t,$$

into the equation, we obtain

$$\begin{aligned} & -4a\cos 2t - 4b\sin 2t + 9a\cos 2t + 9b\sin 2t = \sin 2t \\ \implies \; & 5a\cos 2t + 5b\sin 2t = \sin 2t \implies a = 0, \; b = \frac{1}{5} \end{aligned}$$

A particular solution is $\boxed{y(t) = \frac{1}{5}\sin 2t}$

4.5:16; 8pts: *Find a particular solution of the equation*

$$y'' + 5y' + 6y = 4 - t^2$$

The forcing term is a quadratic polynomial, so we look for a particular solution of the form

$$y_p(t) = at^2 + bt + c \implies y_p'(t) = 2at + b \implies y_p''(t) = 2a.$$

The equation becomes:

$$\begin{aligned} y'' + 5y' + 6y = 4 - t^2 \; & \implies \; 2a + 5(2at + b) + 6(at^2 + bt + c) = 4 - t^2 \\ & \implies \; 6at^2 + (10a + 6b)t + (2a + 5b + 6c) = -t^2 + 4. \end{aligned}$$

Thus, a, b, c must satisfy:

$$6a = -1, \quad 10a + 6b = 0, \quad 2a + 5b + 6c = 4 \implies a = -\frac{1}{6}, \; b = \frac{5}{18}, \; c = \frac{53}{108}.$$

So, a particular solution is $\boxed{y_p(t) = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}}$

4.5:18; 10pts: *For the equation*

$$y'' + 3y' + 2y = 3e^{-4t},$$

first solve the associated homogeneous equation, then find a particular solution. Using Theorem 5.2, form the general solution, and then find the solution satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$.

The characteristic polynomial for the homogeneous equation $y'' + 3y' + 2 = 0$ is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2).$$

Its zeros are $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, the homogeneous solution is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

The trial solution is $y_p = Ae^{-4t}$; then

$$y_p' = -4Ae^{-4t} \qquad \text{and} \qquad y_p'' = 16Ae^{-4t}.$$

Substituting into the inhomogeneous ODE, we get

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t} \quad \Longrightarrow \quad 6A = 3 \quad \Longrightarrow \quad A = \frac{1}{2}$$

Thus, a particular solution is $y_p(t) = \frac{1}{2}e^{-4t}$. By Theorem 5.2, the general solution is

$$\boxed{y = C_1 e^{-t} + C_2 e^{-2t} + \tfrac{1}{2}e^{-4t}}$$

The given initial conditions imply:

$$y(0) = C_1 + C_2 + \frac{1}{2} = 1, \;\; y'(0) = -C_1 - 2C_2 - 2 = 0 \qquad \Longrightarrow \qquad C_1 = 3, \; C_2 = -5/2.$$

So, the solution to the initial value problem is $\boxed{y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}}$

4.5:26; 10pts: *In the equation $y'' + 4y = 4\cos 2t$, the forcing term is also a solution of the associated homogeneous equation. Use this to find a particular solution.*
Our strategy is to look at the equation $z'' + 4z = e^{2it}$, of which the given equation is the real part. The characteristic equation for the homogeneous equation $z'' + 4z = 0$ is $\lambda^2 + 4 = 0$. Its roots are $\pm 2i$. So, the homogeneous solution is:

$$z_h = C_1 e^{2it} + C_2 e^{-2it}.$$

The forcing term of $z'' + 4z = 4e^{2it}$ is also a solution of the homogeneous equation. Thus, we try to find a particular solution of the form $z_p = Ate^{2it}$:

$$z_p = Ate^{2it} \quad \Longrightarrow \quad z_p' = Ae^{2it}(1 + 2it) \quad \Longrightarrow \quad z_p'' = 4Ae^{2it}(i - t).$$

After substituting these into $z'' + 4z = 4e^{2it}$, we get:

$$4Ae^{2it}(i - t) + 4Ate^{2it} = 4e^{2it} \quad \Longrightarrow \quad 4iA = 4 \quad \Longrightarrow \quad A = \frac{1}{i} = -i$$
$$\Longrightarrow \quad z_p = -ite^{2it} = -it(\cos 2t + i\sin 2t) = t\sin 2t - it\cos 2t.$$

Its real part is a particular solution we are looking for: $\boxed{y_p = Re(z_p) = t\sin 2t}$

4.5:30; 8pts: *If $y_f(t)$ and $y_g(t)$ are solutions of*

$$y'' + py' + qy = f(t) \qquad \textit{and} \qquad y'' + py' + qy = g(t),$$

respectively, show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t),$$

where α and β are any real numbers.
We are given that:

$$y_f'' + py_f' + qy_f = f(t) \qquad \text{and} \qquad y_g'' + py_g' + qy_g = g(t).$$

We plug in $z(t)$ into $y''+py'+qy=\alpha f(t)+\beta g(t)$ and use these two properties of y_f and y_g:

$$\begin{aligned} z'' + pz' + qz &= (\alpha y_f + \beta y_g)'' + p(\alpha y_f + \beta y_g)' + q(\alpha y_f + \beta y_g) \\ &= (\alpha y_f'' + \beta y_g'') + p(\alpha y_f' + \beta y_g') + q(\alpha y_f + \beta y_g) \\ &= \alpha(y_f'' + py_f' + qy_f) + \beta(y_g'' + py_g' + qy_g) \\ &= \alpha f(t) + \beta g(t). \end{aligned}$$

Thus, $z(t)=\alpha y_f(t)+\beta y_g(t)$ is a solution of $y''+py'+qy=\alpha f(t)+\beta g(t)$.

4.5:32; 12pts: *Use the previous exercise to find a particular solution of the equation*

$$y'' - y = t - e^{-t}.$$

The forcing term is the linear combination $t-e^{-t} = 1 \cdot t + (-1)e^{-t}$. We first find a particular solution y_{p_1} of $y''-y=t$, and then a particular solution y_{p_2} of $y''-y = -e^{-t}$. By the previous exercise, $y_{p_1}-y_{p_2}$ will be a particular solution to our equation. To find y_{p_1}, substitute $y=at+b$ into

$$y'' - y = t \implies -at - b = t \implies a = -1,\ b = 0 \implies y_{p_1}(t) = -t.$$

To find y_{p_2}, note that the characteristic equation for the homogeneous equation $y''-y=0$ is $\lambda^2-1=0$. Its roots are $\lambda_1 = -1$ and $\lambda_2=1$, giving the homogeneous solution

$$y_h = C_1 e^{-t} + C_2 e^t.$$

It follows that the forcing term e^{-t} is a solution of the homogeneous equation. So we try to find y_{p_2} of the form $y_{p_2}(t)=Ate^{-t}$:

$$y_{p_2} = Ate^{-t} \implies y_{p_2}' = Ae^{-t}(1-t) \implies y_{p_2}'' = Ae^{-t}(t-2).$$

The equation now becomes:

$$e^{-t} = y_{p_2}'' - y_{p_2} = Ae^{-t}(t-2) - Ate^{-t} \implies -2A = 1 \implies A = -\frac{1}{2} \implies y_{p_2}(t) = -\frac{1}{2}te^{-t}.$$

So a particular solution of $y''-y=t-e^{-t}$ is $\boxed{y_p = y_{p_1} - y_{p_2} = -t + \frac{1}{2}te^{-t}}$

4.5:42; 12pts: *Find a particular solution of the equation* $y'' + 5y' + 4y = te^{-t}$.
The characteristic polynomial for the corresponding homogeneous equation $y''+5y'+4=0$ is

$$\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4).$$

Its roots are $\lambda_1 = -1$ and $\lambda_2 = -4$. Thus, the homogeneous solution is

$$y_h = C_1 e^{-4t} + C_2 e^{-t}.$$

In particular, e^{-t} is a solution to the homogeneous equation. Thus, we modify the hint in Exercise 39, and look for a particular solution of the form $y_p = t(at+b)e^{-t}$:

$$\begin{aligned} y_p(t) = t(at+b)e^{-t} \quad &\Longrightarrow \quad y_p'(t) = (-at^2 + (2a-b)t + b)e^{-t} \\ &\Longrightarrow \quad y_p''(t) = (at^2 + (-4a+b)t + (2a-2b))e^{-t} \end{aligned}$$

Substituting, we get:

$$te^{-t} = y'' + 5y' + 4y = (6at + (2a+3b))e^{-t} \quad \Longrightarrow \quad 6a = 1,\ 2a + 3b = 0 \quad \Longrightarrow \quad a = \frac{1}{6},\ b = -\frac{1}{9}.$$

Thus, a solution of $y'' + 5y' + 4y = te^{-t}$ is $\boxed{y_p = \frac{1}{6}t^2e^{-t} - \frac{1}{9}te^{-t}}$

Section 4.6, Problem 13 (12pts)

Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0.$$

Use variation of parameters to find the general solution to

$$t^2y'' + 3ty' - 3y = t^{-1}.$$

For the first part, plug in $y_1(t) = t$ and $y_2(t) = t^{-3}$ into the homogeneous equation:

$$\begin{aligned} y_1 = t,\ \ y_1' = 1,\ \ y_1'' = 0 \quad &\Longrightarrow \quad t^2y_1'' + 3ty_1' - 3y_1 = t^2 \cdot 0 + 3t \cdot 1 - 3 \cdot t = 0; \\ y_1 = t^{-3},\ y_1' = -3t^{-4},\ y_1'' = 12t^{-5} \quad &\Longrightarrow \quad t^2y_2'' + 3ty_2' - 3y_2 = t^2 \cdot (12t^{-5}) + 3t \cdot (-3t^{-4}) - 3t^{-3} = 0, \end{aligned}$$

as needed. We look for a solution to the inhomogeneous equation of the form

$$y_p = v_1y_1 + v_2y_2 = tv_1 + t^{-3}v_2 \qquad \Longrightarrow \qquad y_p' = (tv_1' + t^{-3}v_2') + v_1 - 3t^{-4}v_2.$$

We set the expression in the parenthesis to zero. Thus,

$$y_p' = v_1 - 3t^{-4}v_2 \quad \Longrightarrow \quad y_p'' = v_1' + 12t^{-5}v_2 - 3t^{-4}v_2' \quad \Longrightarrow \quad t^2y_p'' + 3ty_p' - 3y_p = t^2v_1' - 3t^{-2}v_2' = t^{-1}.$$

Since we also assumed that $tv_1' + t^{-3}v_2' = 0$, we need to solve the system

$$\begin{cases} v_1' + t^{-4}v_2' = 0 \\ v_1' - 3t^{-4}v_2' = t^{-3} \end{cases} \qquad \Longrightarrow \qquad v_1' = \frac{1}{4}t^{-3},\ v_2' = -\frac{1}{4}t \quad \Longrightarrow \quad v_1 = -\frac{1}{8}t^{-2},\ v_2 = -\frac{1}{8}t^2.$$

Note that we are looking for only one pair of (v_1, v_2). We conclude that

$$y_p = v_1y_1 + v_2y_2 = -\frac{1}{8}t^{-2} \cdot t - \frac{1}{8}t^2 \cdot t^{-3} = -\frac{1}{4}t^{-1}$$

is a (particular) solution of the inhomogeneous ODE, while the general solution is

$$\boxed{y(t) = C_1t + C_2t^{-3} - \frac{1}{4}t^{-1}}$$