Math53: Ordinary Differential Equations
Autumn 2004

Homework Assignment 4

Problem Set 4 is due by 2:15p.m. on Monday, 11/1, in MuddChem 101

Problem Set 4:

5.1: 14,26; 5.2: 24,32,43; 5.3: 2,30; 5.4: 18,36; 5.5: 20; 5.6: 1,8; 5.7: 14,22,28
Problems C,D (see next page)

Daily Assignments:

Date	*Read*	*Exercises*
10/21 R	5.1, 5.2	5.1:14,26; 5.2:24,32,43
10/22 F	5.3	5.3:2,30
10/25 M	5.4	5.4:18,36
10/26 T	5.5	5.5:20
10/27 W	5.6,5.7	5.6:1,8; 5.7:14,22,28
10/28 R	5.8	Problems C,D

Note 1: You do not need to memorize Table 1 on p250. It will be put on the second midterm and on the final exam. However, you should know how to use it and how to compute the Laplace transforms that appear in this table yourself.

Note 2: While the statement of Problem C is long, its solution uses only Math42. This problem can in principle be done any day, but its purpose is to complement Sections 5.6 and 5.7.
Hint: Read this problem carefully. It is mostly a conceptual problem, like 4.5:30.

Hint for Problem D: Be careful with the limits and variables of integration in the integrals you set up.

Problem C

A smooth function f defined on the entire real line is called *compactly supported* if there exist a and b such that $f(t)=0$ for all $t<a$ and for all $t>b$. In other words, compactly supported functions are nonzero only in the "middle" of the real line. The set of all such functions is denoted by $C_c^\infty(\mathbb{R})$. A *distribution* on $\mathbb{R}$ is any linear operator $T: C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$. In other words, for every compactly supported function f, Tf is a real number, and

$$T(\alpha f_1 + \beta f_2) = \alpha T(f_1) + \beta T(f_2)$$

for all real numbers α and β and compactly supported functions f_1 and f_2. For example, the three operators given by

$$T(f) = 0, \qquad T(f) = f'(1) - f(2), \qquad T_\delta(f) = f(0),$$

are distributions.

(a) If g is a piecewise continuous function on the real line, show that the operators given by

$$T_g f = \int_{-\infty}^{\infty} g(t) f(t)\, dt, \quad f \in C_c^\infty(\mathbb{R}), \qquad \text{and} \qquad T_g' f = -\int_{-\infty}^{\infty} g(t) f'(t)\, dt, \quad f \in C_c^\infty(\mathbb{R}),$$

are well-defined distributions.
(b) Use integration by parts to show that if g is a continuous function with a piecewise continuous first derivative and H is the Heaviside function, then

$$T_g' f = T_{g'} f \quad \text{and} \quad T_H' f = T_\delta f \qquad \text{for all } \ f \in C_c^\infty(\mathbb{R}).$$

If $g_1 \neq g_2$ are piecewise continuous functions, then $T_{g_1} \neq T_{g_2}$ as operators, i.e. the numbers $T_{g_1} f$ and $T_{g_2} f$ are not the same for at least one $f \in C_c^\infty(\mathbb{R})$. Thus, every such function g can be thought of as the distribution T_g. Furthermore, if g is smooth, $T_g' = T_{g'}$ by (b). However, not every distribution is of the form T_g, for some piecewise continuous function g. For example, T_δ is not of this form. In this sense, distributions are *generalized functions*, and in the sense of distributions $H' = \delta$, i.e. $T_H' = T_\delta$. The limit in Definition 6.2 on p282, means that

$$\lim_{\epsilon \longrightarrow 0} T_{\delta_0^\epsilon} f = T_\delta f \qquad \text{for all } \ f \in C_c^\infty(\mathbb{R}).$$

Problem D

Recall from PS2-Problem B, that if p and q are constants,

$$\left(e^{(\lambda_1-\lambda_2)t}(e^{-\lambda_1 t} y)'\right)' = e^{-\lambda_2 t}(y'' + py' + qy), \tag{1}$$

where λ_1 and λ_2 are the two roots of the characteristic polynomial $\lambda^2 + p\lambda + q = 0$ associated to the linear homogeneous second-order ODE $y'' + py' + qy = 0$.

(a) Use (1) to find a solution $y_p = y_p(t)$ to the ODE

$$y'' + py' + qy = f, \qquad f = f(t),$$

in the form $y_p = G * f$ for some function $G = G(t)$. In particular, give an explicit expression for G. Use only (1) to find y_p, and not Section 5.7 in the book. You will need to consider two cases in your computation: $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$. In the latter case, there will be two subcases. In one of these subcases, the expression you obtain for $G = G(t)$ will involve complex numbers, but simplifies to a real expression.
(b) Compare your expression for y_p with that for y_s in Theorem 7.16, on p293. In particular, use the inverse Laplace transform and Theorem 6.10, on p285, to give an explicit expression for the function $e = e(t)$ that appears in the definition of y_s in each of the three cases.