

Math53: Ordinary Differential Equations Autumn 2004

Final Exam from Winter 2004

Problem 1 (20pts)

Find the solution $y=y(t)$, including the interval of existence, to the initial value problem

$$ty' = 6t^2 - 4y, \quad y(1) = 2.$$

Problem 2 (25pts)

Find *all* solutions $y=y(t)$ to the ODE

$$y' = (y - 1)(y - 2), \quad y = y(t),$$

explicitly. Sketch a representative collection of solution curves in the ty -plane, clearly indicating the asymptotic behavior.

Problem 3 (30pts)

Find the solution $y=y(t)$ to the initial value problem

$$y'' - 4y' + 4y = 4e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Problem 4 (30pts)

(a; 5pts) Verify that the functions $y_1(t)=t$ and $y_2(t)=t^2$ are linearly independent solutions of the ODE

$$t^2y'' - 2ty' + 2y = 0, \quad y = y(t).$$

(b; 9pts) Show that finding *a* solution to the ODE

$$t^2y'' - 2ty' + 2y = 2t^2 \ln t, \quad y = y(t),$$

can be reduced to finding a solution $(v_1, v_2) = (v_1(t), v_2(t))$ of the system

$$\begin{cases} v_1' + tv_2' = 0 \\ v_1' + 2tv_2' = 2 \ln t \end{cases}$$

(c; 16pts) Find the general solution to the ODE

$$t^2y'' - 2ty' + 2y = 2t^2 \ln t, \quad y = y(t).$$

Note: You may skip (b), with no penalty, if you can solve (c) without it.

Problem 5 (30pts)

A tank contains V gallons of salt solution of concentration ρ_0 pounds of salt per gallon. Another salt solution containing ρ_1 pounds of salt per gallon is poured into the tank from the top at the constant rate of r gallons per minute. A drain is opened at the bottom of the tank allowing the solution to exit at the same rate r , so that the volume remains unchanged. Assume that the solution in the tank is kept perfectly mixed at all times. Let $\rho(t)$ be the salt concentration in the tank t minutes after the second solution started pouring into the tank.

(a; 10pts) Show that the concentration $\rho(t)$ is the solution to the initial value problem

$$\rho'(t) = \frac{r}{V}(\rho_1 - \rho(t)), \quad \rho(0) = \rho_0.$$

(b; 20pts) Determine the number T of minutes it will take for the solution in the tank to reach the concentration of $(\rho_0 + \rho_1)/2$.

Note: You may skip (a), with no penalty, if you can solve (b) without it.

Problem 6 (30pts)

(a; 18pts) Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

(b; 12pts) Sketch the phase-plane portrait for this ODE, showing all important qualitative information. You may want to state some of it next to the sketch. Your sketch should include *at least two* solution curves in each region cut out by the half-lines. Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

Problem 7 (30pts)

Find the general solution $(x, y) = (x(t), y(t))$ to the system of ODEs

$$\begin{cases} x' = 2x + 4y - 2t - 4t^2 \\ y' = -x - 2y + 2t^2 \end{cases}$$

Problem 8 (25pts)

Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

Problem 9 (50pts)

(a; 30pts) Find all equilibrium points for the system of ODEs

$$\begin{cases} x' = 3(x-1)(y+2) \\ y' = 2(x+1)(y-1) \end{cases} \quad x = x(t), \quad y = y(t).$$

Determine the type of each equilibrium point in detail (i.e. stability, spiral/nodal sink/source, saddle, and, if relevant, direction of rotation, slopes).

Note: All equilibrium points are of generic type.

(b; 20pts) Sketch the phase-plane portrait for the system

$$\begin{cases} x' = 3(x-1)(y+2) \\ y' = 2(x+1)(y-1) \end{cases} \quad x = x(t), \quad y = y(t).$$

Your sketch *must* be drawn on the back of this page, on the preceding page, or on a separate sheet of paper, and it *must* be very large, taking up most of the page. The sketch should show lots of solution curves. Its completeness, accuracy, and clarity will influence its evaluation.

Problem 10 (30pts)

Let $y=y(t)$ be the solution to the initial value problem

$$y' = ty, \quad y(0) = 1.$$

(a; 15pts) Use the *first*-order Euler's numerical method with *four* steps to estimate $y(2)$. (b; 15pts) Use the *second*-order Runge-Kutta numerical method with *two* steps to estimate $y(2)$.

Note: Your final answers should be written as a fraction, i.e. in the form p/q .

Problem 1 (20pts)

Find the solution $y=y(t)$, including the interval of existence, to the initial value problem

$$ty' = 6t^2 - 4y, \quad y(1) = 2.$$

The ODE is linear and we put into the standard form for linear ODEs:

$$ty' = 6t^2 - 4y \quad \iff \quad y' + 4t^{-1}y = 6t.$$

The integrating factor is then

$$P(t) = e^{\int 4t^{-1}dt} = e^{4\ln|t|} = (e^{\ln|t|})^4 = |t|^4 = t^4.$$

Multiplying both sides of the ODE by $P(t)$ and compressing the left hand side, we obtain

$$\begin{aligned} y' + 4t^{-1}y = 6t &\iff t^4y' + 4t^3y = 6t^5 \iff (t^4y)' = 6t^5 \\ &\iff t^4y(t) = \int 6t^5 dt = t^6 + C \iff y(t) = t^2 + Ct^{-4}. \end{aligned}$$

We then use the initial condition to determine the constant C :

$$y(1) = 1 + C = 2 \implies C = 1 \implies y(t) = t^2 + t^{-4}.$$

This function is defined for $t \in (-\infty, 0)$ and $t \in (0, \infty)$. Since the latter interval contains t , the solution of the initial value problem is

$$\boxed{y(t) = t^2 + t^{-4}, \quad t \in (0, \infty)}$$

Problem 2 (25pts)

Find all solutions $y=y(t)$ to the ODE

$$y' = (y-1)(y-2), \quad y = y(t),$$

explicitly. Sketch a representative collection of solution curves in the ty -plane, clearly indicating the asymptotic behavior.

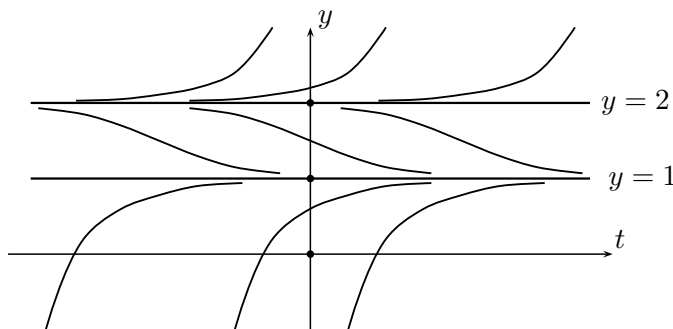
This equation is separable. We write $y' = dy/dt$ and separate variables:

$$\begin{aligned} \frac{dy}{dt} = (y-1)(y-2) &\iff \frac{dy}{(y-1)(y-2)} = dt \iff \frac{1}{(-1)-(-2)} \left(\frac{1}{y-2} - \frac{1}{y-1} \right) dy = dt \\ &\iff \int \left(\frac{1}{y-2} - \frac{1}{y-1} \right) dy = \int dt \iff \ln|y-2| - \ln|y-1| = t + C \\ &\iff \ln \left| \frac{y-2}{y-1} \right| = t + C \iff \frac{y-2}{y-1} = Ae^t \\ &\iff y-2 = Ae^t y - Ae^t \iff y(1-Ae^t) = 2 - Ae^t \\ &\iff y(t) = \frac{2 - Ae^t}{1 - Ae^t} = 1 + \frac{1}{1 - Ae^t} \end{aligned}$$

However, since we have divided by $(y-1)(y-2)$, we may have missed the constant solutions $y=1$ and $y=2$. The latter solution corresponds to $A=0$. On the other hand, the solution $y=1$ cannot correspond to any constant A , since the function $y=y(t)$ is not constant unless $A=0$. Thus, the solutions to the ODE are

$$\boxed{y(t) = 1 + \frac{1}{1 - Ae^t}, \quad A \in \mathbb{R}, \quad \text{and} \quad y(t) = 1}$$

This ODE is autonomous, and we could have sketched solution curves even without solving it. The two constant solutions $y=1$ and $y=2$ correspond to horizontal lines, separating the ty -plane into three regions. In the top region, $y' > 0$ and the solution curves rise to ∞ as t increases and approach the line $y=2$ as $t \rightarrow -\infty$. In the bottom region, $y' > 0$ and the solution curves approach the line $y=2$ as $t \rightarrow \infty$ and drop to $-\infty$ as t decreases. Finally, in the middle region, $y' < 0$ and the solution curves approach the lines $y=1$ and $y=2$ as $t \rightarrow \infty$ and $t \rightarrow -\infty$, respectively. Since the ODE is autonomous, the horizontal shift of a solution curve is a still solution curve. However, the explicit solution does provide some additional information. The top and bottom regions correspond to $A > 0$. From the explicit solution we see that the solution curves in the top region rise to ∞ in a finite time and the solution curves in the bottom region drop to $-\infty$ in a finite time.



Problem 3 (30pts)

Find the solution $y=y(t)$ to the initial value problem

$$y'' - 4y' + 4y = 4e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Since the homogeneous solutions and the forcing terms are *both* exponentials, the easiest approach is likely to be via the Laplace Transform. This is even more likely to be so in this case, since the characteristic equation has a double root, which in addition is the exponent in the forcing term. Using the LT tables to take the Laplace Transform of both sides of the ODE, we obtain

$$y'' - 4y' + 4y = 4e^{2t} \quad \Longrightarrow \quad (s^2Y - sy(0) - y'(0)) - 4(sY - y(0)) + 4Y = 4\frac{1}{s-2},$$

where $Y=Y(s)$ is the Laplace Transform of $y=y(t)$. Applying the initial conditions, we get

$$(s^2 - 4s + 4)Y - 1 = \frac{4}{s-2} \quad \Longrightarrow \quad Y(s) = \frac{4}{(s-2)^3} + \frac{1}{(s-2)^2} \quad \Longrightarrow \quad \boxed{y(t) = 2t^2e^{2t} + te^{2t}}$$

by the first LT table.

The more standard approach is to first find the general solution to the ODE as $y = y_h + y_p$ and then to use the initial conditions to find the two constants. The characteristic polynomial in this case is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

It has repeated roots $\lambda_1 = \lambda_2 = 2$. Thus, the general solution to the associated homogeneous equation is

$$y_h(t) = C_1e^{2t} + C_2te^{2t}.$$

We next use the method of undetermined coefficients to find a particular solution $y_p = y_p(t)$ to the inhomogeneous equation. Since the forcing term is of the form Ae^{2t} , we would normally try $y_p(t) = Ce^{2t}$. However, since e^{2t} and te^{2t} are solutions to the associated homogeneous equation, neither Ce^{2t} nor Cte^{2t} will work for the inhomogeneous equation. Instead we try $y_p(t) = Ct^2e^{2t}$:

$$\begin{aligned} y_p = Ct^2e^{2t} &\Longrightarrow y_p' = 2C(te^{2t} + t^2e^{2t}) \Longrightarrow y_p'' = 2C(e^{2t} + 4te^{2t} + 2t^2e^{2t}) \\ &\Longrightarrow C(2(e^{2t} + 4te^{2t} + 2t^2e^{2t}) - 4 \cdot 2(te^{2t} + t^2e^{2t}) + 4t^2e^{2t}) = 4e^{2t} \\ &\Longrightarrow C \cdot 2e^{2t} = 4e^{2t} \Longrightarrow C = 2 \Longrightarrow y_p(t) = 2t^2e^{2t} \Longrightarrow y(t) = C_1e^{2t} + C_2te^{2t} + 2t^2e^{2t}. \end{aligned}$$

Finally, we use the initial conditions to determine C_1 and C_2 :

$$\begin{cases} y(0) = C_1 = 0 \\ y'(0) = 2C_1 + C_2 = 1 \end{cases} \quad \Longrightarrow \quad C_1 = 0, C_2 = 1 \quad \Longrightarrow \quad \boxed{y(t) = 2t^2e^{2t} + te^{2t}}$$

Problem 4 (30pts)

(a; 5pts) Verify that $y_1(t)=t$ and $y_2(t)=t^2$ are linearly independent solutions of the ODE

$$t^2 y'' - 2ty' + 2y = 0, \quad y = y(t).$$

(b+c; 9+16pts) Find the general solution to the ODE

$$t^2 y'' - 2ty' + 2y = 2t^2 \ln t, \quad y = y(t).$$

(a) We plug in $y_1(t)=t$ and $y_2(t)=t^2$ into the homogeneous ODE:

$$\begin{aligned} t^2 y_1'' - 2ty_1' + 2y_1 &= t^2 \cdot 0 - 2t \cdot 1 + 2 \cdot t = 0 & \checkmark \\ t^2 y_2'' - 2ty_2' + 2y_2 &= t^2 \cdot 2 - 2t \cdot 2t + 2 \cdot t^2 = 0 & \checkmark \end{aligned}$$

Since the function $y_1(t)/y_2(t)=t^{-1}$ is not a constant, $y_1=y_1(t)$ and $y_2=y_2(t)$ are linearly independent.

(b+c) The general solution of the inhomogeneous equation has the form $y(t) = y_h(t) + y_p(t)$. Since $y_1=y_1(t)$ and $y_2=y_2(t)$ are linearly independent solutions of the associated homogeneous ODE, the general solution of the homogeneous ODE is given by

$$y(t) = C_1 t + C_2 t^2.$$

We next use variation of parameters to find a particular solution $y_p = y_p(t)$ to the inhomogeneous equation. In other words, we would like to find $v_1=v_1(t)$ and $v_2=v_2(t)$ such that the function

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t) = tv_1(t) + t^2v_2(t)$$

solves the ODE. We compute $y_p'(t)$, writing the resulting expression symmetrically with respect to v_1 and v_2 :

$$y_p'(t) = (tv_1'(t) + t^2v_2'(t)) + (v_1(t) + 2tv_2(t)).$$

Since the function y_p involves two parameters, v_1 and v_2 , but has to satisfy only one equation, we can impose one condition on v_1 and v_2 :

$$tv_1'(t) + t^2v_2'(t) = 0 \implies y_p'(t) = v_1(t) + 2tv_2(t) \implies y_p''(t) = (v_1'(t) + 2tv_2'(t)) + 2v_2(t).$$

Plugging these expressions into the inhomogeneous ODE, we get

$$t^2(v_1' + 2tv_2' + 2v_2) - 2t(v_1 + 2tv_2) + 2(tv_1 + t^2v_2) = 2t^2 \ln t \implies t^2(v_1' + 2tv_2') = 2t^2 \ln t.$$

Combining this condition on v_1 and v_2 with the one imposed above, we get:

$$\begin{cases} v_1' + tv_2' = 0 \\ v_1' + 2tv_2' = 2 \ln t \end{cases} \implies \begin{cases} v_2' = 2t^{-1} \ln t \\ v_1' = -2 \ln t \end{cases} \implies \begin{cases} v_2 = \int 2t^{-1} \ln t \, dt = \ln^2 t \\ v_1 = -2 \int \ln t \, dt = -2(t \ln t - \int dt) = 2t - 2t \ln t \end{cases}$$

Note that we need to find only one pair (v_1, v_2) that solves the first system. Putting everything together, we conclude that

$$y_p = y_1 v_1 + y_2 v_2 = 2t^2 - 2t^2 \ln t + t^2 \ln^2 t \implies \boxed{y(t) = y_h(t) + y_p(t) = C_1 t + C_2 t^2 - 2t^2 \ln t + t^2 \ln^2 t}$$

Problem 5 (30pts)

A tank contains V gallons of salt solution of concentration ρ_0 pounds of salt per gallon. Another salt solution containing ρ_1 pounds of salt per gallon is poured into the tank from the top at the constant rate of r gallons per minute. A drain is opened at the bottom of the tank allowing the solution to exit at the same rate r , so that the volume remains unchanged. Assume that the solution in the tank is kept perfectly mixed at all times. Let $\rho(t)$ be the salt concentration in the tank t minutes after the second solution started pouring into the tank.

(a; 10pts) Show that the concentration $\rho(t)$ is the solution to the initial value problem

$$\rho'(t) = \frac{r}{V}(\rho_1 - \rho(t)), \quad \rho(0) = \rho_0.$$

(b; 20pts) Determine the number T of minutes it will take for the solution in the tank to reach the concentration of $(\rho_0 + \rho_1)/2$.

(a) By our assumptions, $\rho = \rho(t)$ satisfies the initial condition. Thus, it remains to determine the rate of change of ρ . If $S(t)$ is the amount of salt in the tank at time t ,

$$\begin{aligned} S'(t) &= \text{rate}_{\text{salt}}^{\text{in}}(t) - \text{rate}_{\text{salt}}^{\text{out}}(t) = \rho^{\text{in}}(t) \cdot \text{rate}_{\text{mix}}^{\text{in}}(t) - \rho^{\text{out}}(t) \cdot \text{rate}_{\text{mix}}^{\text{out}}(t) \\ &= \rho_1 \cdot r - \rho(t) \cdot r = r(\rho_1 - \rho(t)). \end{aligned}$$

Since $S(t) = \rho(t) \cdot V(t)$ and $V(t) = V$ is constant, it follows that

$$\rho'(t) \cdot V = S'(t) = r(\rho_1 - \rho(t)) \quad \implies \quad \rho'(t) = \frac{r}{V}(\rho_1 - \rho(t)).$$

(b) We first find the solution $\rho = \rho(t)$ to the initial value problem in (a). The ODE is separable (as well as linear). We write $\rho' = d\rho/dt$ and separate variables:

$$\begin{aligned} \frac{d\rho}{dt} = \frac{r}{V}(\rho_1 - \rho) &\iff \frac{d\rho}{\rho_1 - \rho} = \frac{r}{V}dt \iff \int_{\rho_0}^{\rho} \frac{dy}{\rho_1 - y} = \int_0^t \frac{r}{V}ds \\ &\iff -(\ln|\rho_1 - \rho| - \ln|\rho_1 - \rho_0|) = \frac{r}{V}t \iff \ln\left|\frac{\rho_1 - \rho_0}{\rho_1 - \rho}\right| = \frac{r}{V}t. \end{aligned}$$

The last relation defines $\rho = \rho(t)$ implicitly. We use it to find T such that $\rho(T) = (\rho_0 + \rho_1)/2$:

$$\ln\left|\frac{\rho_1 - \rho_0}{\rho_1 - \rho(T)}\right| = \frac{r}{V}T \iff \ln 2 = \frac{r}{V}T \iff \boxed{T = (\ln 2)V/r}$$

Note 1: Above we use $\rho_1 \neq \rho_0$; otherwise $T = 0$.

Note 2: We could instead find the general solution and then solve for the constant to obtain the solution to the IVP in (a) and finally for T .

Problem 6 (30pts)

(a; 18pts) Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

(b; 12pts) Sketch the phase-plane portrait for this ODE. Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

(a) Since this matrix is upper-triangular, the eigenvalues $\lambda_1 = \lambda_2 = -1$ are the diagonal entries and $\mathbf{v}_1 = (1 \ 0)^t$ is an eigenvector for λ_1 . As a second basis element we can take any vector which is not a multiple of \mathbf{v}_1 , such as $\mathbf{v}_2 = (0 \ 1)^t$. Then,

$$A\mathbf{v}_1 = (-1)\mathbf{v}_1, \quad A\mathbf{v}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (-2)\mathbf{v}_1 + (-1)\mathbf{v}_2 \implies e^{tA}\mathbf{v}_1 = e^{-t}\mathbf{v}_1, \quad e^{tA}\mathbf{v}_2 = -2te^{-t}\mathbf{v}_1 + e^{-t}\mathbf{v}_2.$$

Thus, the general solution to the system is

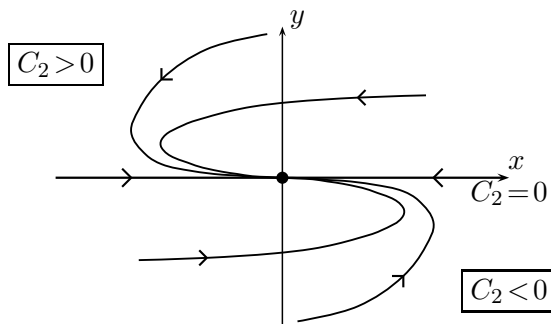
$$\begin{aligned} y(t) = e^{tA}\mathbf{v} &= C_1(e^{tA}\mathbf{v}_1) + C_2(e^{tA}\mathbf{v}_2) = C_1e^{-t}\mathbf{v}_1 + C_2(-2te^{-t}\mathbf{v}_1 + e^{-t}\mathbf{v}_2) \\ &= (C_1 - 2C_2t)e^{-t}\mathbf{v}_1 + C_2e^{-t}\mathbf{v}_2 = \boxed{(C_1 - 2C_2t)e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \end{aligned}$$

In this case, we could instead compute the matrix e^{tA} by splitting off λI from A :

$$\begin{aligned} A = \lambda I + B &= (-1)I + \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\lambda t I)(tB) = (tB)(\lambda t I) \\ \implies e^{tA} &= e^{t\lambda I + tB} = e^{t\lambda I} e^{tB} = e^{t\lambda} I(I + tB) = e^{-t} \begin{pmatrix} 1 & -2t \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The general solution to the ODE is the general linear combination of the columns of the matrix e^{tA} .

(b) Since all eigenvalues of A are negative, all solution curves move toward the origin as t increases. Thus, the origin is an asymptotically stable equilibrium point for the system. For the sketch, the origin is the solution curve corresponding to $C_1 = C_2 = 0$. If $C_1 \neq 0$ and $C_2 = 0$, the corresponding solution curve is either the positive or negative x -axis and approaches the origin. By plugging in $C_1 = 0$ and $t = 0$ into the general solution, we obtain that $C_2 > 0$ above the x -axis and $C_2 < 0$ below it. As $t \rightarrow \pm\infty$, the term involving t in the general solution dominates. Thus, the slope of all solution curves with $C_2 \neq 0$ approaches 0 as $t \rightarrow \pm\infty$, i.e. the curves approach the origin tangent to the x -axis as $t \rightarrow \infty$ and flatten out away from the origin as $t \rightarrow -\infty$. Above the x -axis, i.e. if $C_2 > 0$, the curves approach the origin from the left side as $t \rightarrow \infty$, since the x -component is negative for large t , and ascend to the right, since the x -component is positive for very negative t . Below the x -axis, the picture is reversed.



Problem 7 (30pts)

Find the general solution $(x, y) = (x(t), y(t))$ to the system of ODEs

$$\begin{cases} x' = 2x + 4y - 2t - 4t^2 \\ y' = -x - 2y + 2t^2 \end{cases}$$

This system can be written as

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -2t - 4t^2 \\ 2t^2 \end{pmatrix}, \quad \text{where} \quad \mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The general solution to this inhomogeneous equation has the form $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$, where $\mathbf{y}_h = \mathbf{y}_h(t)$ is the general solution to the associated homogeneous equation and $\mathbf{y}_p = \mathbf{y}_p(t)$ is a solution of the inhomogeneous equation.

The characteristic polynomial of the matrix A is

$$\lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 = 0.$$

Thus, $\lambda_1 = \lambda_2 = 0$. This means that A^2 is the zero matrix, and

$$e^{tA} = I + tA = \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix}.$$

The general solution of the associated homogeneous equation is then

$$\mathbf{y}_h(t) = e^{tA}\mathbf{v} = C_1 e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1+2t \\ -t \end{pmatrix} + C_2 \begin{pmatrix} 4t \\ 1-2t \end{pmatrix}.$$

Alternatively, we can first find an eigenvector \mathbf{v}_1 for λ_1 :

$$\begin{pmatrix} 2-\lambda_1 & 4 \\ -1 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2c_1 + 4c_2 = 0 \\ -c_1 - 2c_2 = 0 \end{cases} \iff c_1 = -2c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

We then pick a second basis element, such as $\mathbf{v}_2 = (1 \ 0)^t$, and find that

$$\begin{aligned} A\mathbf{v}_2 &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \mathbf{v}_1 \implies e^{tA}\mathbf{v}_1 = e^{0t}\mathbf{v}_1 = \mathbf{v}_1, \quad e^{tA}\mathbf{v}_2 = te^{0t}\mathbf{v}_1 + e^{0t}\mathbf{v}_2 = t\mathbf{v}_1 + \mathbf{v}_2 \\ \implies y_h(t) &= C_1 e^{tA}\mathbf{v}_1 + C_2 e^{tA}\mathbf{v}_2 = (C_1 + tC_2)\mathbf{v}_1 + C_2\mathbf{v}_2 = (C_1 + tC_2) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

This expression reduces to the previous one by substituting $-C_2$ and $C_1 + 2C_2$ for C_1 and C_2 , respectively; this is just a change of constants.

One way of finding $\mathbf{y}_p = \mathbf{y}_p(t)$ is by using e^{tA} :

$$y_p(t) = e^{tA} \int e^{-tA} \mathbf{f}(t) dt, \quad \text{where} \quad e^{-tA} \mathbf{f}(t) = \begin{pmatrix} 1-2t & -4t \\ t & 1+2t \end{pmatrix} \begin{pmatrix} -2t-4t^2 \\ 2t^2 \end{pmatrix} = \begin{pmatrix} -2t \\ 0 \end{pmatrix},$$

$$\implies e^{tA} \int e^{-tA} \mathbf{f}(t) dt = \begin{pmatrix} 1-2t & -4t \\ t & 1+2t \end{pmatrix} \begin{pmatrix} -t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -t^2 - 2t^3 \\ t^3 \end{pmatrix}.$$

Thus, the general solution to the system is

$$\mathbf{y}(t) = \mathbf{y}_h(t) + \mathbf{y}_p(t) = \boxed{C_1 \begin{pmatrix} 1+2t \\ -t \end{pmatrix} + C_2 \begin{pmatrix} 4t \\ 1-2t \end{pmatrix} + \begin{pmatrix} -t^2 - 2t^3 \\ t^3 \end{pmatrix}}$$

Another way of finding $\mathbf{y}_p = \mathbf{y}_p(t)$ is by using the fundamental matrix arising in the second approach to finding \mathbf{y}_h :

$$Y(t) = (\mathbf{y}_1(t) \ \mathbf{y}_2(t)) = (e^{tA} \mathbf{v}_1 \ e^{tA} \mathbf{v}_2) = \begin{pmatrix} 2 & 1+2t \\ -1 & -t \end{pmatrix} \implies Y(t)^{-1} = \begin{pmatrix} -t & -1-2t \\ 1 & 2 \end{pmatrix}.$$

We can find $\mathbf{y}_p = \mathbf{y}_p(t)$ from

$$y_p(t) = Y(t) \int Y(t)^{-1} \mathbf{f}(t) dt, \quad \text{where} \quad Y(t)^{-1} \mathbf{f}(t) = \begin{pmatrix} -t & -1-2t \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2t-4t^2 \\ 2t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \end{pmatrix},$$

$$\implies Y(t) \int Y(t)^{-1} \mathbf{f}(t) dt = \begin{pmatrix} 2 & 1+2t \\ -1 & -t \end{pmatrix} \begin{pmatrix} 0 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t^2 - 2t^3 \\ t^3 \end{pmatrix}$$

$$\implies \mathbf{y}(t) = \mathbf{y}_h(t) + \mathbf{y}_p(t) = \boxed{(C_1 + tC_2) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -t^2 - 2t^3 \\ t^3 \end{pmatrix}}$$

The method of undetermined coefficients could also be used to find \mathbf{y}_p , but it would not work well in the present case because we would have to solve for a lot of coefficients. The Laplace Transform would be messy too.

Problem 8 (25pts)

Find the general solution to the ODE

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t).$$

Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium and why.

Since this matrix is upper-triangular, the eigenvalues are the diagonal entries $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -1$. Furthermore, $\mathbf{v}_1 = (1 \ 0 \ 0)^t$ is an eigenvector for λ_1 . We next find eigenvectors for $\lambda_2 = \lambda_3$:

$$\begin{pmatrix} 2-\lambda_2 & 1 & 1 \\ 0 & -1-\lambda_2 & 0 \\ 0 & 0 & -1-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff 3c_1 + c_2 + c_3 = 0.$$

Two linearly independent solutions of the last equation are

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}.$$

Thus, the general solution of the system is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

Since one of the eigenvalues is positive, there are solutions that move away from the origin, i.e. if $C_1 \neq 0$ above. Thus, the origin is an unstable equilibrium point.

Problem 9 (50pts)

(a; 30pts) Find all equilibrium points for the system of ODEs

$$\begin{cases} x' = 3(x-1)(y+2) \\ y' = 2(x+1)(y-1) \end{cases} \quad x = x(t), \quad y = y(t),$$

and determine their type in detail.

(b; 20pts) Sketch the phase-plane portrait for this system of ODEs.

(a) The equilibrium points are the solutions (x, y) of the system:

$$\begin{cases} x' = 3(x-1)(y+2) = 0 \\ y' = 2(x+1)(y-1) \end{cases} \implies \begin{cases} x = 1 \text{ or } y = -2 \\ x = -1 \text{ or } y = 1 \end{cases}$$

Thus, the equilibrium points are $\boxed{(1, 1) \text{ and } (-1, -2)}$ The jacobian in this case is:

$$J(x, y) = \begin{pmatrix} 3(y+2) & 3(x-1) \\ 2(y-1) & 2(x+1) \end{pmatrix} \implies$$
$$J(1, 1) = \begin{pmatrix} 9 & 0 \\ 0 & 6 \end{pmatrix} \implies \lambda_1 = 9, \lambda_2 = 6, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $\boxed{(1, 1) \text{ is a nodal source; larger-eigenvalue slope}=0; \text{smaller-eigenvalue slope}=\infty}$ Similarly,

$$J(-1, -2) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \implies \lambda^2 - 0\lambda - 36 = 0 \implies \lambda_1 = 6, \lambda_2 = -6.$$

We next find eigenvectors \mathbf{v}_1 and \mathbf{v}_2 for λ_1 and λ_2 :

$$\lambda_1 = 6: \quad \begin{pmatrix} 0 - \lambda_1 & -6 \\ -6 & -\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -6c_1 - 6c_2 = 0 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda_2 = -6: \quad \begin{pmatrix} 0 - \lambda_2 & -6 \\ -6 & -\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 6c_1 - 6c_2 = 0 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, $\boxed{(-1, -2) \text{ is a saddle point; slope-in}=1; \text{slope-out}=\infty}$

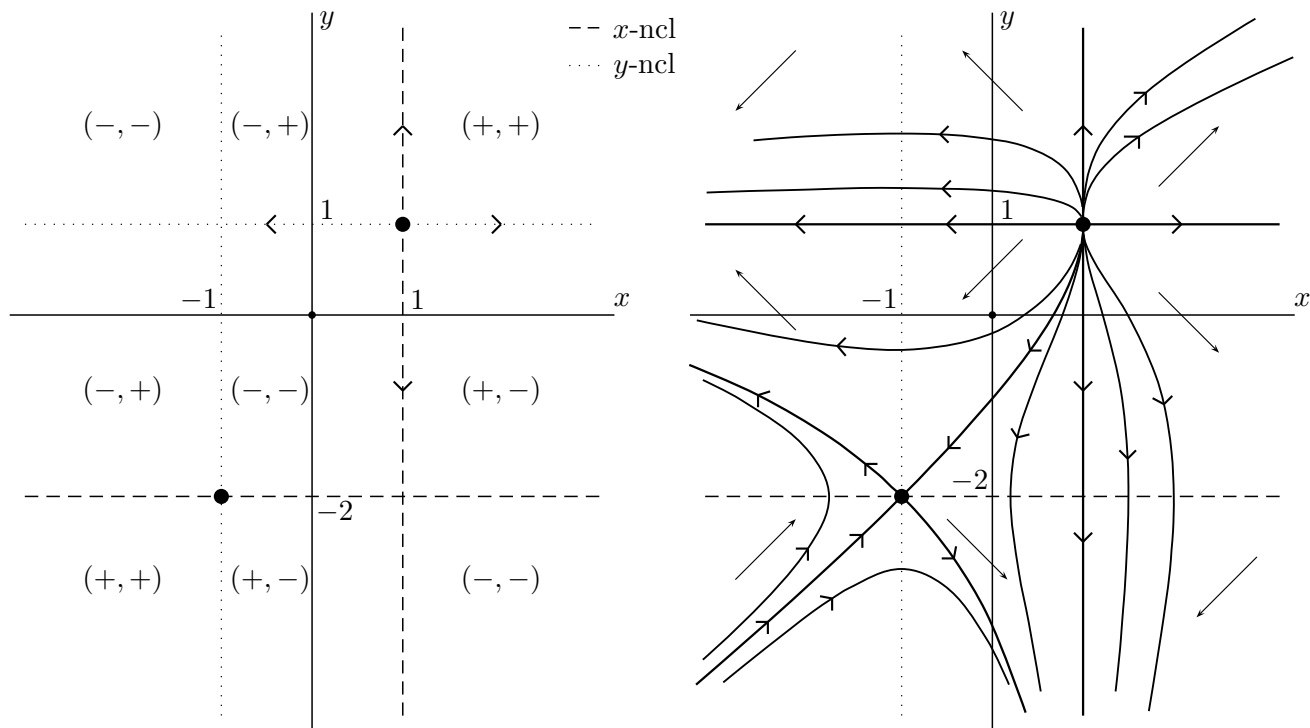
(b) We first indicate the two equilibrium points, $(1, 1)$ and $(-2, 1)$, with large dots. The next step is to sketch the nullclines. The x -nullcline is described by the equation $x' = 0$; it consists of the lines $x = 1$ and $y = -2$. The y -nullcline is described by the equation $y' = 0$; it consists of the lines $x = -1$ and $y = 1$. The two equilibrium points are the intersections of the curves making up the x -nullcline with the curves making up the y -nullcline.

Since $(1, 1)$ is a nodal source, the flow direction in the top right region is up and to the right. We indicate this by labeling the region with $(+, +)$. Every time, we cross the x -nullcline, the x -sign changes; every time, we cross the y -nullcline, the y -sign changes. In this way, we label all the regions, cut out by the nullclines, with (\pm, \pm) on the first sketch below. In particular, the flow stays on the lines $x = 1$ and $y = 1$, pointing away from $(1, 1)$. Thus, these lines split into solution curves, and these solution curves correspond to the two slopes at $(1, 1)$. All of this information can also be obtained by looking at

the sign of x' on each segment of the y -nullcline and at the sign of y' on each segment of the x -nullcline.

We translate the (\pm, \pm) labels into arrows on the second sketch. These arrows indicate the general direction of the flow. We also show that the lines $x=1$ and $y=1$ are made up of solution curves. We next sketch the pair of incoming solution curves and the pair of outgoing solution curves at $(-1, -2)$. By (a), the slope of the incoming curves at $(-1, -2)$ is 1. Thus, one of these curves must come from the bottom left region and one from the center region. The first curve, traced backwards and thus against the flow, descends to the left forever. The second curve, traced backwards, must rise to the right and cannot cross the lines $x=1$ and $y=1$. Thus, this curve must have come from the source at $(1, 1)$. Since the eigenvectors for the smaller eigenvalue λ_2 at $(-1, -2)$ are vertical, this solution curve must "leave" $(1, 1)$ downward. On the other hand, the slope of the outgoing curves at $(-1, -2)$ is -1 , also by (a). Thus, one of these curves must move into the bottom middle region and one into the middle right region. The first curve descends to the right, as indicated by the flow directions, and becomes asymptotic the vertical line $x=1$, as this line cannot be crossed. The second curve ascends to the left and becomes asymptotic the horizontal line $y=1$, as this line cannot be crossed either. The eight distinguished solution curves, four for each equilibrium point, are shown with the thickest lines in the second below.

We now sketch additional solution curves in the various regions of the plane. All solution curves, except for the two horizontal half-lines, leave the source $(1, 1)$ vertically. The ones that move into the top right region simply ascend to the right. The curves ascending to the left of $(1, 1)$ rise until they reach the y -nullcline $x=-1$, after which they start descending, still to the left, and approach the line $y=1$. Similarly, the curves descending to the right of $(1, 1)$ move right until they reach the x -nullcline $y=-2$, after which start moving left, still descending, and approach the line $x=1$. Of the curves that "leave" $(1, 1)$ down and to the left, the ones that lie above the distinguished solution curve sinking into the saddle point descend until they reach the line $x=-1$, after which they start ascending, still to the left, and approach the line $y=1$. On the other hand, the ones that lie below the distinguished curve move left until they reach the line $y=-2$, after which start moving right, still descending, and approach the line $x=1$. Finally, we sketch solution curves in the bottom left region. All these curves rise to the right at the first. The ones that lie below the distinguished solution curve sinking into the saddle point eventually cross the line $x=-1$ and then start descending, still to the right, and approach the line $x=1$. The ones that lie above the distinguished curve eventually cross the line $y=-2$ and then start moving right, still ascending, and approach the line $y=1$.



Problem 10 (30pts)

Let $y=y(t)$ be the solution to the initial value problem

$$y' = ty, \quad y(0) = 1.$$

(a; 15pts) Use the first-order Euler's numerical method with four steps to estimate $y(2)$.

(b; 15pts) Use the second-order Runge-Kutta numerical method with two steps to estimate $y(2)$.

(a) The step size is $h=(2-0)/4 = \frac{1}{2}$, and the first-order method gives

$$\begin{array}{llll} t_0 = 0 & y_0 = 1 & s_0 = t_0 y_0 = 0 & s_0 h = 0 \\ t_1 = \frac{1}{2} & y_1 = y_0 + s_0 h = 1 & s_1 = t_1 y_1 = \frac{1}{2} & s_1 h = \frac{1}{4} \\ t_2 = 1 & y_2 = y_1 + s_1 h = \frac{5}{4} & s_2 = t_2 y_2 = \frac{5}{4} & s_2 h = \frac{5}{8} \\ t_3 = \frac{3}{2} & y_3 = y_2 + s_2 h = \frac{15}{8} & s_3 = t_3 y_3 = \frac{45}{16} & s_3 h = \frac{45}{32} \\ t_4 = 2 & y_4 = y_3 + s_3 h = \frac{105}{32} & & \end{array}$$

Thus, the resulting estimate for $y(2)$ is $\boxed{105/32}$

(b) In this case $h=(2-0)/2 = 1$, but we need to find two slopes at each step and average them:

$$\begin{array}{llllll} t_0 = 0 & y_0 = 1 & s_{0,1} = t_0 y_0 = 0 & s_{0,1} h = 0 & & \\ & & s_{0,2} = t_1(y_0 + s_{0,1} h) = 1 & s_0 = \frac{s_{0,1} + s_{0,2}}{2} = \frac{1}{2} & s_0 h = \frac{1}{2} & \\ t_1 = 1 & y_1 = y_0 + s_0 h = \frac{3}{2} & s_{1,1} = t_1 y_1 = \frac{3}{2} & s_{1,1} h = \frac{3}{2} & & \\ & & s_{1,2} = t_2(y_1 + s_{1,1} h) = 6 & s_1 = \frac{s_{1,1} + s_{1,2}}{2} = \frac{15}{4} & s_1 h = \frac{15}{4} & \\ t_2 = 2 & y_2 = y_1 + s_1 h = \frac{21}{4} & & & & \end{array}$$

Thus, the resulting estimate for $y(2)$ is $\boxed{21/4}$