Notes on J-Holomorphic Maps

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Abstract

These notes present a systematic treatment of local properties of J-holomorphic maps and of Gromov's convergence for sequences of such maps, specifying the assumptions needed for all statements. In particular, only one auxiliary statement depends on the manifold being symplectic. The content of these notes roughly corresponds to Chapters 2 and 4 of McDuff-Salamon's book on the subject.

Contents

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1 Introduction

Gromov's introduction [17] of pseudoholomorphic curves techniques into symplectic topology has revolutionized this field and led to its numerous connections with algebraic geometry. The ideas put forward in [17] have been further elucidated and developed in [36, 44, 28, 38, 39, 26, 12] and in many other works. The most comprehensive introduction to the subject of pseudoholomorphic curves is without a doubt the monumental book [29]. Chapters 2 and 4 of this book concern two of the three fundamental building blocks of this subject, the local structure of J-holomorphic maps and Gromov's convergence for sequences of J-holomorphic maps. The present notes contain an alternative systematic exposition of these two topics with generally sharper specification of the assumptions needed for each statement. Chapter 3 and Sections 6.2 and 6.3 in [29] concern the third fundamental building block of the subject, transversality for J-holomorphic maps. A more streamlined and general treatment of this topic is the concern of [45].

The present notes build on the lecture notes on J-holomorphic maps written for the class the author taught at Stony Brook University in Spring 2014. The lectures themselves were based on the hand-written notes he made while studying [28] back in graduate school and were also influenced by the more thorough exposition of the same topics in [29]. The author would like to thank D. McDuff and D. Salamon for the time and care taken in preparing and updating these books, the students in the Spring 2014 class for their participation that guided the preparation of the original version of the present notes, and M. Albanese, S. Cattalani, and X. Chen for thoughtful comments during the revision process.

1.1 Stable maps

A (smooth) Riemann surface (without boundary) is a pair $(\Sigma, \mathfrak{j})$ consisting of a smooth two-dimensional manifold Σ (without boundary) and a complex structure $\mathfrak{j}$ in the fibers of $T\Sigma$. A nodal Riemann surface is a pair $(\Sigma, \mathfrak{j})$ obtained from a Riemann surface $(\widetilde{\Sigma}, \mathfrak{j})$ by identifying pairs of distinct points in a discrete subset $S_\Sigma \subset \widetilde{\Sigma}$ (with no point identified with more than one other point); see the left-hand sides of Figures 1 and 2. The pair $(\widetilde{\Sigma}, \mathfrak{j})$ is called the normalization of $(\Sigma, \mathfrak{j})$; the images of the points of S_Σ in Σ are called the nodes of Σ. We denote their complement in Σ by Σ^*. An irreducible component of $(\Sigma, \mathfrak{j})$ is the image of a topological component of $\widetilde{\Sigma}$ in Σ. Let

$$\mathfrak{a}(\Sigma) = \frac{2 - \chi(\widetilde{\Sigma}) + |S_\Sigma|}{2},$$

where $\chi(\widetilde{\Sigma})$ is the Euler characteristic of $\widetilde{\Sigma}$, be the (arithmetic) genus of Σ. An equivalence between nodal Riemann surfaces $(\Sigma, \mathfrak{j})$ and $(\Sigma', \mathfrak{j}')$ is a homeomorphism $h: \Sigma \longrightarrow \Sigma'$ induced by a biholomorphic map $\widetilde{h}$ from $(\widetilde{\Sigma}, \mathfrak{j})$ to $(\widetilde{\Sigma}', \mathfrak{j}')$. We denote by $\mathrm{Aut}(\Sigma, \mathfrak{j})$ the group of automorphisms, i.e. self-equivalences, of a Riemann surface $(\Sigma, \mathfrak{j})$. A nodal Riemann surface $(\Sigma, \mathfrak{j})$ is called stable if $(\Sigma, \mathfrak{j})$ is compact and $\mathrm{Aut}(\Sigma, \mathfrak{j})$ is a finite group.

Let (X, J) be an almost complex manifold. If $(\Sigma, \mathfrak{j})$ is a Riemann surface, a smooth map $u: \Sigma \longrightarrow X$ is called J-holomorphic map if

$$\mathrm{d}u \circ \mathfrak{j} = J \circ \mathrm{d}u : T\Sigma \longrightarrow u^*TX.$$

A J-holomorphic map from a nodal Riemann surface $(\Sigma, \mathfrak{j})$ is a tuple $(\Sigma, \mathfrak{j}, u)$, where $u: \Sigma \longrightarrow X$ is a continuous map induced by a J-holomorphic map $\widetilde{u}: \widetilde{\Sigma} \longrightarrow X$; see Figures 1 and 2. An equivalence between J-holomorphic maps $(\Sigma, \mathfrak{j}, u)$ and $(\Sigma', \mathfrak{j}', u')$ is an equivalence

$$h: (\Sigma, \mathfrak{j}) \longrightarrow (\Sigma', \mathfrak{j}')$$

between the underlying Riemann surfaces such that $u = u' \circ h$. We denote by $\mathrm{Aut}(\Sigma, \mathfrak{j}, u)$ the group of automorphisms of a J-holomorphic map $(\Sigma, \mathfrak{j}, u)$. A J-holomorphic map $(\Sigma, \mathfrak{j}, u)$ is called stable if $(\Sigma, \mathfrak{j})$ is compact and $\mathrm{Aut}(\Sigma, \mathfrak{j}, u)$ is a finite group.

Example 1.1. The Riemann surface $(\Sigma, \mathfrak{j})$ on the left-hand side of Figure 1 is obtained by identifying the marked points of two copies of a smooth elliptic curve $(\Sigma_0, \mathfrak{j}_0, z_1^*)$, i.e. a torus with a complex structure and a marked point. The Riemann surface $(\Sigma_0, \mathfrak{j}_0)$ with the marked point z_1^* is biholomorphic to $\mathbb{C}/\Lambda$ with the marked point 0 for some lattice $\Lambda \subset \mathbb{C}$ and thus has an automorphism of order 2 that preserves z_1^* (it is induced by the map $z \longrightarrow -z$ on $\mathbb{C}$); see [18, Prp 1.4]. This is the only non-trivial automorphism of $(\Sigma_0, \mathfrak{j}_0)$ preserving z_1^* if $\mathfrak{j}_0$ is generic; in special cases, the group of such automorphisms is either $\mathbb{Z}_4$ or $\mathbb{Z}_6$. Each automorphism of $(\Sigma_0, \mathfrak{j}_0)$ preserving z_1^* gives rise to an automorphism of $(\Sigma, \mathfrak{j})$ fixing one of the irreducible components. There is also an automorphism of $(\Sigma, \mathfrak{j})$ which interchanges the two irreducible components of Σ. Since it does not commute with the automorphisms preserving one of the components, $\mathrm{Aut}(\Sigma, \mathfrak{j}) \approx D_4$ in most cases

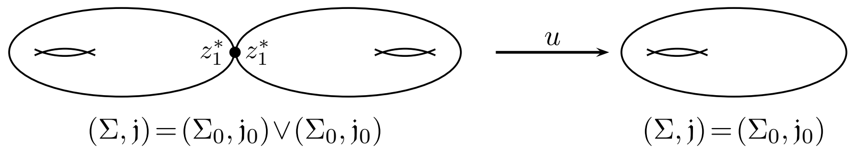

Figure 1: A stable J-holomorphic map

and contains D_4 in the special cases. If $u\colon \Sigma \longrightarrow \Sigma_0$ is the identity on each irreducible component, $(\Sigma, \mathfrak{j}, u)$ is a stable J-holomorphic map; the interchange of the two irreducible components is then the only non-trivial automorphism of $(\Sigma, \mathfrak{j}, u)$. The J-holomorphic maps $u\colon \Sigma \longrightarrow \Sigma_0$ obtained by sending either or both irreducible components of Σ to z_1^* instead are also stable, but have different automorphism groups. If $(\Sigma_0, \mathfrak{j}_0)$ were taken to be the Riemann sphere $\mathbb{P}^1$, the J-holomorphic map $u\colon \Sigma \longrightarrow \Sigma_0$ restricting to the identity on each copy of Σ_0 would still be stable. However, a map $u\colon \Sigma \longrightarrow \Sigma_0$ sending either copy of Σ_0 to z_1^* would not be stable, since the group of automorphisms of $\mathbb{P}^1$ fixing a point is a complex two-dimensional submanifold of PSL_2.

Let $(\Sigma, \mathfrak{j})$ be a compact connected Riemann surface of genus g. If $g \geq 2$, then $\mathrm{Aut}(\Sigma, \mathfrak{j})$ is a finite group. If $g=1$, then $\mathrm{Aut}(\Sigma, \mathfrak{j})$ is an infinite group, but its subgroup fixing any point is finite. If $g=0$, then the subgroup of $\mathrm{Aut}(\Sigma, \mathfrak{j})$ fixing any pair of points is infinite, but the subgroup fixing any triple of points is trivial. If in addition (X, J) is an almost complex manifold and $u\colon \Sigma \longrightarrow X$ is a non-constant J-holomorphic map, then the subgroup of $\mathrm{Aut}(\Sigma, \mathfrak{j})$ consisting of the automorphisms such that $u = u \circ h$ is finite; this is an immediate consequence of Corollary 3.4. If $(\Sigma, \mathfrak{j})$ is a compact nodal Riemann surface, a J-holomorphic map $(\Sigma, \mathfrak{j}, u)$ is thus stable if and only if

- every genus 1 topological component of the normalization $\widetilde{\Sigma}$ of Σ such that u restricts to a constant map on its image in Σ contains at least 1 element of S_Σ and
- every genus 0 topological component of $\widetilde{\Sigma}$ such that u restricts to a constant map on its image in Σ contains at least 3 elements of S_Σ.

1.2 Gromov's topology

Given a Riemann surface $(\Sigma, \mathfrak{j})$, a Riemannian metric g on a smooth manifold X determines the energy $E_g(f)$ for every smooth map $f : \Sigma \longrightarrow X$; see (2.16) and (2.17). The fundamental insight in [17] that laid the foundations for the pseudoholomorphic curves techniques in symplectic topology and for the moduli spaces of stable maps and related curve-parametrizing objects in algebraic geometry is that a sequence of stable J-holomorphic maps $(\Sigma_i, \mathfrak{j}_i, u_i)$ into a compact almost complex manifold (X, J) with

$$\liminf_{i \longrightarrow \infty} \Big(\big|\pi_0(\Sigma_i)\big| + \mathfrak{a}(\Sigma_i) + E_g(u_i) \Big) < \infty \tag{1.1}$$

has a subsequence converging in a suitable sense to another stable J-holomorphic map.

The notion of Gromov's convergence of a sequence of stable J-holomorphic maps $(\Sigma_i, \mathfrak{j}_i, u_i)$ to another stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ comes down to

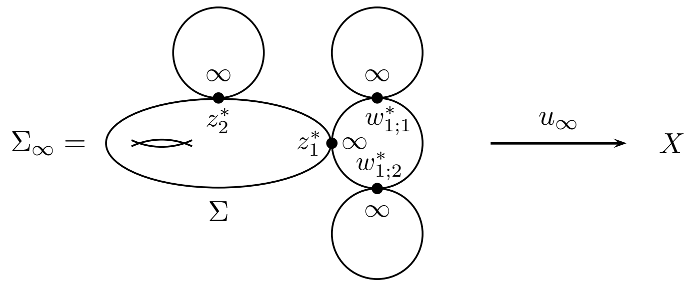

Figure 2: Gromov's limit of a sequence of J-holomorphic maps $u_i\colon \Sigma \longrightarrow X$

(GC1) $|\pi_0(\Sigma_i)| = |\pi_0(\Sigma_\infty)|$ and $\mathfrak{a}(\Sigma_i) = \mathfrak{a}(\Sigma_\infty)$ for all i large,

(GC2) $(\Sigma_\infty, \mathfrak{j}_\infty)$ is at least as singular as $(\Sigma_i, \mathfrak{j}_i)$ for all i large,

(GC3) the energy is preserved, i.e. $E_g(u_i) \longrightarrow E_g(u_\infty)$ as $i \longrightarrow \infty$, and

(GC4) u_i converges to u_∞ uniformly in the C^∞-topology on compact subsets of Σ_∞^*.

Most applications of the pseudoholomorphic curves techniques in symplectic topology involve J-holomorphic maps from the Riemann sphere $\mathbb{P}^1$. This is a special case of the situation when the complex structures $\mathfrak{j}_i$ on the domains Σ_i of u_i are fixed. The condition (GC4) can then be formally stated in a way clearly indicative of the rescaling procedure of [17]. We call a triple (X, J, g) an almost complex Riemannian manifold if J is an almost complex structure on X and g is a Riemannian metric on X, not necessarily compatible with J.

Definition 1.2 (Gromov's Convergence I)**.** Let (X, J, g) be an almost complex Riemannian manifold and $(\Sigma, \mathfrak{j})$ be a compact Riemann surface. A sequence $(\Sigma, \mathfrak{j}, u_i)$ of stable J-holomorphic maps converges to a stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ if

(1) $(\Sigma_\infty, \mathfrak{j}_\infty)$ is obtained from $(\Sigma, \mathfrak{j})$ by identifying a point on each of ℓ trees of Riemann spheres $\mathbb{P}^1$, for some $\ell \in \mathbb{Z}^{\geq 0}$, with distinct points $z_1^*, \ldots, z_\ell^* \in \Sigma$,

(2) $E_g(u_\infty) = \lim\limits_{i\longrightarrow\infty} E_g(u_i)$,

(3) there exist $h_i \in \mathrm{Aut}(\Sigma, \mathfrak{j})$ with $i \in \mathbb{Z}^+$ such that $u_i \circ h_i$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of $\Sigma - \{z_1^*, \ldots, z_\ell^*\}$,

(4) for each $z_1^*, \ldots, z_\ell^* \in \Sigma \subset \Sigma_\infty$ and all $i \in \mathbb{Z}^+$ sufficiently large, there exist a neighborhood $U_j \subset \Sigma$ of z_j^*, an open subset $U_{j;i} \subset \mathbb{C}$, and a biholomorphic map $\psi_{j;i}\colon U_{j;i} \longrightarrow U_j$ such that

    (4a) $U_{j;i} \subset U_{j;i+1}$ and $\mathbb{C} = \bigcup_{i=1}^\infty U_{j;i}$ for every $j = 1, \ldots, \ell$,

    (4b) $u_i \circ h_i \circ \psi_{j;i}$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty, w_{j;1}^*, \ldots, w_{j;k_j}^*$ in the sphere $\mathbb{P}^1_j$ attached at $z_j^* \in \Sigma$,

    (4c) condition (4) applies with Σ, $(z_1^*, \ldots, z_\ell^*)$, and $u_i \circ h_i$ replaced by $\mathbb{P}^1$, $(w_{j;1}^*, \ldots, w_{j;k_j}^*)$, and $u_i \circ h_i \circ \psi_{j;i}$, respectively, for each $j = 1, \ldots, \ell$.

An example of a possible limiting map with $\ell = 2$ trees of spheres is shown in Figure 2. The recursive condition (4) in Definition 1.2 is equivalent to the *Rescaling* axiom in [29, Definition 5.2.1] on sequences of automorphisms ϕ_α^i of $\mathbb{P}^1$; they correspond to compositions of the maps $\psi_{j;i}$ associated with different irreducible components of Σ_∞. The single energy condition (2) in Definition 1.2 is replaced in [29, Definition 5.2.1] by multiple conditions of the *Energy* axiom. These multiple conditions are equivalent to (2) if the other three axioms in [29, Definition 5.2.1] are satisfied.

Theorem 1.3 (Gromov's Compactness I)**.** *Let (X,J,g) be a compact almost complex Riemannian manifold, $(\Sigma,\mathfrak{j})$ be a compact Riemann surface, and $u_i:\Sigma\longrightarrow X$ be a sequence of non-constant J-holomorphic maps. If $\liminf E_g(u_i)<\infty$, then the sequence $(\Sigma,\mathfrak{j},u_i)$ contains a subsequence converging to some stable J-holomorphic map $(\Sigma_\infty,\mathfrak{j}_\infty,u_\infty)$ in the sense of Definition 1.2.*

This theorem is established in Section 5.3 by assembling together a number of geometric statements obtained earlier in these notes. In Section 5.4, we relate the convergence notion of Definition 1.2 in the case of holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^n$, which can always be represented by $(n+1)$-tuples of homogeneous polynomials in two variables, to the behavior of the linear factors of the associated polynomials.

The convergence notion of Definition 1.2 can be equivalently reformulated in terms of deformations of the limiting domain $(\Sigma_\infty,\mathfrak{j}_\infty)$ so that it readily extends to sequences of stable J-holomorphic maps with varying complex structures $\mathfrak{j}_i$ on the domains Σ_i. This was formally done in the algebraic geometry category by [13], several years after this perspective had been introduced into the field informally, and adapted to the almost complex category by [26]. We summarize this perspective below.

Let $(\Sigma,\mathfrak{j})$ be a nodal Riemann surface. A flat family of deformations of $(\Sigma,\mathfrak{j})$ is a holomorphic map $\pi:\mathcal{U}\longrightarrow\Delta$, where $\mathcal{U}$ is a complex manifold and $\Delta\subset\mathbb{C}^N$ is a neighborhood of 0, such that

- $\pi^{-1}(\lambda)$ is a nodal Riemann surface for each $\lambda\in\Delta$ and $\pi^{-1}(0)=(\Sigma,\mathfrak{j})$,
- π is a submersion outside of the nodes of the fibers of π,
- for every $\lambda^*\equiv(\lambda_1^*,\ldots,\lambda_N^*)\in\Delta$ and every node $z^*\in\pi^{-1}(\lambda^*)$, there exist $i\in\{1,\ldots,N\}$ with $\lambda_i^*=0$, neighborhoods Δ_{λ^*} of λ^* in Δ and $\mathcal{U}_{z^*}$ of z^* in $\mathcal{U}$, and a holomorphic map
$$\Psi:\mathcal{U}_{z^*}\longrightarrow\big\{\big((\lambda_1,\ldots,\lambda_N),x,y\big)\in\Delta_{\lambda^*}\times\mathbb{C}^2:xy=\lambda_i\big\}$$
such that Ψ is a homeomorphism onto a neighborhood of $(\lambda^*,0,0)$ and the composition of Ψ with the projection to Δ_{λ^*} equals $\pi|_{\mathcal{U}_{z^*}}$.

If $\pi:\mathcal{U}\longrightarrow\Delta$ is a flat family of deformations of $(\Sigma,\mathfrak{j})$ and Σ is compact, there exists a neighborhood $\mathcal{U}^*\subset\mathcal{U}$ of $\Sigma^*\subset\pi^{-1}(0)$ such that
$$\pi|_{\mathcal{U}^*}:\mathcal{U}^*\longrightarrow\Delta_0\equiv\pi(\mathcal{U}^*)\subset\Delta$$
is a trivializable Σ^*-fiber bundle in the smooth category. For each $\lambda\in\Delta_0$, let
$$\psi_\lambda:\Sigma^*\longrightarrow\pi^{-1}(\lambda)\cap\mathcal{U}^*$$
be the corresponding smooth identification. If $\lambda_i\in\Delta$ is a sequence converging to $0\in\Delta$ and $u_i:\pi^{-1}(\lambda_i)\longrightarrow X$ is a sequence of continuous maps that are smooth on the complements of the

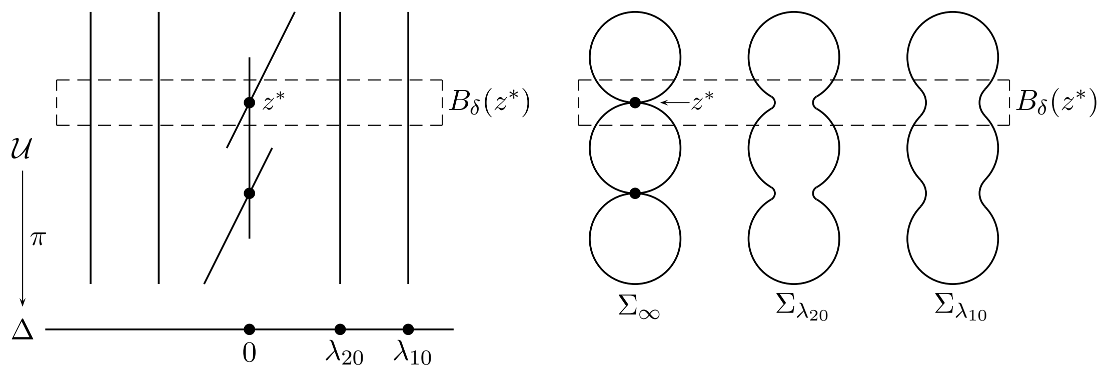

Figure 3: A complex-geometric presentation of a flat family of deformations of $(\Sigma_\infty, \mathfrak{j}_\infty) = \pi^{-1}(0)$ and a differential-geometric presentation of the domains of the maps u_i in Definition 1.4.

nodes of $\pi^{-1}(\lambda_i)$, we say that the sequence u_i converges to a smooth map $u: \Sigma^* \longrightarrow X$ u.c.s. (uniformly on compact subsets) if the sequence of maps

$$u_i \circ \psi_{\lambda_i} : \Sigma^* \longrightarrow X$$

converges to u uniformly in the C^∞-topology on compact subsets of Σ^*. This notion is independent of the choices of $\mathcal{U}^*$ and trivialization of $\pi|_{\mathcal{U}^*}$.

Definition 1.4 (Gromov's Convergence II)**.** Let (X, J, g) be an almost complex Riemannian manifold. A sequence $(\Sigma_i, \mathfrak{j}_i, u_i)$ of stable J-holomorphic maps converges to a stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ if $E_g(u_i) \longrightarrow E_g(u_\infty)$ as $i \longrightarrow \infty$ and there exist

(a) a flat family of deformations $\pi: \mathcal{U} \longrightarrow \Delta$ of $(\Sigma_\infty, \mathfrak{j}_\infty)$,

(b) a sequence $\lambda_i \in \Delta$ converging to $0 \in \Delta$, and

(c) equivalences $h_i: \pi^{-1}(\lambda_i) \longrightarrow (\Sigma_i, \mathfrak{j}_i)$

such that $u_i \circ h_i$ converges to $u_\infty|_{\Sigma^*_\infty}$ u.c.s.

By the compactness of Σ_∞, the notion of convergence of Definition 1.4 is independent of the choice of metric g on X. It is illustrated in Figure 3. If the Riemann surfaces $(\Sigma_i, \mathfrak{j}_i)$ are smooth, the limiting Riemann surface $(\Sigma_\infty, \mathfrak{j}_\infty)$ is obtained by pinching some disjoint embedded circles in the smooth two-dimensional manifold Σ underlying these Riemann surfaces.

If $(\Sigma_i, \mathfrak{j}_i) = (\Sigma, \mathfrak{j})$ for all i as in Definition 1.2, only contractible circles are pinched to produce Σ_∞; it then consists of Σ with trees of spheres attached. The family $\pi: \mathcal{U} \longrightarrow \Delta$ is obtained by starting with the family

$$\pi_0 : \mathcal{U}_0 \equiv \mathbb{C} \times \Sigma \longrightarrow \mathbb{C},$$

then blowing up $\mathcal{U}_0$ at a point of $\{0\} \times \Sigma$ to obtain a family $\pi_1 : \mathcal{U}_1 \longrightarrow \mathbb{C}$ with the central fiber $\Sigma_1 \equiv \pi_1^{-1}(0)$ consisting of Σ with $\mathbb{P}^1$ attached, then blowing up a smooth point of Σ_1, and so on. The number of blowups involved is precisely the number of nodes of Σ_∞, i.e. four in the case of Figure 2 and two in the case of Figure 3. The pinched annuli on the right-hand side of Figure 3 correspond to $\phi_\alpha(B_\delta(z_{\alpha\beta})) \cup \phi_\beta(B_\delta(z_{\beta\alpha}))$ in the notation of [29, Chapters 4,5].

With the setup of Definition 1.4, let $B_\delta(z^*)\subset\mathcal{U}$ denote the ball of radius $\delta\in\mathbb{R}^+$ around a point $z^*\in\mathcal{U}$ with respect to some metric on $\mathcal{U}$. Then,

$$\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}\mathrm{diam}_g\big(u_i\big(h_i(\pi^{-1}(\lambda_i)\cap B_\delta(z^*))\big)\big)=0 \qquad \forall\ z^*\in\Sigma_\infty\,. \tag{1.2}$$

This is immediate from the last condition in Definition 1.4 if $z^*\in\Sigma_\infty^*$. If $z^*\in\Sigma_\infty-\Sigma_\infty^*$ is a node of Σ_∞, (1.2) is a consequence of both convergence conditions of Definition 1.4 and the maps u_i being J-holomorphic. It is a reflection of the fact that bubbling or any other kind of erratic C^0-behavior of a sequence of J-holomorphic maps requires a nonzero amount of energy in the limit, but the two convergence conditions of Definition 1.4 ensure that all limiting energy is absorbed by $u|_{\Sigma_\infty^*}$ and thus none is left for bubbling around the nodes of Σ_∞. An immediate implication of (1.2) is that $u_i(h_i(\pi^{-1}(\lambda_i)\cap B_\delta(z^*)))$ is contained in a geodesic ball around $u_\infty(z^*)$ in X. Thus,

$$u_{i*}[\Sigma_i]=u_{\infty *}[\Sigma_\infty]\in H_2(X;\mathbb{Z})$$

for all $i\in\mathbb{Z}^+$ sufficiently large. If Σ_∞ is a tree of spheres (and thus so is each Σ_i), then u_i with i sufficiently large lies in the equivalence class in $\pi_2(X)$ determined by u_∞ for the same reason.

Theorem 1.5 (Gromov's Compactness II)**.** *Let (X,J,g) be a compact almost complex Riemannian manifold and $(\Sigma_i,\mathfrak{j}_i,u_i)$ be a sequence of stable J-holomorphic maps. If it satisfies (1.1), then it contains a subsequence converging to some stable J-holomorphic map $(\Sigma_\infty,\mathfrak{j}_\infty,u_\infty)$ in the sense of Definition 1.4.*

This theorem is obtained by combining the compactness of the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{1,1}$ of stable (possibly) nodal elliptic curves and $\overline{\mathcal{M}}_g$ of stable nodal genus $g\geq 2$ curves with the proof of Theorem 1.3 in Section 5.3. One first establishes Theorem 1.5 under the assumption that each $(\Sigma_i,\mathfrak{j}_i)$ is a smooth connected Riemann surface of genus $g\geq 1$ (the $g=0$ case is treated by Theorem 1.3). If $g=1$, we add a marked point to each domain $(\Sigma_i,\mathfrak{j}_i)$ and take a subsequence converging in $\overline{\mathcal{M}}_{1,1}$ to the equivalence class of some stable nodal elliptic curve $(\Sigma'_\infty,\mathfrak{j}'_\infty,z'_\infty)$. If $g\geq 2$, we take a subsequence of $(\Sigma_i,\mathfrak{j}_i)$ converging in $\overline{\mathcal{M}}_g$ to the equivalence class of some stable nodal genus g curve $(\Sigma'_\infty,\mathfrak{j}'_\infty)$. This ensures the existence of a flat family of deformations $\pi':\mathcal{U}'\longrightarrow\Delta'$ of $(\Sigma'_\infty,\mathfrak{j}'_\infty)$, of a sequence $\lambda'_i\in\Delta'$ converging to $0\in\Delta'$, and of equivalences $h_i:\pi'^{-1}(\lambda'_i)\longrightarrow(\Sigma_i,\mathfrak{j}_i)$. The associated neighborhood $\mathcal{U}'^*$ of Σ'^*_∞ in $\mathcal{U}'$ can be chosen so that $\pi'^{-1}(\lambda)$–$\mathcal{U}'^*$ consists of finitely many circles for every $\lambda'\in\Delta'$ sufficiently small. The complement of the image of the associated identifications

$$\psi'_\lambda:\Sigma'^*_\infty\longrightarrow\pi'^{-1}(\lambda)\cap\mathcal{U}'^*$$

in $\pi'^{-1}(\lambda)$ has the same property.

One then applies the construction in the proof of Theorem 1.3 to the sequence of J-holomorphic maps

$$u_i\circ h'_i:\Sigma'^*_\infty\longrightarrow X$$

to obtain a J-holomorphic map $\widetilde{u}'_\infty$ from the normalization $\widetilde{\Sigma}'_\infty$ of Σ_∞ and finitely J-holomorphic maps from trees of $\mathbb{P}^1$. Each of these trees will have one or two special points that are associated with points of $\widetilde{\Sigma}'_\infty$ (the latter happens if bubbling occurs at a preimage of a node of Σ'_∞ in $\widetilde{\Sigma}'_\infty$). Identifying these trees with the corresponding points of $\widetilde{\Sigma}'_\infty$ as in the proof of Theorem 1.3, we obtain a J-holomorphic map $(\Sigma_\infty,\mathfrak{j}_\infty,u_\infty)$ satisfying the requirements of Definition 1.4. It is necessarily stable if $g\geq 2$, or Σ'_∞ is smooth, or Σ_∞ contains a separating node. Otherwise, the identifications h'_i may first need to be reparametrized to ensure that either the limiting map $\widetilde{u}'_\infty$ is not constant or the sequence $u_i \circ h_i$ produces a bubble at least one smooth point of $\widetilde{\Sigma}'_\infty$.

A k-marked Riemann surface is a tuple $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k)$ such that $(\Sigma, \mathfrak{j})$ is a Riemann surface and $z_1, \ldots, z_k \in \Sigma^*$ are distinct points. If (X, J) is an almost complex manifold, a k-marked J-holomorphic map into X is a tuple $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k, u)$, where $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k)$ is k-marked Riemann surface and $(\Sigma, \mathfrak{j}, u)$ is a J-holomorphic map into X. The degree of such a map is the homology class

$$A = u_*[\Sigma] \in H_2(X; \mathbb{Z}) .$$

The notions of equivalence, stability, and convergence as in Definition 1.4 and the above convergence argument for smooth domains $(\Sigma_i, \mathfrak{j}_i)$ readily extend to k-marked J-holomorphic maps. The general case of Theorem 1.5, including its extension to stable marked maps, is then obtained by

- passing to a subsequence of $(\Sigma_i, \mathfrak{j}_i, u_i)$ with the same topological structure of the domain,
- viewing it as a sequence of tuples of J-holomorphic maps with smooth domains with an additional marked point for each preimage of the nodes in the normalization, and
- applying the conclusion of the above argument to each component of the tuple.

1.3 Moduli spaces

The natural extension of Definition 1.4 to marked J-holomorphic maps topologizes the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ of equivalence classes of stable degree A k-marked genus g J-holomorphic maps into X for each $A \in H_2(X; \mathbb{Z})$. The evaluation maps

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow X, \quad (\Sigma, \mathfrak{j}, z_1, \ldots, z_k, u) \longrightarrow u(z_i),$$

are continuous with respect to this topology. If $2g+k \geq 3$, there is a continuous map

$$\mathfrak{f} : \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow \overline{\mathcal{M}}_{g,k}$$

to the Deligne-Mumford moduli space of stable k-marked genus g nodal curves obtained by forgetting the map u and then contracting the unstable components of the domain.

There is a continuous map

$$\mathfrak{f}_{k+1} : \overline{\mathfrak{M}}_{g,k+1}(X, A; J) \longrightarrow \overline{\mathfrak{M}}_{g,k}(X, A; J) \tag{1.3}$$

obtained by forgetting the last marked point z_{k+1} and then contracting the components of the domain to stabilize the resulting k-marked J-holomorphic map. For each $i = 1, \ldots, k$, this fibration has a natural continuous section

$$s_i : \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow \overline{\mathfrak{M}}_{g,k+1}(X, A; J)$$

described as follows. For a k-marked nodal Riemann surface $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k)$, let $(\Sigma', \mathfrak{j}', z_1, \ldots, z_{k+1})$ be the $(k+1)$-marked nodal Riemann surface so that $(\Sigma', \mathfrak{j}')$ consists of $(\Sigma, \mathfrak{j})$ with $\mathbb{P}^1$ attached at z_i, $z'_1, z'_i \in \mathbb{P}^1$, and $z'_j = z_j \in \Sigma$ for all $j = 1, \ldots, k$ different from k; see Figure 4. We define

$$s_i\big([\Sigma, \mathfrak{j}, z_1, \ldots, z_k, u]\big) = \big[\Sigma', \mathfrak{j}', z'_1, \ldots, z'_{k+1}, u'\big],$$

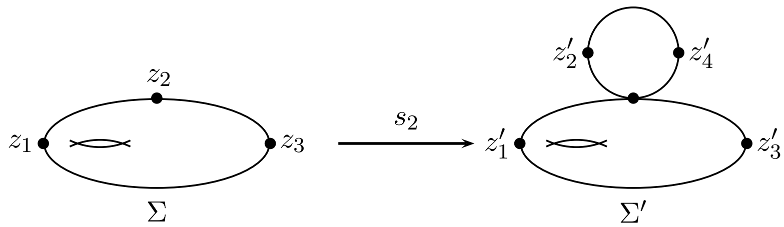

Figure 4: Section s_2 of the fibration (1.3) with $k=3$

with $(\Sigma', \mathfrak{j}', z'_1, \ldots, z'_{k+1})$ as described and u' extending u over the extra $\mathbb{P}^1$ by the constant map with value $u(z_i)$. The pullback

$$L_i \longrightarrow \overline{\mathfrak{M}}_{g,k}(X, A; J)$$

of the vertical tangent line bundle of (1.3) by s_i is called the universal tangent line bundle at the i-th marked point. Let $\psi_i = c_1(L_i^*)$ be the i-th descendant class.

A remarkable property of Gromov's topology which lies behind most of its applications is that the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ is Hausdorff and has a particularly nice deformation-obstruction theory. In the algebraic-geometry category, the latter is known as a perfect two-term deformation-obstruction theory. In the almost complex category, this is reflected in the existence of an atlas of finite-dimensional approximations in the terminology of [26] or of an atlas of Kuranishi charts in the terminology of [26].

If (X, J) is an almost complex manifold and J is tamed by a symplectic form ω, then the energy $E_g(u)$ of degree A J-holomorphic map u with respect to the metric g determined by J and ω is $\omega(A)$; see (2.18). In particular, it is the same for all elements of the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$. If in addition X is compact, then Theorem 1.5 implies that this moduli space is also compact. Combining this with the remarkable property of the previous paragraph, the constructions of [4, 25, 26, 12] endow $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ with a virtual fundamental class. It depends only on ω, in a suitable sense, and not an almost complex structure J tamed by ω. This class in turn gives rise to Gromov-Witten invariants of (X, ω):

$$\langle \tau_{a_1}\alpha_1, \ldots, \tau_{a_k}\alpha_k \rangle_{g,A}^X \equiv \left\langle \left(\psi_1^{a_1} \mathrm{ev}_1^* \alpha_1\right) \ldots \left(\psi_k^{a_k} \mathrm{ev}_k^* \alpha_k\right), \left[\overline{\mathfrak{M}}_{g,k}(X, A; J)\right]^{\mathrm{vir}} \right\rangle \in \mathbb{Q}$$

for all $a_i \in \mathbb{Z}^{\geq 0}$ and $\alpha_i \in H^*(X; \mathbb{Q})$.

2 Preliminaries

An outline of these notes with an informal description of the key statements appears in Section 2.1; Figure 5 indicates primary connections between these statements. Sections 2.3 introduces the most frequently used notation and terminology and makes some basic observations.

2.1 Overview of the main statements

The main technical statement of Section 3 of these notes and of Chapter 2 in [29] is the Carleman Similarity Principle; see Proposition 3.1. It yields a number of geometric conclusions about the local behavior of a J-holomorphic map $u\colon \Sigma \longrightarrow X$ from a smooth Riemann surface $(\Sigma, \mathfrak{j})$ into an almost complex manifold (X,J). For example, it implies that for every point z of a topological component of Σ on which u is not constant, the ℓ-th derivative of u at z in a chart around $u(z)$ does not vanish for some $\ell\in\mathbb{Z}^+$; see Corollary 3.3. We denote by $\mathrm{ord}_z u\in\mathbb{Z}^+$ the minimum of such integers ℓ and call it the order of u at z; it is independent of the choice of a chart around $u(z)$. If u is constant on the component of Σ containing z, we set $\mathrm{ord}_z u=0$; this convention (rather than $\mathrm{ord}_z u=\infty$) fits nicely with Corollary 3.11 and Proposition 3.13. A point $z\in\Sigma$ is a singular point of u, i.e. $\mathrm{d}_z u=0$, if and only if $\mathrm{ord}_z u\neq 1$.

If u is not constant on every connected component of Σ, the singular points of u and the preimages of a point $x\in X$ are discrete subsets of Σ; see Corollary 3.4. In the case Σ is compact, the second statement of Corollary 3.4 implies that

$$\mathrm{ord}_x u\equiv\sum_{z\in u^{-1}(x)}\mathrm{ord}_z u \quad\in\mathbb{Z}^{\ge 0}\qquad\forall\,x\in X; \tag{2.1}$$

we call this number the order of u at x. If $x\notin\mathrm{Im}(u)$, $\mathrm{ord}_x u=0$. By Corollary 3.11, the number (2.1) is seen by the behavior of the energy (2.16) of u and its restrictions to open subsets of Σ. This observation underpins the Monotonicity Lemma for J-holomorphic maps, which bounds below the energy required to "escape" from a small ball in X; see Proposition 3.13.

The main technical statement of Section 4 of these notes and of Chapter 4 in [29] is the Mean Value Inequality. It bounds the pointwise differentials $\mathrm{d}_z u$ of a J-holomorphic map u from $(\Sigma,\mathfrak{j})$ into (X,J) of sufficiently small energy $E_g(u)$ by $E_g(u)$, i.e. by the L^2-norm of $\mathrm{d}u$, from above and immediately yields a bound on the energy of non-constant J-holomorphic maps from S^2 into (X,J) from below; see Proposition 4.1 and Corollary 4.2, respectively. The Mean Value Inequality also implies that the energy of a J-holomorphic map u from a cylinder $[-R,R]\times S^1$ carried by $[-R+T,R-T]\times S^1$ and the diameter of the image of this middle segment decay at least exponentially with T, provided the overall energy of u is sufficiently small. As shown in the proof of Proposition 5.5, this technical implication ensures that the energy is preserved under Gromov's convergence and the resulting bubbles connect.

Another important implication of Proposition 4.1 is that a continuous map from a Riemann surface $(\Sigma,\mathfrak{j})$ into an almost complex manifold (X,J) which is holomorphic outside of a discrete collection of points and has bounded energy is in fact holomorphic on all of Σ; see Proposition 4.8. This conclusion plays a central role in the proof of Lemma 5.4. Theorem 1.3 is deduced from Lemma 5.4 and Proposition 5.5 in Section 5.3.

Combined with Proposition 3.1 and some of its corollaries, Proposition 4.1 implies that every non-constant J-holomorphic map from a compact Riemann surface $(\Sigma,\mathfrak{j})$ factors through a somewhere injective J-holomorphic map from a compact Riemann surface $(\Sigma',\mathfrak{j}')$; see Proposition 4.11. The proof of this statement with X compact appears in Chapter 2 of [29], but uses the Removal Singularities Theorem proved in Chapter 4 of [29]. Proposition 4.1 is the key technical step in establishing transversality for the moduli spaces of simple J-holomorphic maps and constructing pseudocycles out of these spaces; see [45].

$$
\begin{tikzcd}
\mathbf{3.1} \arrow[d] \arrow[dr] & & & & & \mathbf{4.1} \arrow[dl] \arrow[ddl] \arrow[dd] \arrow[dr] & \\
3.11 \arrow[d] & 3.4 \arrow[r] & \mathbf{4.11} & 4.8 \arrow[l] \arrow[d] & 4.9 \arrow[l] & & 4.2 \arrow[dl] \arrow[ddl] \\
\mathbf{3.13} \arrow[d] \arrow[rrr] \arrow[rrrr, bend right] & & & \mathbf{5.1} \arrow[rr, bend right] & 4.17 \arrow[r] & 5.2\text{-}5.4 \arrow[d] \arrow[dr] & \\
3.14 & & & & & 5.5 \arrow[r] & \mathbf{1.3}
\end{tikzcd}
$$

1.3 Gromov's Compactness
3.4 Singularities of J-holomorphic maps
3.13 Monotonicity Lemma
4.1 Mean Value Inequality
4.9 Isoperimetric Inequality
4.17 Bounds on long cylinders
5.2-5.4 Bubbling

3.1 Carleman Similarity Principle
3.11 Local energy vs. $\mathrm{ord}_x u$
3.14,4.2 Lower energy bounds
4.8 Regularity of J-holomorphic maps
4.11 Global structure of J-holomorphic maps
5.1 Removal of Singularity
5.5 Gromov's convergence

Figure 5: Connections between the main statements leading to Theorem 1.3 and Proposition 4.11

2.2 Almost complex structures

An almost complex structure J on a smooth manifold X is a complex structure on (the fibers of) the real vector bundle TX over X, i.e.

$$J \in \Gamma\big(X; \mathrm{End}_{\mathbb{R}}(TX)\big) \qquad \text{and} \qquad J^2 = -\mathrm{Id}_{TX}\,.$$

An almost complex manifold is a pair (X, J) consisting of a smooth manifold X and an almost complex structure J on X. Since there is a canonical identification

$$T\mathbb{C}^n \approx \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

of real vector bundles over $\mathbb{C}^n$, the scalar multiplication by $\mathfrak{i}$ on the vector space $\mathbb{C}^n$ determines an almost complex structure $J_{\mathbb{C}^n}$ on $\mathbb{C}^n$. A complex structure on a manifold X, i.e. an atlas of coordinate charts that overlap holomorphically, determines an almost complex structure J on X: if

$$\varphi_\alpha : U_\alpha \longrightarrow \varphi(U_\alpha) \subset \mathbb{C}^n$$

is a chart in the chosen atlas, then

$$J|_{TU_\alpha} = \big\{\mathrm{d}\varphi_\alpha\big\}^{-1} \circ J_{\mathbb{C}^n} \circ \mathrm{d}\varphi_\alpha : TU_\alpha \longrightarrow TU_\alpha\,. \tag{2.2}$$

Such an almost complex structure J is called integrable or simply a complex structure on X.

Exercise 2.1. Let X a complex manifold with a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of coordinate charts as above. Show that the definitions of the almost complex structure J in (2.2) agree on the overlaps $U_\alpha \cap U_\beta$.

Let (X,J) be an almost complex manifold. We call a k-tensor g on X J-invariant if

$$g\big(Jv_1,\ldots,Jv_k\big)=g(v_1,\ldots,v_k) \quad \forall\, v_1,\ldots,v_k\in T_xX,\ x\in X.$$

We call such a tensor J-anti-invariant

$$g\big(Jv_1,\ldots,Jv_k\big)=-g(v_1,\ldots,v_k) \quad \forall\, v_1,\ldots,v_k\in T_xX,\ x\in X.$$

The Nijenhuis tensor A_J of (X,J) is defined by

$$A_J(\xi_1,\xi_2)=\frac{1}{4}\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big) \quad \forall\, \xi_1,\xi_2\in\Gamma(X;TX). \tag{2.3}$$

Exercise 2.2. Let (X,J) be an almost complex manifold. Show that

(1) equation (2.3) determines a J-anti-invariant alternating 2-tensor

$$A_J\in\Gamma\big(X;\mathrm{Hom}_{\mathbb{C}}\big((\Lambda^2_{\mathbb{C}}(TX,J))\otimes_{\mathbb{C}}(TX,-J)\big)\big)\subset\Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(TX\otimes_{\mathbb{R}}TX,TX)\big);$$

(2) if J is integrable, then $N_J=0$.

The converse of Exercise 2.2(2) is [34, Theorem 1.1]. Since the Nijenhuis tensor $N_{\mathfrak{j}}$ of any two-dimensional almost complex manifold $(\Sigma,\mathfrak{j})$ vanishes, it follows that every almost complex structure $\mathfrak{j}$ on a two-dimensional manifold is integrable. We call such a pair $(\Sigma,\mathfrak{j})$ a Riemann surface.

Let (X,J) be an almost complex manifold. For a 2-form ω on a manifold X, we define a J-invariant symmetric 2-tensor and a J-anti-invariant 2-form on X by

$$\begin{aligned} g_J^{\omega}(v,v')&=\frac{1}{2}\big(\omega(v,Jv')-\omega(Jv,v')\big),\\ \omega_J(v,v')&=\frac{1}{2}\big(\omega(Jv,Jv')-\omega(v,v')\big)\end{aligned} \qquad \forall\, v,v'\in T_xX,\ x\in X, \tag{2.4}$$

respectively. We note that

$$g_J^{\omega}(v,v)+g_J^{\omega}(v',v')=2\omega(v,v')+g_J^{\omega}(v+Jv',v+Jv')+2\omega_J(v,v') \quad \forall\, v,v'\in T_xX,\ x\in X. \tag{2.5}$$

A 2-form ω on X tames J if $g_J^{\omega}(v,v)>0$ for all $v\in TX$ nonzero; in such a case, ω is nondegenerate and g_J^{ω} is a metric. Conversely, if g_J^{ω} is a metric, then ω tames J and is thus nondegenerate. The almost complex structure J is ω-compatible if ω tames J and $\omega_J=0$. If g is a J-invariant metric g, then J is compatible with the 2-form on X defined by

$$\omega_J^g(v,v')=g(Jv,v') \quad \forall\, v,v'\in T_xX,\ x\in X\,.$$

We note that $g_J^{\omega_J^g}=g$ and $\omega_J^{g_J^{\omega}}=\omega$ for any 2-form ω on X compatible with J.

A symplectic form on a smooth manifold X is a nondegenerate 2-form ω on X which is closed, i.e. $\mathrm{d}\omega=0$. If ω is any nondegenerate 2-form on X and the dimension of X is $2n$, then ω^n is a volume form on X. If X is compact, then $\langle\omega^n,[X]\rangle\neq 0$. If in addition ω is closed, this implies that

$$[\omega]\neq 0\in H^2_{\mathrm{deR}}(X). \tag{2.6}$$

We call a triple (X,ω,J) an almost Kähler manifold (resp. Kähler manifold) if ω is a symplectic form and J is an almost complex structure (resp. a complex structure) on X compatible with ω. If X is a compact manifold admitting a Kähler structure, then

$$\mathrm{rk}_{\mathbb{Z}} H^{2k-1}(X;\mathbb{Z}) \in 2\mathbb{Z} \qquad \forall\, k\in\mathbb{Z} \tag{2.7}$$

by the Hodge Decomposition Theorem; see [15, p116].

The prototypical example of a compact Kähler manifold is the n-dimensional complex projective space $\mathbb{P}^n$ with its standard complex structure $J_{\mathbb{P}^n}$ and the Fubini-Study symplectic form ω_{FS}; see [15, p30]. A degree a homogeneous polynomial P on $\mathbb{C}^{n+1}$ determines a complex hypersurface

$$X_a \equiv \big\{[Z_0,Z_1,\ldots,Z_n]\in\mathbb{P}^n\colon P(Z_0,Z_1,\ldots,Z_n)=0\big\}$$

in $\mathbb{P}^n$. This hypersurface is smooth if and only if the partial derivatives $\partial P/\partial Z_i$ do not have a common point of vanishing on $\mathbb{C}^{n+1}-\{0\}$. In such a case, $(X_a,\omega_{\mathrm{FS}}|_{TX_a},J_{\mathbb{P}^n}|_{TX_a})$ is also a Kähler manifold. Below we provide more exotic examples of almost complex and symplectic manifolds.

By [6, Proposition 2.3], any almost complex structure J on a smooth manifold X of (real) dimension at least 4 can be deformed inside a non-empty open subset to an almost complex structure not tamed by any symplectic form; see Example 2.14 for more details. Below we give other examples of almost complex structures that are not compatible with any symplectic form and of symplectic forms that are not compatible with any integrable almost complex structure. On the other hand, the spaces of almost complex structure compatible and tamed by a fixed symplectic form ω are nonempty and contractible; see Proposition 2.6.

Example 2.3. The action of the group $\mathbb{Z}$ on the complex manifold $\mathbb{C}^2-\{0\}$ given by

$$\mathbb{Z}\times\big(\mathbb{C}^2-\{0\}\big)\longrightarrow\mathbb{C}^2-\{0\}, \qquad k\cdot z=2^k z,$$

is properly discountinuous (every point $z\in\mathbb{C}^2-\{0\}$ has a neighborhood U disjoint from $k\cdot U$ for every $k\in\mathbb{Z}-\{0\}$). Thus, the quotient

$$X\equiv\big(\mathbb{C}^2-\{0\}\big)/\mathbb{Z}\approx S^3\times S^1$$

inherits a complex structure from $\mathbb{C}^2-\{0\}$. Since $H^2(X;\mathbb{Z})=\{0\}$, the sentence containing (2.6) implies that X admits no *symplectic* form. Since $H^1(X;\mathbb{Z})\approx\mathbb{Z}$, the sentence containing (2.7) also implies that X admits no *Kähler* structure.

Example 2.4. We denote by $\mathbb{H}$ and $\mathbb{O}\equiv\mathbb{H}\oplus\mathbb{H}\epsilon$ the $\mathbb{R}$-algebras of quaternions and octonions (or Cayley numbers), respectively. The conjugation, (non-associative) multiplication, and the Euclidean inner-product on $\mathbb{O}\approx\mathbb{R}^8$ are given by

$$\begin{gathered}\overline{a+b\epsilon}=\bar a-b\epsilon, \quad (a+b\epsilon)(c+d\epsilon)=(ac-\bar d b)+(da+b\bar c)\epsilon,\\ \langle a+b\epsilon,c+d\epsilon\rangle=\frac12\big(a\bar c+c\bar a+b\bar d+d\bar b\big)\end{gathered} \qquad \forall\, a,b,c,d\in\mathbb{H}.$$

These operations satisfy

$$\begin{gathered}x+\bar x\in\mathbb{R}\subset\mathbb{O}\ \ \forall\,x\in\mathbb{O}, \quad xr=rx\ \ \forall\,x\in\mathbb{O},\,r\in\mathbb{R}, \quad \overline{xy}=\bar y\,\bar x,\ (xy)y=xy^2\ \ \forall\,x,y\in\mathbb{O},\\ \langle x,y\rangle=\frac12\big(x\bar y+y\bar x\big)\ \ \forall\,x,y\in\mathbb{O}, \qquad \langle xu,yu\rangle=\langle x,y\rangle|u|^2\ \ \forall\,x,y,u\in\mathbb{O}.\end{gathered} \tag{2.8}$$

Let

$$\operatorname{Im}\mathbb{O} = \big\{x\in\mathbb{O}\colon \overline{x}=-x\big\} = \big\{x\in\mathbb{O}\colon \langle x,1\rangle=0\big\} = (\operatorname{Im}\mathbb{H})\oplus\mathbb{H}\epsilon$$
$$\text{and}\qquad X = \big\{x\in\operatorname{Im}\mathbb{O}\colon |x|=1\big\}\approx S^6$$

be the subspaces of purely imaginary octonions and of purely imaginary unit octonions, respectively. For each $u\in X$, define

$$J_u\colon \mathbb{O}\longrightarrow\mathbb{O}, \qquad J_u(x)=xu.$$

Since $\bar{u}=-u$, (2.8) implies that this linear automorphism of $\mathbb{O}$ satisfies

$$J_u^2=-\mathrm{Id}_{\mathbb{O}} \qquad\text{and}\qquad \big\langle J_u(x),J_u(y)\big\rangle=\langle x,y\rangle \quad \forall\, x,y\in\mathbb{O}.$$

Thus, J_u preserves the subspace

$$\big\{v\in\operatorname{Im}\mathbb{O}\colon \langle v,u\rangle=0\big\}=T_uX.$$

It follows that the family $\{J_u\}_u$ determines an almost complex structure on $\mathbb{O}-\mathbb{R}$, which restricts to an almost complex structure J on X. The Nijenhuis tensor of J is given by

$$A_J(v_1,v_2)=\frac{1}{4}\big(v_1(v_2u)-(v_1v_2)u-v_2(v_1u)+(v_2v_1)u\big) \qquad \forall\, u\in X,\ v_1,v_2\in T_uX.$$

For example, $A_J(\mathfrak{i},\mathfrak{j})|_{u=\epsilon}=(\mathfrak{j}\mathfrak{i})\epsilon$ if $\mathfrak{i},\mathfrak{j}\in\mathbb{H}$ are orthonormal purely imaginary quaternions. Thus, the almost complex structure J is non-integrable. Since $H^2(X;\mathbb{Z})=\{0\}$, the sentence containing (2.6) implies that X admits *no* symplectic form.

Example 2.5 ([22, 41])**.** Let G be the group with the underlying set $\mathbb{Z}^4$ and the product given by

$$(k,\ell,m,n)(k',\ell',m',n')=(k+k',kk'+\ell+\ell',m+m',km'+n+n').$$

This group acts on $\mathbb{R}^4$ on the left by the diffeomorphisms

$$(k,\ell,m,n)(x_1,x_2,y_1,y_2)=\big(x_1+k,x_2+kx_1+\ell,y_1+m,y_2+ky_1+n\big)$$

properly discontinuously. Thus, the quotient X is a smooth manifold, known as the Kodaira-Thurston manifold. It is a non-trivial $\mathbb{T}^2$-bundle over $\mathbb{T}^2$ and thus compact. Since the commutator $[G,G]$ of G is $\{0\}^3\times\mathbb{Z}$, $H_1(X;\mathbb{Z})\approx\mathbb{Z}^3$. The above G-action on $\mathbb{R}^4$ preserves the symplectic form

$$\omega_{\mathbb{R}^4}\equiv \mathrm{d}x_1\wedge\mathrm{d}x_2+\mathrm{d}y_1\wedge\mathrm{d}y_2$$

and the complex structure $J_{\mathbb{R}^4}$ given by

$$J_{\mathbb{R}^4}\frac{\partial}{\partial x_1}=\frac{\partial}{\partial y_1} \qquad\text{and}\qquad J_{\mathbb{R}^4}\frac{\partial}{\partial x_2}=\frac{\partial}{\partial y_2}\,.$$

Thus, they descend to a symplectic form ω and a complex structure J on X. On the other hand, the sentence containing (2.7) implies that X admits *no* Kähler structure.

A simply connected closed ten-dimensional smooth manifold X admitting a symplectic form, but no Kähler structure, is constructed in [27] as a symplectic blowup, in the sense of [29, Section 7.1], of $\mathbb{P}^5$ along an embedded copy of the Kodaira-Thurston manifold. Simply connected closed four-dimensional smooth manifolds X admitting a symplectic form, but no complex structure, are constructed in [16, Sections 3,6]. On the other hand, every simply connected closed complex manifold (X,J) of real dimension 4 admits a Kähler structure; see [2, Theorem IV.3.1]. A simply connected closed six-dimensional smooth manifold X admitting a symplectic form and a complex structure, but no Kähler structure, is constructed in [3].

Let X be a smooth manifold. We denote by $\mathcal{J}(X)$ and $\Omega^2_\bullet(X)$ the spaces of (smooth) almost complex structure on X and of nondegenerate 2-forms on X, respectively, with the C^∞-topologies. These are Fréchet manifolds and

$$T_J\mathcal{J}(X)=\big\{A\in\Gamma(X;\mathrm{End}_{\mathbb{R}}(TX)):\,JA=-AJ\big\}\qquad\forall\,J\in\mathcal{J}(X).$$

For a 2-form ω on X, we denote by

$$\mathcal{J}_{\mathrm{tm}}(\omega),\mathcal{J}_{\mathrm{cm}}(\omega)\subset\mathcal{J}(X)$$

the subspaces of ω-tame and ω-compatible almost complex structures on X; both are empty unless $\omega\in\Omega^2_\bullet(X)$. The former is then an open subspace of $\mathcal{J}(X)$, while the latter is a Fréchet submanifold with

$$T_J\mathcal{J}_{\mathrm{cm}}(X)=\big\{A\in T_J\mathcal{J}(X):\,\omega(JA\cdot,\cdot)=-\omega(\cdot,JA\cdot)\big\}\qquad\forall\,J\in\mathcal{J}_{\mathrm{cm}}(X).$$

For a metric g on X, we similarly define

$$\mathcal{J}(g)\equiv\big\{J\in\mathcal{J}(X):\,g\big(J(v),J(v')\big)=g(v,v')\ \forall\,v,v'\in T_xX,\,x\in X\big\}$$

to be the subspace of almost complex structures on X preserving g.

Let g be a metric on X and ω a 2-form on X. Following [21, Appendix], we define

$$A_{g,\omega}\in\Gamma\big(X;\mathrm{End}_{\mathbb{R}}(TX)\big)\qquad\text{by}\quad g\big(A_{g,\omega}(v),v'\big)=\omega(v,v')\ \ \forall\,v,v'\in T_xX,\,x\in X.\tag{2.9}$$

Since ω is anti-symmetric in the two inputs,

$$g\big(A_{g,\omega}(v),v'\big)=-g\big(v,A_{g,\omega}(v')\big)\ \ \forall\,v,v'\in T_xX,\,x\in X,$$

i.e. the transpose $A^{\mathrm{tr}}_{g,\omega}$ with respect to g equals $-A_{g,\omega}$. If $\omega\in\Omega^2_\bullet(X)$, $A_{g,\omega}$ is an automorphism of TX. The Polar Decomposition Theorem [19, Proposition 2.19(1)] then implies that there exist unique automorphisms $J_{g,\omega},S_{g,\omega}$ of TX such that

$$\begin{gathered}A_{g,\omega}=J_{g,\omega}\circ S_{g,\omega},\qquad g\big(S_{g,\omega}(v),v\big)>0\ \ \forall\,v\in TX-X,\\ g\big(J_{g,\omega}(v),J_{g,\omega}(v')\big)=g(v,v')\text{ and }g\big(S_{g,\omega}(v),v'\big)=g\big(v,S_{g,\omega}(v')\big)\ \ \forall\,v,v'\in T_xX,\,x\in X.\end{gathered}\tag{2.10}$$

We denote by $J^{\mathrm{tr}}_{g,\omega}$ the transpose of $J_{g,\omega}$ with respect to g.

By the penultimate equation in (2.10),

$$J^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega},\,J_{g,\omega}\circ J^{\mathrm{tr}}_{g,\omega}=-\mathrm{Id}_{TX}.$$

Combined with the first and last equations in (2.10) and $A^{\mathrm{tr}}_{g,\omega}=-A_{g,\omega}$, this gives

$$\begin{split}\big(J_{g,\omega}\circ J_{g,\omega}\big)\circ\big(J^{\mathrm{tr}}_{g,\omega}\circ S_{g,\omega}\circ J_{g,\omega}\big)&=-J_{g,\omega}\circ S_{g,\omega}\circ J_{g,\omega}\\ &=-A_{g,\omega}\circ J_{g,\omega}=A^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega}=S_{g,\omega}\circ J^{\mathrm{tr}}_{g,\omega}\circ J_{g,\omega}=-S_{g,\omega}\,.\end{split}$$

From the last equation in (2.10) and the uniqueness of polar decomposition, it then follows that

$$J_{g,\omega}^2 = -\mathrm{Id}_{TX}, \quad J_{g,\omega}^{\mathrm{tr}} = -J_{g,\omega}, \quad J_{g,\omega}^{\mathrm{tr}} \circ S_{g,\omega} \circ J_{g,\omega} = S_{g,\omega}\,.$$

Along with (2.9) and the first three equations in (2.10), this gives

$$\begin{gathered}\omega\big(v, J_{g,\omega}(v)\big) > 0 \;\; \forall\, v \in TX - X, \quad \omega\big(J_{g,\omega}(v), J_{g,\omega}(v')\big) = \omega\big(v, v'\big) \;\; \forall\, v, v' \in T_xX,\, x \in X,\\ g^{\omega}_{J_{g,\omega}}(v, v') = \omega\big(v, J_{g,\omega}(v')\big) = -g\big(J_{g,\omega}\big(A_{g,\omega}(v)\big), v'\big) = g\big(A_{g,\omega}(v), v'\big) \;\; \forall\, v, v' \in T_xX,\, x \in X.\end{gathered}$$

In particular, $J_{g,\omega} \in \mathcal{J}(g) \cap \mathcal{J}_{\mathrm{cm}}(\omega)$. If either $J \in \mathcal{J}_{\mathrm{cm}}(\omega)$ and $g = g^{\omega}_J$ or $J \in \mathcal{J}_{\mathrm{cm}}(g)$ and $\omega = \omega^g_J$, then $A_{g,\omega} = J$ and thus $J_{g,\omega} = J$.

Suppose B is a topological space, $(\omega_s)_{s\in B}$ is a continuous family of nondegenerate 2-forms on X, and $(J_{s;0})_{s\in B}$ and $(J_{s;1})_{s\in B}$ are two families of almost complex structures on X with $J_{s;0}, J_{s;1} \in \mathcal{J}_{\mathrm{cm}}(\omega_s)$ for every $s \in B$. The map

$$B \times [0,1] \longrightarrow \mathcal{J}(X), \qquad (s,t) \longrightarrow J_{(1-t)g^{\omega_s}_{J_{s;0}} + t g^{\omega_s}_{J_{s;1}}, \omega_s}\,,$$

is then a homotopy between $(J_{s;0})_{s\in B}$ and $(J_{s;1})_{s\in B}$ such that $J_{s;0}, J_{s;1} \in \mathcal{J}_{\mathrm{cm}}(\omega_s)$ for all $s \in B$ and $t \in [0,1]$. In particular, the space $\mathcal{J}_{\mathrm{cm}}(\omega)$ is contractible for every $\omega \in \Omega^2_{\bullet}(X)$, provided $\mathcal{J}_{\mathrm{cm}}(\omega) \neq \emptyset$. Since $\mathcal{J}_{\mathrm{cm}}(\omega) \neq \emptyset$ if X is a point, the contractability of $\mathcal{J}_{\mathrm{cm}}(\omega)$ when nonempty implies that $\mathcal{J}_{\mathrm{cm}}(\omega) \neq \emptyset$ for any manifold X and $\omega \in \Omega^2_{\bullet}(X)$. This yields the claim concerning $\mathcal{J}_{\mathrm{cm}}(\omega)$ below. By Proof 2 of [30, Proposition 2.5.13], the inclusion

$$\mathcal{J}_{\mathrm{cm}}(\omega) \longrightarrow \mathcal{J}_{\mathrm{tm}}(\omega)$$

is a homotopy equivalence. This implies the claim concerning $\mathcal{J}_{\mathrm{tm}}(\omega)$ below.

Proposition 2.6. *Let X be a smooth manifold and $\omega \in \Omega^2_{\bullet}(X)$. The spaces $\mathcal{J}_{\mathrm{tm}}(\omega)$ and $\mathcal{J}_{\mathrm{cm}}(\omega)$ of ω-tamed and ω-compatible almost complex structures on X are nonempty and contractible.*

Any almost complex structure J on a smooth manifold X of real dimension 2 is necessarily integrable and compatible with a symplectic form. On the other hand, [6, Proposition 2.3] implies there can be no topological condition on an almost complex manifold (X, J) of real dimension 4 or higher that ensures the existence of a symplectic form ω even just taming J. Conjecture 2.7 below instead surmises that every deformation equivalence class of almost complex structures on a closed smooth manifold X of real dimension 6 or higher contains an almost complex structure which is compatible with some symplectic form ω in a given deformation equivalence class of "nondegenerate" elements of $H^2_{\mathrm{deR}}(X)$. If true, this would in particular imply that every deformation equivalence class of "nondegenerate" elements of $H^2_{\mathrm{deR}}(X)$ can be represented by a symplectic form if X is a closed smooth manifold of dimension at least 6 that admits an almost complex struture. The desired conclusion of this conjecture does not hold for closed smooth manifolds of dimension 4, as illustrated by Examples 2.9 and 2.10 below; a weaker claim is proposed by Conjecture 2.8 in this case.

Conjecture 2.7 ([9, Conjecture 6.1])**.** Suppose X is a closed smooth manifold of dimension $2n$, $\omega_0 \in \Omega^2_{\bullet}(X)$, and $\eta_0 \in H^2_{\mathrm{deR}}(X)$ is such that $\eta_0^n \neq 0$. If $n \geq 3$, there exist paths $(\omega_t)_{t\in[0,1]}$ in $\Omega^2_{\bullet}(X)$ and $(\eta_t)_{t\in[0,1]}$ in $H^2_{\mathrm{deR}}(X)$ so that $\eta_t^n \neq 0$ for every $t \in [0,1]$, $\mathrm{d}\omega_1 = 0$, and $[\omega_1] = \eta_1$.

Conjecture 2.8 ([9, Conjecture 6.2])**.** Suppose X is a closed smooth fourfold, $\omega_0\in\Omega^2_\bullet(X)$, and $\eta_0\in H^2(X;\mathbb{Z})$ is such that $\eta_0^2\neq0$. There exist $N\in\mathbb{Z}$, a closed oriented surface $\Sigma\subset X$ representing the Poincaré dual to $\pm N\eta_0$, a smooth covering $\pi\colon\widetilde{X}\longrightarrow X$ branched only over Σ so that $\pi^{-1}(\Sigma)\subset\widetilde{X}$ is a submanifold, and paths $(\widetilde{\omega}_t)_{t\in[0,1]}$ in $\Omega^2_\bullet(\widetilde{X})$ and $(\widetilde{\eta}_t)_{t\in[0,1]}$ in $H^2_{\mathrm{deR}}(\widetilde{X})$ so that

$$\widetilde{\omega}_0=\pi^*\omega_0,\quad \widetilde{\eta}_0=\pi^*\eta_0,\quad \widetilde{\eta}_t^2\neq0\ \ \forall\,t\in[0,1],\quad \mathrm{d}\widetilde{\omega}_1=0,\quad [\widetilde{\omega}_1]=\widetilde{\eta}_1,$$

and $\widetilde{\omega}_1|_{T\pi^{-1}(\Sigma)}$ does not vanish on $\pi^{-1}(\Sigma)$.

If J is an almost complex structure on a smooth manifold X, then X is oriented by J,

$$w_1(X)=0,\qquad\text{and}\qquad w_{2i}(X)=c_i(X,J)_{\mathbb{Z}_2}\in H^{2i}(X;\mathbb{Z}_2)\quad\forall\,i\in\mathbb{Z}\,,\tag{2.11}$$

where $c_i(X,J)_{\mathbb{Z}_2}$ is the mod 2 reduction the i-th Chern class of the complex vector bundle (TX,J). If in addition X is closed and of real dimension 4, then

$$\chi(X)=\big\langle c_2(X,J),[X]\big\rangle\qquad\text{and}\qquad \big\langle c_1(X,J)^2,[X]\big\rangle=3\sigma(X)+2\chi(X),\tag{2.12}$$

where $\chi(X)$ is the Euler characteristic of X and $\sigma(X)\equiv b_2^+(X)-b_2^-(X)$ is the signature of X, i.e. the difference between the numbers $b_2^+(X)$ and $b_2^-(X)$ of positive and negative eigenvalues of the quadratic form

$$H^2(X;\mathbb{R})^{\otimes2}\longrightarrow\mathbb{R},\qquad(\alpha,\beta)\longrightarrow\big\langle\alpha\beta,[X]\big\rangle.$$

The first equality in (2.12) follows from Corollary 11.12 and the definition of $c_n(X,J)$ on page 158 in [31], while the second from the Hirzebruch Signature Theorem [31, Theorem 19.4], [31, Theorem 15.5], and the first equality.

By [43, Théorème 10], a closed oriented smooth fourfold X admits an almost complex structure if and only if

$$\exists\,c\in H^2(X;\mathbb{Z})\quad\text{s.t.}\quad c_{\mathbb{Z}_2}=w_2(X)\in H^2(X;\mathbb{Z}_2)\ \ \text{and}\ \ \big\langle c^2,[X]\big\rangle=3\sigma(X)+2\chi(X)\,;\tag{2.13}$$

the *only if* part is immediate from the $i=1$ case of the second equation in (2.11) and the second equation in (2.12). If J is an almost complex structure on a closed smooth fourfold X , then

$$\big\langle c_2(X,J)+c_1(X,J)^2,[X]\big\rangle\in12\mathbb{Z}\,;$$

this follows from [23, Corollary D.18] with $c=c_1(X,J)$. Along with (2.12), this implies that

$$\chi(X)+\sigma(X)\in4\mathbb{Z}\qquad\text{and}\qquad b_2^+(X)-b_1(X)\notin2\mathbb{Z}\tag{2.14}$$

if X is a closed oriented smooth fourfold admitting an almost complex structure.

Example 2.9 ([1])**.** The manifold $X=(S^1\times S^3)\#(S^1\times S^3)\#(S^2\times S^2)$ is oriented and satisfies

$$\begin{gathered}\chi(X)=2\chi(S^1\times S^3)+\chi(S^2\times S^2)-2\big(2-\chi\big(S^3\times[0,1]\big)\big)=0,\\ \sigma(X)=2\sigma(S^1\times S^3)+\sigma(S^2\times S^2)=0.\end{gathered}$$

Since $w_2(S^1\times S^3),w_2(S^2\times S^2)=0$, $w_2(X)=0$ (this condition is equivalent to X being spin, i.e. TX being trivializable over a 2-skeleton of X; see [8, Sections 1.1,1.2]). By [43, Théorème 10] with $c=0$,

X thus admits an almost complex structure. Since $H^2(X;\mathbb{R})\neq 0$, $\eta_0^2\neq 0$ for some $\eta_0\in H^2(X;\mathbb{Z})$. The smooth manifold

$$\widetilde{X}\equiv(S^1\times S^3)\#2(S^1\times S^3)\#2(S^2\times S^2)=\big((S^1\times S^3)\#2(S^1\times S^3)\#(S^2\times S^2)\big)\#(S^2\times S^2)$$

is a double cover of X. Since b_2^+ of each of the two summands on the right-hand side above is nonzero, all Seiberg-Witten invariants of $\widetilde{X}$ vanish; see [35, Theorem 4.6.1]. By [40], $\widetilde{X}$ thus does not admit a symplectic form. It follows that X does not admit a symplectic form either. Along with [2, Theorem IV.3.1], this implies that X does not admit an integrable almost complex structure.

Example 2.10. We denote by $a\in H^2(\mathbb{P}^2;\mathbb{Z})$ the positive generator and by $a_2\in H^2(\mathbb{P}^2;\mathbb{Z}_2)$ its mod 2 reduction. For $k\in\mathbb{Z}^+$, let $X_k\equiv\#k\mathbb{P}^2$ be the connected sum of k copies of the complex projective plane. This simply connected manifold is oriented and satisfies

$$\begin{gathered}\chi(X_k)=k\chi(\mathbb{P}^2)-(k-1)\big(2-\chi\big(S^3\times[0,1]\big)\big)=k+2, \qquad \chi(X_k), b_2^+(X_k)=kb_2^+(\mathbb{P}^2)=k,\\ H^2(X_k;R)\approx kH^2(\mathbb{P}^2;R);\end{gathered}$$

the last isomorphism holds for any commutative ring R with unity. By (2.14), X_k with $k\in 2\mathbb{Z}^+$ thus does not admit an almost complex structure. Under the above isomorphism with $R=\mathbb{Z}_2$, $w_2(X_k)$ corresponds to $(a_2,\ldots,a_2)$. If $r\in\mathbb{Z}^+$, [43, Théorème 10] with

$$c=\big(\underbrace{3a,\ldots,3a}_{r},\underbrace{a,\ldots,a}_{r-1}\big)$$

thus implies that X_{2r-1} admits an almost complex structure. If $r\geq 2$, $X_{2r-1}=X_{2r-2}\#\mathbb{P}^2$. Since b_2^+ of each of the two summands of X_{2r-1} is nonzero, all Seiberg-Witten invariants of X_{2r-1} vanish; see [35, Theorem 4.6.1]. By [40], X_{2r-1} with $r\geq 2$ thus does not admit a symplectic form. Along with [2, Theorem IV.3.1], this implies that X_{2r-1} with $r\geq 2$ does not admit an integrable almost complex structure.

2.3 Key notation and terminology

Let $(\Sigma,\mathfrak{j})$ be a Riemann surface, V be a vector bundle over Σ,

$$\mu,\eta\in\Gamma\big(\Sigma;T^*\Sigma\otimes_{\mathbb{R}}V\big), \qquad\text{and}\qquad g\in\Gamma\big(\Sigma;V^{*\otimes 2}\big).$$

For a local coordinate $z=s+\mathfrak{i}t$, define

$$\begin{aligned} g(\mu\otimes_{\mathfrak{j}}\eta)&=\big(g\big(\mu(\partial_s),\eta(\partial_s)\big)+g\big(\mu(\partial_t),\eta(\partial_t)\big)\big)\mathrm{d}s\wedge\mathrm{d}t\,,\\ g(\mu\wedge_{\mathfrak{j}}\eta)&=\big(g\big(\mu(\partial_s),\eta(\partial_t)\big)-g\big(\mu(\partial_t),\eta(\partial_s)\big)\big)\mathrm{d}s\wedge\mathrm{d}t\,.\end{aligned}\tag{2.15}$$

By a direct computation, the 2-forms $g(\mu\otimes_{\mathfrak{j}}\eta)$ and $g(\mu\wedge_{\mathfrak{j}}\eta)$ are independent of the choice of local coordinate $z=s+\mathfrak{i}t$. Thus, (2.15) determines global 2-forms on Σ (which depend on the choice of $\mathfrak{j}$).

Let X be a manifold, $(\Sigma,\mathfrak{j})$ be a Riemann surface, and $f:\Sigma\longrightarrow X$ be a smooth map. We denote the pullbacks of a 2-tensor g and a 2-form ω on X to the vector bundle f^*TX over Σ also by g and ω. If g is a Riemannian metric on X and $U\subset\Sigma$ is an open subset, let

$$E_g(f)\equiv\frac{1}{2}\int_\Sigma g(\mathrm{d}f\otimes_{\mathfrak{j}}\mathrm{d}f)\in[0,\infty] \qquad\text{and}\qquad E_g(f;U)\equiv E_g\big(f|_U\big)\tag{2.16}$$

be the energy of f and of its restriction to U. By the first equation in (2.15),

$$E_g(f) = \frac{1}{2}\int_\Sigma |\mathrm{d}f|^2_{g_\Sigma,g} \tag{2.17}$$

is the square of the L^2-norm of $\mathrm{d}f$ with respect to the metric g on X and a metric g_Σ compatible with $\mathfrak{j}$. In particular, the right-hand side of (2.17) depends on the metric g on X and on the complex structure $\mathfrak{j}$ on Σ, but *not* on the metric g_Σ on Σ compatible with $\mathfrak{j}$.

Let J be an almost complex structure on a manifold X and $(\Sigma,\mathfrak{j})$ be a Riemann surface. For a smooth map $f:\Sigma\longrightarrow X$, define

$$\bar\partial_J f = \frac{1}{2}\big(\mathrm{d}f+J\circ\mathrm{d}f\circ\mathfrak{j}\big) \in \Gamma\big(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}f^*(TX,J)\big)\,.$$

If ω is a 2-form on X taming J and $u\colon\Sigma\longrightarrow X$ is J-holomorphic, then

$$E_{g_J^\omega}(f) = \int_\Sigma \big(f^*\omega+g_J^\omega(\bar\partial_J f\otimes_{\mathfrak{j}}\bar\partial_J f)+f^*\omega_J\big) \tag{2.18}$$

by (2.16) and (2.5). If J is ω-compatible, the last term above vanishes. A smooth map $u\colon\Sigma\longrightarrow X$ is J-holomorphic if $\bar\partial_J u=0$. For such a map, the last two terms in (2.18) vanish.

The next lemma summarizes key properties of the energy function; they follow from (2.17) and (2.18). In the case the 2-form ω in Lemma 2.11(2) is closed, (2.19) imposes a restriction on the elements of $H_2(X;\mathbb{Z})$ that can be represented by J-holomorphic maps from closed Riemann surfaces for an ω-tame almost complex structure J on X.

Lemma 2.11. *Suppose X is a smooth manifold, $(\Sigma,\mathfrak{j})$ is a Riemann surface, and $f\colon\Sigma\longrightarrow X$ is a smooth map.*

(1) Let g be a Riemann metric on X. The map f is constant if and only if $E_g(f)=0$.

(2) Let ω be a 2-form taming J. If f is non-constant and J-holomorphic, then

$$\int_\Sigma f^*\omega > 0. \tag{2.19}$$

By Lemma 2.11(2),

$$u_*[\Sigma]\neq 0\in H_2(X;\mathbb{Q}) \tag{2.20}$$

for any non-constant J-holomorphic map $u\colon\Sigma\longrightarrow X$ from a closed Riemann surface *if* J is tamed by a symplectic form ω. The next three examples show that a non-constant J-holomorphic map $u\colon\Sigma\longrightarrow X$ from a closed Riemann surface could represent the zero element of $H_2(X;\mathbb{Z})$ otherwise. By Example 2.14, any almost complex structure J can be deformed locally to an almost complex structure not tamed by any symplectic form.

Example 2.12. Let $X\approx S^3\times S^1$ be the complex manifold of Example 2.3. The $\mathbb{Z}$-action on Example 2.3 preserves the complex linear subspaces of $\mathbb{C}^2$. For every complex linear one-dimensional subspace $L\subset\mathbb{C}^2$, the subspace

$$\big(L-\{0\}\big)/\mathbb{Z}\subset X$$

is a complex submanifold diffeomorphic to the 2-torus $S^1\times S^1$. Since $H_2(X;\mathbb{Z})=0$, the homology class of this complex torus in X vanishes.

Example 2.13. Let (X,J) be the almost complex manifold of Example 2.4. For every linear three-dimensional subspace $L\subset\mathrm{Im}\,\mathbb{O}$ so that $\mathbb{R}\oplus L$ is preserved under the octonion multiplication, such as $\mathrm{Im}\,\mathbb{H}$, the intersection $L\cap X$ in $\mathrm{Im}\,\mathbb{O}$ is an almost complex submanifold of (X,J) diffeomorphic to $\mathbb{P}^1$. Since $H_2(X;\mathbb{Z})=0$, the homology class of this J-holomorphic sphere in X vanishes.

Example 2.14. [[6, Proposition 2.3]] Let $\mathbb{T}^2\subset B_1^3$ be a 2-torus embedded inside of the unit ball in $\mathbb{R}^3$. For $n\geq 2$, the standard complex structure $J_{\mathbb{C}^n}$ on

$$\mathbb{C}^n=\mathbb{R}^3\times\mathbb{R}\times\mathbb{C}$$

can deformed within the ball $B_2^{2n}\subset\mathbb{C}^n$ of radius 2 through almost complex structures to an almost complex structure $J'_{\mathbb{C}^n}$ preserving $T\mathbb{T}^2\subset T\mathbb{C}^n|_{\mathbb{T}^2}$. Suppose now that (X,J) is any almost complex manifold so that the real dimension of X is $2n\geq 4$ and $U\subset X$ is an nonempty open subset. The latter contains a coordinate 3-ball B_3^{2n} inside of which J can be deformed to an almost complex structure restricting to $J_{\mathbb{C}^n}$ in the coordinates on B_2^{2n}. The new almost complex structure can then be deformed within B_2^{2n} to an almost complex structure J' on X restricting to $J'_{\mathbb{C}^n}$ on B_2^{2n}. The 2-torus $\mathbb{T}^2\subset B_1^{2n}$ is then a J'-holomorphic submanifold representing the zero class in $H_2(X;\mathbb{Z})$.

For each $R\in\mathbb{R}^+$, denote by $B_R\subset\mathbb{C}$ the open ball of radius R around the origin and let

$$B_R^*=B_R-\{0\}.$$

If in addition (X,g) is a Riemannian manifold and $x\in X$, let $B_\delta^g(x)\subset X$ be the ball of radius δ around x in X with respect to the metric g.

Exercise 2.15. Let (X,J,g) be an almost complex Riemannian manifold. Show that there exists a continuous function $\delta\colon X\longrightarrow\mathbb{R}^+$ with the following property. If $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map from a closed Riemann surface with $u(\Sigma)\subset B^g_{\delta(x)}(x)$ for some $x\in X$, then u is constant.

Let (X,J) be an almost complex manifold and $(\Sigma,\mathfrak{j})$ be a connected closed orientable surface. A smooth map $u\colon\Sigma\longrightarrow X$ is called

- somewhere injective if there exists $z\in\Sigma$ such that $u^{-1}(u(z))=\{z\}$ and $\mathrm{d}_z u\neq 0$,
- multiply covered if $u=u'\circ h$ for some connected closed orientable surface Σ', branched cover $h\colon\Sigma\longrightarrow\Sigma'$ of degree different from ±1, and a smooth map $u'\colon\Sigma'\longrightarrow X$,
- simple if it is not multiply covered.

By Proposition 4.11, every J-holomorphic map from a compact Riemann surface is simple if and only if it is somewhere injective (the *if* implication is trivial).

3 Local Properties

We begin by studying local properties of J-holomorphic maps u from Riemann surfaces $(\Sigma,\mathfrak{j})$ into almost complex manifolds (X,J) that resemble standard properties of holomorphic maps. None of the statements in Section 3 depending on X being compact; very few depend on Σ being compact.

3.1 Carleman Similarity Principle

Carleman Similarity Principle, i.e. Proposition 3.1 below, is a local description of solutions of a non-linear differential equation which generalizes the equation $\bar{\partial}_J u = 0$. It states that such solutions look similar to holomorphic maps and implies that they exhibit many local properties one would expect of holomorphic maps.

Proposition 3.1 (Carleman Similarity Principle, [11, Theorem 2.2])**.** *Suppose $n \in \mathbb{Z}^+$, $p, \epsilon \in \mathbb{R}^+$ with $p > 2$, $J \in L^p_1(B_\epsilon; \mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$, $C \in L^p(B_\epsilon; \mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$, and $u \in L^p_1(B_\epsilon; \mathbb{C}^n)$ are such that*

$$u(0) = 0, \qquad J(z)^2 = -\mathrm{Id}_{\mathbb{C}^n}, \quad u_s(z) + J(z)u_t(z) + C(z)u(z) = 0 \quad \forall\ z = s+\mathfrak{i}t \in B_\epsilon\,. \tag{3.1}$$

Then, there exist $\delta \in (0, \epsilon)$, $\Phi \in L^p_1(B_\delta; \mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta \longrightarrow \mathbb{C}^n$ such that

$$\sigma(0) = 0, \qquad J(z)\Phi(z) = \Phi(z)J_{\mathbb{C}^n}, \quad u(z) = \Phi(z)\sigma(z) \quad \forall\ z \in B_\delta\,. \tag{3.2}$$

By the Sobolev Embedding Theorem of Corollary D.3, the assumption $p > 2$ implies that u is a continuous function. In particular, all equations in (3.1) and in (3.2) make sense. This assumption also implies that the left-hand sides of the third equation in (3.1) and of the second equation in (3.2) and the right-hand side of the third equations in (3.2) lie in L^p_1.

Example 3.2. Let $\mathfrak{c}\colon \mathbb{C} \longrightarrow \mathbb{C}$ denote the usual conjugation. Define

$$\widehat{J}(z_1, z_2) = \begin{pmatrix} \mathfrak{i} & 0 \\ -2\mathfrak{i}s_1\mathfrak{c} & \mathfrak{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_1\mathfrak{c} & 1 \end{pmatrix} J_{\mathbb{C}^2} \begin{pmatrix} 1 & 0 \\ s_1\mathfrak{c} & 1 \end{pmatrix}^{-1} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \quad \forall\, z_i = s_i + \mathfrak{i}t_i,$$
$$u\colon \mathbb{C} \longrightarrow \mathbb{C}^2, \qquad u(s+\mathfrak{i}t) = \big(z, s^2\big).$$

Thus, $\widehat{J}$ is an almost complex structure on $\mathbb{C}^2$ and u is a $\widehat{J}$-holomorphic map, i.e. it satisfies the last condition in (3.1) with $J(z) = \widehat{J}(u(z))$ and $C(z) = 0$. The functions

$$\sigma\colon \mathbb{C} \longrightarrow \mathbb{C}^2, \quad \sigma(z) = (z, 0), \qquad \Phi\colon \mathbb{C} \longrightarrow \mathrm{GL}_4\mathbb{R}, \quad \Phi(s+\mathfrak{i}t) = \begin{pmatrix} 1 & 0 \\ s\mathfrak{c} + \frac{\mathfrak{i}st}{z} & 1 \end{pmatrix},$$

satisfy (3.2).

Corollary 3.3. *Let n, p, ϵ, J, C, and u be as in Proposition 3.1. If in addition $J_0 = J_{\mathbb{C}^n}$ and u does not vanish identically on a neighborhood of 0, then there exist $\ell \in \mathbb{Z}^+$ and $\alpha \in \mathbb{C}^n - 0$ such that*

$$\lim_{z \longrightarrow 0} \frac{u(z) - \alpha z^\ell}{z^\ell} = 0\,.$$

Proof. This follows from (3.2) and from the existence of such ℓ and α for σ. □

Corollary 3.4. *Suppose (X, J) is an almost complex manifold, $(\Sigma, \mathfrak{j})$ is a Riemann surface, and $u\colon \Sigma \longrightarrow X$ is a J-holomorphic map. If u is not constant on every connected component of Σ, then the subset*

$$u^{-1}\big(\{u(z)\colon\ z \in \Sigma,\ \mathrm{d}_z u = 0\}\big) \subset \Sigma$$

is discrete. If in addition $x \in X$, the subset $u^{-1}(x) \subset \Sigma$ is also discrete.

Proof. The first and third equations in (3.2) immediately imply the second claim (but not the first, since Φ may not be in C^1). The first claim follows from Corollary 3.3 and Taylor's formula for u (as well as from Corollary 3.6). $\square$

Before establishing the full statement of Proposition 3.1, we consider a special case.

Lemma 3.5. *Suppose $n\in\mathbb{Z}^+$ and $p,\epsilon\in\mathbb{R}^+$ are as in Proposition 3.1, $A\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$, and $u\in L^p_1(B_\epsilon;\mathbb{C}^n)$ are such that*

$$u(0)=0, \qquad u_s+J_{\mathbb{C}^n}u_t(z)+A(z)u(z)=0 \ \ \forall\ z=s+\mathfrak{i}t\in B_\epsilon\,. \tag{3.3}$$

Then, there exist $\delta\in(0,\epsilon)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_n\mathbb{C})$, a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$\sigma(0)=0, \qquad \Phi(0)=\mathrm{Id}_{\mathbb{C}^n}, \qquad u(z)=\Phi(z)\sigma(z) \ \ \forall\ z\in B_\delta\,. \tag{3.4}$$

Proof. For each $\delta\in[0,\epsilon]$, we define

$$A_\delta\in L^p(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad A_\delta(z)=\begin{cases}A(z), & \text{if } z\in B_\delta;\\ 0, & \text{otherwise;}\end{cases}$$
$$D_\delta: L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad D_\delta\Theta=\big(\Theta_s+J_{\mathbb{C}^n}\Theta_t+A_\delta\Theta\big)\mathrm{d}\bar z\,.$$

Since the cokernel of $D_0=2\bar\partial$ is isomorphic $H^1(S^2;\mathbb{C})\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n$, D_0 is surjective and the homomorphism

$$\widetilde{D}_0\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus\mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow\big(D_0\Theta,\Theta(0)\big),$$

is an isomorphism. Since

$$\big\|D_\delta\Theta-D_0\Theta\big\|_{L^p}\le\|A_\delta\|_{L^p}\|\Theta\|_{C^0}\le C\|A_\delta\|_{L^p}\|\Theta\|_{L^p_1} \qquad \forall\ \Theta\in L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$$

and $\|A_\delta\|_{L^p}\longrightarrow0$ as $\delta\longrightarrow0$, the homomorphism

$$\widetilde{D}_\delta\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus\mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow\big(D_\delta\Theta,\Theta(0)\big),$$

is also an isomorphism for $\delta>0$ sufficiently small. Let $\Theta_\delta=D_\delta^{-1}(0,\mathrm{Id}_{\mathbb{C}^n})$. Since D_δ is an isomorphism,

$$\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{C^0}\le C\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{L^p_1}\le C'\big\|D_\delta(\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n})\big\|_{L^p}=C'\big\|A_\delta\big\|_{L^p}\,.$$

Since $\|A_\delta\|_{L^p}\longrightarrow0$ as $\delta\longrightarrow0$, $\Theta_\delta\in L^p_1(B_\delta;\mathrm{GL}_n\mathbb{C})$. By (3.3) and $D_\delta\Theta_\delta=0$, the function $\sigma\equiv\Theta_\delta^{-1}u$ satisfies

$$\sigma(0)=0, \qquad \sigma_s+J_{\mathbb{C}^n}\sigma_t=0 \quad \forall\ z\in B_\delta,$$

i.e. σ is $J_{\mathbb{C}^n}$-holomorphic, as required. $\square$

Proof of Proposition 3.1. (1) Since B_ϵ is contractible, the complex vector bundles $u^*(T\mathbb{C}^n,J_{\mathbb{C}^n})$ and $u^*(T\mathbb{C}^n,J)$ over B_ϵ are isomorphic. Thus, there exists

$$\Psi\in L^p_1(B_\epsilon;\mathrm{GL}_{2n}\mathbb{R}) \qquad \text{s.t.} \quad J(z)\Psi(z)=\Psi(z)J_{\mathbb{C}^n} \quad \forall\ z\in B_\epsilon\,.$$

Let $v=\Psi^{-1}u$. By the assumptions on u, $v\in L^p_1(B_\epsilon;\mathbb{C}^n)$ and

$$v(0)=0, \qquad v_s(z)+J_{\mathbb{C}^n}v_t(z)+\widetilde{C}(z)v(z)=0 \quad \forall\ z=s+\mathfrak{i}t\in B_\epsilon, \tag{3.5}$$
$$\text{where} \quad \widetilde{C}=\Psi^{-1}\cdot\big(\Psi_s+J\Psi_t+C\Psi\big)\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)\,.$$

Thus, we have reduced the problem to the case $J = J_{\mathbb{C}^n}$.

(2) Let $\widetilde{C}^{\pm} = \frac{1}{2}(\widetilde{C} \mp J_{\mathbb{C}^n}\widetilde{C}J_{\mathbb{C}^n})$ be the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of $\widetilde{C}$, i.e. $\widetilde{C}^{\pm}J_{\mathbb{C}^n} = \pm J_{\mathbb{C}^n}\widetilde{C}^{\pm}$. With $\langle\cdot,\cdot\rangle$ denoting the Hermitian inner-product on $\mathbb{C}^n$ which is $\mathbb{C}$-antilinear in the second input, define

$$D \in L^{\infty}(B_\epsilon; \mathrm{End}_{\mathbb{R}}\mathbb{C}^n), \quad D(z)w = \begin{cases} |v(z)|^{-2}\langle v(z), w\rangle v(z), & \text{if } v(z) \neq 0; \\ 0, & \text{otherwise;} \end{cases} \quad A = \widetilde{C}^{+} + \widetilde{C}^{-}D\,.$$

Since $DJ_{\mathbb{C}^n} = -J_{\mathbb{C}^n}D$ and $Dv = v$, $A \in L^p(B_\epsilon; \mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$ and $Av = \widetilde{C}v$. Thus, by (3.5),

$$v_s + J_{\mathbb{C}^n}v_t + Av = 0\,.$$

The claim now follows from Lemma 3.5. $\square$

Corollary 3.6. *Suppose $n \in \mathbb{Z}^+$, $\epsilon \in \mathbb{R}^+$, J is a smooth almost complex structure on $\mathbb{C}^n$ with $J_0 = J_{\mathbb{C}^n}$, and $u: B_\epsilon \longrightarrow \mathbb{C}^n$ is a J-holomorphic map with $u(0) = 0$. Then, there exist $\delta \in (0,\epsilon)$, $C \in \mathbb{R}^+$, $\Phi \in C^0(B_\delta; \mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma: B_\delta \longrightarrow \mathbb{C}^n$ such that Φ is smooth on B_δ^*,*

$$\sigma(0) = 0, \quad \Phi(0) = \mathrm{Id}_{\mathbb{C}^n}, \quad J(u(z))\Phi(z) = \Phi(z)J_{\mathbb{C}^n}, \quad u(z) = \Phi(z)\sigma(z), \quad \big|\mathrm{d}_z\Phi\big| \le C \;\forall\, z \in B_\delta^*\,.$$

Proof. We can assume that u is not identically 0 on some neighborhood of $0 \in B_\epsilon$. Similarly to (1) in the proof of Proposition 3.1, there exists

$$\Psi \in C^{\infty}(\mathbb{C}^n; \mathrm{GL}_{2n}\mathbb{R}) \qquad \text{s.t.} \qquad \Psi(0) = \mathrm{Id}_{\mathbb{C}^n}, \quad J(x)\Psi(x) = \Psi(x)J_{\mathbb{C}^n} \quad \forall\, x \in \mathbb{C}^n\,.$$

Let $v(z) = \Psi(u(z))^{-1}u(z)$. By Corollary 3.3, we can choose complex linear coordinates on $\mathbb{C}^n$ so that

$$v(z) = \big(f(z), g(z)\big)h(z) \in \mathbb{C}\oplus\mathbb{C}^{n-1} \qquad \forall\, z \in B_{\epsilon'}$$

for some $\epsilon' \in (0,\epsilon)$, holomorphic function h on $B_{\epsilon'}$ with $h(0) = 0$, and continuous functions f and g on $B_{\epsilon'}$ with $f(0) = 1$ and $g(0) = 0$. By Lemma 3.7 below applied with f above and with each component of g separately, there exists $\delta \in (0,\epsilon')$ so that the function

$$\Phi\colon B_\delta \longrightarrow \mathrm{GL}_{2n}\mathbb{R}, \qquad \Phi(z) = \Psi\big(u(z)\big)\begin{pmatrix} f(z) & 0 \\ g(z) & 1 \end{pmatrix},$$

is continuous on B_δ and smooth on $B_\delta - 0$ with $|\mathrm{d}_z\Phi|$ uniformly bounded on $B_\delta - 0$. Taking $\sigma(z) = (h(z), 0)$, we conclude the proof. $\square$

Lemma 3.7. *Suppose $\epsilon \in \mathbb{R}^+$, and $f, h\colon B_\epsilon \longrightarrow \mathbb{C}$ are continuous functions such that h is holomorphic, $h(z) \neq 0$ for some $z \in B_\epsilon$, and the function*

$$B_\epsilon \longrightarrow \mathbb{C}, \qquad z \longrightarrow f(z)h(z), \tag{3.6}$$

is smooth. Then there exist $\delta \in (0,\epsilon)$ and $C \in \mathbb{R}^+$ such that f is differentiable on $B_\epsilon - 0$ and

$$\big|\mathrm{d}_z f\big| \le C \qquad \forall\, z \in B_\delta - 0\,. \tag{3.7}$$

Proof. After a holomorphic change of coordinate on $B_{2\delta}\subset B_\epsilon$, we can assume that $h(z)=z^\ell$ for some $\ell\in\mathbb{Z}^{\ge 0}$. Define
$$g\colon B_{2\delta}\longrightarrow\mathbb{C},\qquad g(z)=f(z)z^\ell-f(0)z^\ell\,.$$
By Taylor's Theorem and the smoothness of the function (3.6), there exists $C>0$ such that the smooth function g satisfies
$$\big|g(z)\big|\le C|z|^{\ell+1}\qquad\forall\,z\in B_\delta\,.$$
Dividing g by z^ℓ, we thus obtain (3.7). □

Remark 3.8. Corollary 3.6 refines the conclusion of Proposition 3.1 for J-holomorphic maps. In contrast to the output (Φ,σ) of Proposition 3.1, the output of Corollary 3.6 does not depend continuously on the input u with respect to the L^p_1-norms. This makes Corollary 3.6 less suitable for applications in settings involving families of J-holomorphic maps.

3.2 Local structure of J-holomorphic maps

We now obtain three corollaries from Proposition 3.1. They underpin important geometric statements established later in these notes, such as Propositions 3.13 and 4.11 and Lemma 5.4.

Corollary 3.9 (Unique Continuation)**.** *Suppose (X,J) is an almost complex manifold, $(\Sigma,\mathfrak{j})$ is a connected Riemann surface, and*
$$u,u'\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$$
are J-holomorphic maps. If u_0 and u_0' agree to infinite order at $z_0\in\Sigma$, then $u'=u'$.

Proof. Since the subset of the points of Σ at which u and u' agree is closed to infinite order, it is enough to show that $u=u'$ on some neighborhood of z_0. By the continuity of u, we can assume that $X=\mathbb{C}^n$, $\Sigma=B_1$, $z_0=0$, and $u(0),u'(0)=0$. Let
$$w=u'-u:B_\epsilon\longrightarrow\mathbb{C}^n\,.$$
Since J is C^1,
$$J(x+y)=J(x)+\int_0^1\frac{\mathrm{d}J(x+ty)}{\mathrm{d}t}\mathrm{d}t=J(x)+\sum_{i=1}^n y_i\int_0^1\frac{\partial J}{\partial y_i}\bigg|_{x+ty}\mathrm{d}t\,.\tag{3.8}$$
Since u and u' are J-holomorphic, (3.8) implies that
$$\partial_s w+J\big(u(z)\big)\partial_t w+C(z)w(z)=0,\qquad\text{where}\quad C\in L^p\big(B_1;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n\big),$$
$$C(z)y=\sum_{i=1}^n y_i\bigg(\int_0^1\frac{\partial J}{\partial y_i}\bigg|_{v(z)+tw(z)}\mathrm{d}t\bigg)\partial_t w|_z\,.$$
By Proposition 3.1, there thus exist $\delta\in(0,1)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and holomorphic map $\widetilde{w}\colon B_\delta\longrightarrow\mathbb{C}^n$ such that
$$w(z)=\Phi(z)\widetilde{w}(z)\qquad\forall\,z\in B_\delta\,.$$
Since w vanishes to infinite order at 0, it follows that $\widetilde{w}(z)=0$ for all $z\in B_\delta$ (otherwise, w would satisfy the conclusion of Corollary 3.3) and thus $w(z)=0$ for all $z\in B_\delta$. □

Corollary 3.10. *Suppose (X,J) is an almost complex manifold,*

$$u,u'\colon (\Sigma,\mathrm{j}),(\Sigma',\mathrm{j}')\longrightarrow (X,J)$$

are J-holomorphic maps, $z_0\in\Sigma$ is such that $\mathrm{d}_{z_0}u\neq 0$, and $z_0'\in\Sigma'$ is such that $u'(z_0')=u(z_0)$. If there exist sequences $z_i\in\Sigma-z_0$ and $z_i'\in\Sigma'-z_0'$ such that

$$\lim_{i\longrightarrow\infty} z_i = z_0\,, \qquad \lim_{i\longrightarrow\infty} z_i' = z_0'\,, \quad \textit{and} \quad u(z_i)=u'(z_i') \;\; \forall\, i\in\mathbb{Z}^+\,,$$

then there exists a holomorphic map $\sigma\colon U'\longrightarrow\Sigma$ from a neighborhood of z_0' in Σ' such that $\sigma(z_0')=z_0$ and $u'|_{U'}=u\circ\sigma$.

Proof. It can be assumed that $(\Sigma,\mathrm{j},z_0),(\Sigma',\mathrm{j}',z_0')=(B_1,\mathrm{j}_0,0)$, where $B_1\subset\mathbb{C}$ is the unit ball with the standard complex structure. Since $\mathrm{d}_{z_0}u\neq 0$ and u is J-holomorphic, u is an embedding near $0\in B_1$ and so is a slice in a coordinate system. Thus, we can assume that

$$u,u'\equiv(v,w)\colon (B_1,0)\longrightarrow (\mathbb{C}\times\mathbb{C}^{n-1},0), \qquad u(z)=(z,0)\in\mathbb{C}\times\mathbb{C}^{n-1}\,,$$

and u,u' are J-holomorphic with respect to some almost complex structure

$$J(x,y)=\begin{pmatrix} J_{11}(x,y) & J_{12}(x,y)\\ J_{21}(x,y) & J_{22}(x,y)\end{pmatrix}\colon \mathbb{C}\times\mathbb{C}^{n-1}\longrightarrow\mathbb{C}\times\mathbb{C}^{n-1}\,, \qquad (x,y)\in\mathbb{C}\times\mathbb{C}^{n-1}\,.$$

Since J is C^1,

$$J_{ij}(x,y)=J_{ij}(x,0)+\int_0^1\frac{\mathrm{d}J_{ij}(x,ty)}{\mathrm{d}t}\mathrm{d}t = J_{ij}(x,0)+\sum_{i=1}^{n-1}y_i\int_0^1\frac{\partial J_{ij}}{\partial y_i}\bigg|_{(x,ty)}\mathrm{d}t\,. \tag{3.9}$$

Since u is J-holomorphic,

$$J_{21}(x,0)=0, \quad J_{22}(x,0)^2=-\mathrm{Id} \qquad \forall\; x\in B_1\subset\mathbb{C}. \tag{3.10}$$

Since u' is J-holomorphic,

$$\partial_s w + J_{22}\big(v(z),w(z)\big)\partial_t w + J_{21}\big(v(z),w(z)\big)\partial_t v = 0.$$

Combining this with (3.9) and the first equation in (3.10), we find that

$$\partial_s w + J_{22}\big(v(z),0\big)\partial_t w + C(z)w(z)=0, \qquad \text{where} \quad C\in L^p\big(B_1;\mathrm{End}_{\mathbb{R}}\mathbb{C}^{n-1}\big),$$
$$C(z)y=\sum_{i=1}^{n-1}y_i\Bigg(\bigg(\int_0^1\frac{\partial J_{22}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_t w|_z+\bigg(\int_0^1\frac{\partial J_{21}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_t v|_z\Bigg)\,.$$

By Proposition 3.1 and the second identity in (3.10), there thus exist $\delta\in(0,1)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n-2}\mathbb{R})$, and holomorphic map $\widetilde{w}\colon B_\delta\longrightarrow\mathbb{C}^{n-1}$ such that

$$w(z)=\Phi(z)\widetilde{w}(z) \qquad \forall\; z\in B_\delta\,.$$

Since $u'(z_i')=u(z_i)$, $\widetilde{w}(z_i')=0$ for all $i\in\mathbb{Z}^+$. Since $z_i'\longrightarrow 0$ and $z_i'\neq 0$, it follows that $w=0$. This implies the claim with $U'=B_\delta$ and $\sigma=v$. $\square$

Corollary 3.11. *Let (X,J,g) be an almost complex Riemannian manifold and $x\in X$ be such that g is compatible with J at x. If $u:\Sigma\longrightarrow X$ is a J-holomorphic map from a compact Riemann surface with boundary so that $x\notin u(\partial\Sigma)$, then*

$$\lim_{\delta\longrightarrow 0}\frac{E_g(u;u^{-1}(B^g_\delta(x)))}{\pi\delta^2}=\mathrm{ord}_x u\,.$$

Exercise 3.12. Let $u\colon\mathbb{C}\longrightarrow\mathbb{C}^n$ be given by $u(z)=\alpha z^\ell$ with $\alpha\in\mathbb{C}^n-\{0\}$ and $\ell\in\mathbb{Z}^+$. Show that

$$E_{g_{\mathbb{C}^n}}\big(u;u^{-1}(B^{g_{\mathbb{C}^n}}_\delta(0))\big)=\ell\pi\delta^2,$$

where $g_{\mathbb{C}^n}$ is the standard metric on $\mathbb{C}^n$.

Proof of Corollary 3.11. By the continuity of u, we can assume that $X=\mathbb{C}^n$, J agrees with the standard complex structure $J_{\mathbb{C}^n}$ at the origin, g agrees with the standard metric $g_{\mathbb{C}^n}$ at the origin, $\Sigma=\overline{B_R}$ for some $R\in\mathbb{R}^+$, and $u(0)=0$. In particular, there exists $C\geq 1$ such that

$$\big|g_x-g_{\mathbb{C}^n}\big|\leq C|x|\qquad\forall\,x\in\mathbb{C}^n\text{ s.t. }|x|\leq 1,\tag{3.11}$$

where $|\cdot|$ denotes the usual norm of x (i.e. the distance to the origin with respect to $g_{\mathbb{C}^n}$). We can also assume u does not vanish identically on a neighborhood of 0.

Let $\ell\equiv\mathrm{ord}_0u$ and $\alpha\in\mathbb{C}^n-0$ be as in Corollary 3.3, where $0\in B_R$ is the origin in the domain of u. Thus, there exist $\epsilon\in(0,1)$, $C\in\mathbb{R}^+$, and a smooth function $f:\mathbb{C}\longrightarrow\mathbb{C}^n$ such that

$$u(z)=\alpha z^\ell+f(z),\quad|\alpha|\big|f(z)\big|\leq C|z|^{\ell+1}\qquad\forall\,z\in B_\epsilon\,.\tag{3.12}$$

With $z=s+\mathrm{i}t$ as before,

$$u_s(z)=\alpha\ell z^{\ell-1}+f_s(z),\quad u_t(z)=\alpha\ell\mathrm{i}z^{\ell-1}+f_t(z)\quad\forall\,z\in B_\epsilon\,.$$

By (3.12), there exists $C\in\mathbb{R}^+$ such that

$$|\alpha|\ell\big|f_s(z)\big|,|\alpha|\ell\big|f_t(z)\big|\leq C|z|^\ell\quad\forall\,z\in B_\epsilon\,.\tag{3.13}$$

We can also assume that the three constants C in (3.11), (3.12), and (3.13) are the same, $C\geq 1$,

$$C_\alpha\epsilon\equiv\big(C+C|\alpha|+C^2|\alpha|\big)\epsilon\leq 1\,,$$

and $|u(z)|\leq 1$ for all $z\in B_\epsilon$. By (3.11)-(3.13),

$$\begin{split}\left|\frac{|u(z)|_g}{|\alpha||z|^\ell}-1\right|,\left|\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}}-1\right|,\left|\frac{|u_t(z)|_g}{|\alpha|\ell|z|^{\ell-1}}-1\right|&\leq C|z|+C|\alpha||z|^\ell+C^2|\alpha||z|^{\ell+1}\\&\leq C_\alpha|z|\quad\forall\;z\in B_\epsilon,\end{split}\tag{3.14}$$

where $|\cdot|_g$ denotes the distance to the origin in $\mathbb{C}^n$ with respect to the metric g and the corresponding norm on $T\mathbb{C}^n$.

Given $r\in(0,1)$, let $\delta_r\in(0,\epsilon)$ be such that

$$C_\alpha\bigg(\frac{2\delta_r}{(1-r)|\alpha|}\bigg)^{1/\ell}\leq r\,.\tag{3.15}$$

For any $\delta\in[0,\delta_r]$, (3.14) and (3.15) give

$$\begin{aligned}
|z|\le\Big(\frac{\delta}{(1+r)|\alpha|}\Big)^{1/\ell} \qquad &\Longrightarrow \qquad u(z)\in B_\delta^g(0)\,,\\
u(z)\in B_\delta^g(0) \qquad &\Longrightarrow \qquad |z|\le\Big(\frac{\delta}{(1-r)|\alpha|}\Big)^{1/\ell},\\
|z|\le\Big(\frac{\delta}{(1-r)|\alpha|}\Big)^{1/\ell} \qquad &\Longrightarrow \qquad 1-r\le\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}},\frac{|u_t(z)|_g}{|\alpha|\ell|z|^{\ell-1}}\le1+r.
\end{aligned}$$

Combining these, we obtain

$$\begin{aligned}
\int_{|z|\le\big(\frac{\delta}{(1+r)|\alpha|}\big)^{\frac1\ell}}(1-r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2 &\le \frac12\int_{u^{-1}(B_\delta^g(0))}\big(|u_s|_g^2+|u_t|_g^2\big)\\
&\le \int_{|z|\le\big(\frac{\delta}{(1-r)|\alpha|}\big)^{\frac1\ell}}(1+r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2\,.
\end{aligned}$$

Evaluating the outer integrals, we find that

$$\Big(\frac{1-r}{1+r}\Big)^2\ell\pi\delta^2\le E_g\big(u;u^{-1}(B_\delta^g(0))\big)\le\Big(\frac{1+r}{1-r}\Big)^2\ell\pi\delta^2\,.$$

These inequalities hold for all $r\in(0,1)$ and $\delta\in(0,\delta_r)$; the claim is obtained by sending $r\longrightarrow0$. □

3.3 The Monotonicity Lemma

Proposition 3.13 below is a key step in the continuity part of the proof of the Removal of Singularity Proposition 5.1. The precise nature of the lower energy bound on the right hand-side of (3.16) does not matter, as long as it is positive for $\delta>0$.

Proposition 3.13 (Monotonicity Lemma)**.** *If (X,J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $C_{g,J}:X\longrightarrow\mathbb{R}^+$ with the following property. If $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary, $u:\Sigma\longrightarrow X$ is a J-holomorphic map, $x\in X$, and $\delta\in\mathbb{R}^+$ are such that $u(\partial\Sigma)\cap B_\delta^g(x)=\emptyset$, then*

$$E_g(u)\ge\big(\mathrm{ord}_x u\big)\frac{\pi\delta^2}{1+C_{g,J}(x)\delta}\,. \tag{3.16}$$

If $\omega(\cdot,\cdot)\equiv g(J\cdot,\cdot)$ is a symplectic form on X, then the above fraction can be replaced by $\pi\delta^2\mathrm{e}^{-C_{g,J}(x)\delta^2}$. If in addition the metric g is flat and $\delta<r_g(x)$, then the above fraction can be replaced by $\pi\delta^2$.

Corollary 3.14 (Lower Energy Bound)**.** *Suppose (X,J,g) is a compact almost complex Riemannian manifold. There exists $\hbar_{J,g}\in\mathbb{R}^+$ such that $E_g(u)\ge\hbar_{J,g}$ for every non-constant J-holomorphic map $u:\Sigma\longrightarrow X$ from a closed Riemann surface.*

Proof. By the compactness of X, we can assume that g is compatible with J. Let $C\in\mathbb{R}^+$ be the maximum value of a function $C_{g,J}$ provided by Proposition 3.13 and

$$\hbar_{J,g}=\max_{\delta\in\mathbb{R}^+}\frac{\pi\delta^2}{1+C\delta}\,.$$

The desired energy bound for non-constant J-holomorphic maps $u:\Sigma\longrightarrow X$ from compact Riemann surfaces with $\partial\Sigma=\emptyset$ then follows from (3.16). □

According to Proposition 3.13, "completely getting out" of the ball $B_\delta(x)$ via a J-holomorphic map requires an energy bounded below by a little less than $\pi\delta^2$. Thus, the L^2_1-norm of a J-holomorphic map u exerts some control over the C^0-norm of u. If $p>2$, the L^p_1-norm of any smooth map f from a two-dimensional manifold controls the C^0-norm of f; see Corollary D.3. However, this is not the case of the L^2_1-norm, as the following example illustrates.

Example 3.15 ([29, Lemma 10.4.1])**.** The function

$$f_\epsilon\colon \mathbb{R}^2 \longrightarrow [0,1], \qquad f_\epsilon(z)=\begin{cases}1, & \text{if } |z|\le\epsilon;\\ \frac{\ln|z|}{\ln\epsilon}, & \text{if } \epsilon\le|z|\le 1;\\ 0, & \text{if } |z|\ge 1;\end{cases}$$

with any $\epsilon\in(0,1)$ is continuous and satisfies

$$\int_{\mathbb{R}^2}|f_\epsilon|^2=\frac{\pi}{(2\ln\epsilon)^2}\Big(1-\epsilon^2\big(2(\ln\epsilon)^2-2(\ln\epsilon)+1\big)\Big), \qquad \int_{\mathbb{R}^2}|\mathrm{d}f_\epsilon|^2=-\frac{2\pi}{\ln\epsilon}\,.$$

This function is arbitrarily close in the L^2_1-norm to a smooth function $\widetilde{f}_\epsilon$. Thus, it is possible to "completely get out" of $B^g_\delta(x)$ using a smooth function with arbitrarily small energy ($\widetilde{f}_\epsilon$ does this for the ball $B_1(1)$ in $\mathbb{R}$).

Suppose (X,J) is an almost complex manifold and ω is a symplectic form taming J. By (2.18), the holomorphic maps from a closed Riemann surface $(\Sigma,\mathfrak{j})$ are the local minima of the functional

$$C^\infty(\Sigma;X)\longrightarrow\mathbb{R}, \qquad f\longrightarrow E_{g^\omega_J}(f)-\int_\Sigma f^*\omega_J\,.$$

This fact underpins Lemma 3.19, the key ingredient in the proof of the Monotonicity Lemma. Lemma 3.19 implies that the ratio of $E_g(u;u^{-1}(B^g_\delta(x)))$ and the fraction on the right-hand side (3.16) is a non-decreasing function of δ, as long as $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. By Corollary 3.11, this ratio approaches $\mathrm{ord}_x u$ as δ approaches 0. These two statements imply Proposition 3.13.

We first make some general Riemannian geometry observations. Let (X,g) be a Riemannian manifold. Denote by $\exp\colon\mathcal{W}_g\longrightarrow X$, the exponential map from a neighborhood of X in TX with respect to the Levi-Civita connection ∇ of g. For each $v\in TX$, we denote by

$$\gamma_v\colon[0,1]\longrightarrow X, \qquad \gamma_v(\tau)=\exp_x(\tau v),$$

the geodesic with $\gamma_v'(0)=v$. Let

$$r_g\colon X\longrightarrow\mathbb{R}^+ \qquad\text{and}\qquad d_g\colon X\times X\longrightarrow\mathbb{R}^{\ge0}$$

be the injectivity radius of exp and the distance function. For each $x\in X$, define

$$\zeta_x\in\Gamma\big(B^g_{r_g(x)}(x);TX\big)\quad\text{by}\quad \exp_y\big(\zeta_x(y)\big)=x,\ \ g\big(\zeta_x(y),\zeta_x(y)\big)<r_g(x)^2\ \ \forall\,y\in B^g_{r_g(x)}(x).$$

Lemma 3.16. *Let (X,g) be a Riemannian manifold and $x\in X$. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow X$ is a smooth curve such that $\alpha(0)\in B^g_{r_g(x)}(x)$, then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\bigg|_{\tau=0}=-g\big(\alpha'(0),\zeta_x(\alpha(0))\big)\,.$$

Proof. If $\beta(\tau)=\exp_x^{-1}\alpha(\tau)$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\bigg|_{\tau=0}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}|\beta(\tau)|^2\bigg|_{\tau=0}=g\big(\beta'(0),\beta(0)\big)\,.$$

By Gauss's Lemma,

$$g\big(\beta'(0),\beta(0)\big)=g\big(\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta'(0)),\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta(0))\big)=g\big(\alpha'(0),-\zeta_x(\alpha(0))\big)\,.$$

This establishes the claim. □

Lemma 3.17. *If (X,g) is a Riemannian manifold, there exists a continuous function $C_g\colon X\longrightarrow\mathbb{R}^+$ with the following property. If $x\in X$, $v\in T_xX$ with $|v|_g<\frac{1}{2}r_g(x)$, and $\tau\longrightarrow J(\tau)$ is a Jacobi vector field along the geodesic γ_v with $J(0)=0$, then*

$$\big|J'(1)-J(1)\big|_g\le C_g(x)|v|_g^2\big|J(1)\big|_g\,.$$

If the metric g is flat on $B^g_{r_g(x)/2}(x)$, then C_g can be chosen to vanish at x.

Proof. Let R_g be the Riemann curvature tensor of g and $f(\tau)=|\tau J'(\tau)-J(\tau)|_g$. Then, $f(0)=0$ and

$$\begin{aligned}f(\tau)f'(\tau)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau)^2=g\big(\tau J''(\tau),\tau J'(\tau)-J(\tau)\big)&=\tau g\big(R(\gamma'(\tau),J(\tau))\gamma'(\tau),\tau J'(\tau)-J(\tau)\big)\\&\le C_g(x)|v|_g^2|J(\tau)|_g\tau f(\tau).\end{aligned}$$

If C_g is sufficiently large, then $|J(\tau)|_g\le C_g(x)|J(1)|_g$. Thus,

$$f(\tau)f'(\tau)\le C_g(x)|v|_g^2|J_v(\tau)|_g\tau f(\tau)\le C_g(x)^2|v|_g^2|J(1)|_g\tau f(\tau),\quad f'(\tau)\le C_g(x)^2|v|_g^2|J(1)|_g\tau.$$

The claim follows from the last inequality. □

Corollary 3.18. *If (X,g) is a Riemannian manifold, there exists a continuous function $C_g\colon X\longrightarrow\mathbb{R}^+$ with the following property. If $x\in X$, then*

$$\big|\nabla_w\zeta_x|_y+w\big|_g\le C_g(x)d_g(x,y)^2|w|_g\qquad\forall\ w\in T_yX,\ y\in B^g_{r_g(x)/2}(x).$$

If the metric g is flat on $B^g_{r_g(x)/2}(x)$, then C_g can be chosen to vanish at x.

Proof. Let $\tau\longrightarrow u(s,\tau)$ be a family of geodesics such that

$$u(s,0)=x,\qquad u(0,1)=y,\qquad \frac{\mathrm{d}}{\mathrm{d}s}u(s,1)\bigg|_{s=0}=w.$$

Since $\tau\longrightarrow u(s,\tau)$ is a geodesic,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}u(s,\tau)\bigg|_{\tau=1}&=\big\{\mathrm{d}_{u_\tau(s,0)}\exp_x\big\}\big(u_\tau(s,0)\big)=-\zeta_x\big(u(s,1)\big),\\ \frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}s}\bigg|_{(s,\tau)=(0,1)}&=\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}\tau}\bigg|_{(s,\tau)=(0,1)}=-\nabla_w\zeta_x|_y\,.\end{aligned}$$

Furthermore, $J(\tau)\equiv\frac{\mathrm{d}}{\mathrm{d}s}u(s,\tau)\big|_{s=0}$ is a Jacobi vector field along the geodesic $\tau\longrightarrow u(0,\tau)$ with

$$J(0)=0,\quad J(1)=w,\quad J'(1)=\frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}s}\bigg|_{(s,\tau)=(0,1)}=-\nabla_w\zeta_x|_y\,.$$

Thus, the claim follows from Lemma 3.17. □

Lemma 3.19. *Suppose* (X,ω) *is a symplectic manifold,* J *is an almost complex structure on* X *tamed by* ω*, and* ∇ *is the Levi-Civita connection of the metric* g_J^ω *as in (2.4). If* $(\Sigma,\mathfrak{j})$ *is a compact Riemann surface with boundary and* $u\colon\Sigma\longrightarrow X$ *is a* J*-holomorphic map, then*

$$\int_\Sigma g_J^\omega\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi\big)=\int_\Sigma\big(u^*\{\nabla_\xi\omega_J\}+\omega_J(\mathrm{d}u\wedge_{\mathfrak{j}}\nabla\xi)\big)\qquad\forall\ \xi\in\Gamma(\Sigma;u^*TX)\ \textit{s.t.}\ \xi|_{\partial\Sigma}=0.$$

Proof. For $\tau\in\mathbb{R}$ sufficiently close to 0, define

$$u_\tau\colon\Sigma\longrightarrow X,\qquad u_\tau(z)=\exp_{u(z)}(\tau\xi(z)).$$

Since $\xi|_{\partial\Sigma}=0$, $u_\tau|_{\partial\Sigma}=u|_{\partial\Sigma}$. Denote by $\widehat\Sigma$ the closed oriented surface obtained by gluing two copies of Σ along the common boundary and reversing the orientation on the second copy. Let

$$\widehat{u}_\tau\colon\widehat\Sigma\longrightarrow X$$

be the map restricting to u_τ on the first copy of Σ and to u on the second.

By (2.18),

$$E(\tau)\equiv E_{g_J^\omega}(u_\tau)-\int_\Sigma u_\tau^*\omega_J-E_{g_J^\omega}(u)=\int_{\widehat\Sigma}\widehat{u}_\tau^*\omega+2\int_\Sigma g_J^\omega\big(\bar\partial u_\tau\otimes_{\mathfrak{j}}\bar\partial u_\tau\big)\ge0\quad\forall\tau.$$

Since ω is closed and $\widehat{u}_*$ represents the zero class in $H_2(X;\mathbb{Z})$, the first integral on the right-hand side above vanishes. Thus, the function $\tau\longrightarrow E(\tau)$ is minimized at $\tau=0$ (when it equals 0) and so

$$\begin{aligned}0=E'(0)&=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(E_{g_J^\omega}(u_\tau)-\int_\Sigma u_\tau^*\omega_J\bigg)\bigg|_{\tau=0}\\&=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(\frac12\int_\Sigma g_J^\omega(\mathrm{d}u_\tau\otimes_{\mathfrak{j}}\mathrm{d}u_\tau)-\int_\Sigma u_\tau^*\omega_J\bigg)\bigg|_{\tau=0};\end{aligned}\tag{3.17}$$

the last equality above uses the definition of $E(u_\tau)$ in (2.16).

Let $z=s+\mathfrak{i}t$ be a local coordinate on $(\Sigma,\mathfrak{j})$. Since ∇ is torsion-free,

$$\frac{D}{\mathrm{d}\tau}(u_\tau)_s\Big|_{\tau=0}\equiv\frac{D}{\mathrm{d}\tau}\frac{\mathrm{d}u_\tau}{\mathrm{d}s}\bigg|_{\tau=0}=\frac{D}{\mathrm{d}s}\frac{\mathrm{d}u_\tau}{\mathrm{d}\tau}\bigg|_{\tau=0}=\frac{D}{\mathrm{d}s}\xi\equiv\nabla_s\xi,\qquad\frac{D}{\mathrm{d}\tau}(u_\tau)_t\Big|_{\tau=0}=\nabla_t\xi\,.$$

Since ∇ is also g_J^ω-compatible,

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}\tau}g_J^\omega(\mathrm{d}u_\tau\otimes_{\mathfrak{j}}\mathrm{d}u_\tau)\bigg|_{\tau=0}&=\bigg(g_J^\omega\bigg(u_s,\frac{D}{\mathrm{d}\tau}(u_\tau)_s\Big|_{\tau=0}\bigg)+g_J^\omega\bigg(u_t,\frac{D}{\mathrm{d}\tau}(u_\tau)_t\Big|_{\tau=0}\bigg)\bigg)\mathrm{d}s\wedge\mathrm{d}t\\&=g_J^\omega(u_s,\nabla_s\xi)+g_J^\omega(u_t,\nabla_t\xi)=g_J^\omega\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi\big)\,,\\\frac{\mathrm{d}}{\mathrm{d}\tau}u_\tau^*\omega_J\bigg|_{\tau=0}&=\bigg(\big\{\nabla_\xi\omega_J\big\}(u_s,u_t)+\omega_J\bigg(\frac{D}{\mathrm{d}\tau}(u_\tau)_s\Big|_{\tau=0},u_t\bigg)+\omega_J\bigg(u_s,\frac{D}{\mathrm{d}\tau}(u_\tau)_t\Big|_{\tau=0}\bigg)\bigg)\mathrm{d}s\wedge\mathrm{d}t\\&=u^*\{\nabla_\xi\omega_J\}+\omega_J\big(\mathrm{d}u\wedge_{\mathfrak{j}}\nabla\xi\big)\,.\end{aligned}$$

Combining this with (3.17), we obtain the claim. □

Proof of Proposition 3.13. Let $\delta_g: X \longrightarrow \mathbb{R}^+$ be a continuous function such that for every $x\in X$ there exists a symplectic form ω_x on $B^g_{2\delta_g(x)}(x)$ so that J is tamed by ω_x on $B^g_{2\delta_g(x)}(x)$ and the metric $g_J^{\omega_x}$ as in (2.4) agrees with g at x. We assume that $2\delta_g(x)\le r_g(x)$ for every $x\in X$. It is sufficient to establish the proposition for each $x\in X$ and each $\delta\le\delta_g(x)$ under the assumption that the metric g agrees with $g_J^{\omega_x}$ on $B^g_{\delta_g(x)}(x)$.

Choose a C^∞-function $\eta\colon \mathbb{R}\longrightarrow[0,1]$ such that

$$\eta(\tau)=\begin{cases}1, & \text{if } \tau\le\frac{1}{2};\\ 0, & \text{if } \tau\ge 1;\end{cases} \qquad \eta'(\tau)\le 0. \tag{3.18}$$

For a compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, a smooth map $u\colon\Sigma\longrightarrow X$, $x\in X$, and $\delta\in\mathbb{R}^+$, define

$$\eta_{u,x,\delta}\in C^\infty(\Sigma;\mathbb{R}), \qquad \eta_{u,x,\delta}(z)=\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg),$$
$$E_{u,x,\eta}(\delta)=\frac{1}{2}\int_\Sigma \eta_{u,x,\delta}(z)g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\mathrm{d}u\big)\,, \quad E_{u,x}(\delta)=E_g\big(u;u^{-1}(B^g_\delta(x))\big).$$

We show in the remainder of this proof that there exists a continuous function $C_{g,J}\colon X\longrightarrow\mathbb{R}^+$ such that

$$-\delta E'_{u,x,\eta}(\delta)+2E_{u,x,\eta}(\delta)\le 2C_{g,J}(x)\delta E_{u,x,\eta}(\delta)+C_{g,J}(x)\delta^2E'_{u,x,\eta}(\delta) \tag{3.19}$$

for every compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, J-holomorphic map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$ such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. This inequality is equivalent to

$$\bigg(E_{u,x,\eta}(\delta)\bigg/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}\bigg)'\ge 0.$$

By Lebesgue's Dominated Convergence Theorem, $E_{u,x,\eta}(\delta)$ approaches $E_{u,x}(\delta)$ from below as η approaches the characteristic function $\chi_{(-\infty,1)}$ of $(-\infty,1)$. Thus, the function

$$\delta\longrightarrow E_{u,x}(\delta)\bigg/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}$$

is non-decreasing as long as $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. By Corollary 3.11,

$$\lim_{\delta\longrightarrow0}\bigg(E_{u,x}(\delta)\bigg/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}\bigg)=\lim_{\delta\longrightarrow0}\frac{E_{u,x}(\delta)}{\delta^2}=\big(\mathrm{ord}_x u\big)\pi.$$

This implies the first claim.

Fix $x\in X$. We note that

$$E'_{u,x,\eta}(\delta)=-\frac{1}{2}\int_\Sigma\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{d_g(x,u(z))}{\delta^2}g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\mathrm{d}u\big). \tag{3.20}$$

For a compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, a smooth map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$, let

$$\xi_{u,x,\delta}\in\Gamma(\Sigma;u^*TX), \qquad \xi_{u,x,\delta}(z)=-\eta_{u,x,\delta}(z)\zeta_x\big(u(z)\big);$$

the vanishing assumption in (3.18) implies that $\xi_{u,x,\delta}$ is well-defined. If $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$, then $\xi_{u,x,\delta}|_{\partial\Sigma}=0$. By Lemma 3.16,

$$\nabla\xi_{u,x,\delta}|_z=\eta'\biggl(\frac{d_g(x,u(z))}{\delta}\biggr)\frac{1}{\delta\,d_g(x,u(z))}g\bigl(\mathrm{d}_zu,\zeta_x(u(z))\bigr)\zeta_x(u(z))-\eta_{u,x,\delta}(z)\nabla\zeta_x\circ\mathrm{d}_zu. \tag{3.21}$$

Along with Corollary 3.18, (3.20), and the last assumption in (3.18), this implies that

$$\int_\Sigma d_g(x,u(z))\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\le 2\delta^2E'_{u,x,\eta}(\delta)+2\bigl(1+C_g(x)\delta^2\bigr)\delta E_{u,x,\eta}(\delta). \tag{3.22}$$

By the ω_x-compatibility assumption on J at x, there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\int_\Sigma\big|(\omega_x)_J(\mathrm{d}u\wedge_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\le C(x)\int_\Sigma d_g\bigl(x,u(z)\bigr)\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|$$

for all u and δ as above. Along with this, Lemma 3.19 implies that there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\bigg|\int_\Sigma g\bigl(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta}\bigr)\bigg|\le C(x)\int_\Sigma\Bigl(g\bigl(\mathrm{d}u\otimes_{\mathfrak{j}}\mathrm{d}u\bigr)|\xi_{u,x,\delta}|+d_g(x,u(z))\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\Bigr)$$

for every compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, J-holomorphic map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$ such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. Combining this with (3.22), we conclude that there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\bigg|\int_\Sigma g\bigl(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta}\bigr)\bigg|\le C(x)\bigl(\delta E_{u,x,\eta}(\delta)+\delta^2E'_{u,x,\eta}(\delta)\bigr) \tag{3.23}$$

for all u and δ as above.

Suppose $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary, $u\colon\Sigma\longrightarrow X$ is a smooth map, and $\delta\in(0,\delta_g(x))$. Let $z=s+\mathfrak{i}t$ be a coordinate on $(\Sigma,\mathfrak{j})$. By (3.21),

$$\begin{split}g\bigl(u_s,\nabla_s\xi_{u,x,\delta}\bigr)=\eta'\biggl(\frac{d_g(x,u(z))}{\delta}\biggr)\frac{1}{\delta\,d_g(x,u(z))}g\bigl(u_s,\zeta_x(u(z))\bigr)^2\\+\eta_{u,x,\delta}(z)g\bigl(u_s,\nabla_s(-\zeta_x)|_z\bigr).\end{split} \tag{3.24}$$

By Corollary 3.18,

$$|u_s|^2\le g\bigl(u_s,\nabla_s(-\zeta_x)|_z\bigr)+C_g(x)d_g(x,u(z))^2|u_s|^2\quad\forall\,z\in u^{-1}\bigl(B^g_{\delta_g(x)}(x)\bigr)\,. \tag{3.25}$$

If u is J-holomorphic, then $|u_s|=|u_t|$, $\langle u_s,u_t\rangle=0$, and

$$\begin{split}\frac12\bigl(|u_s|^2+|u_t|^2\bigr)d_g(x,u(z))^2&=\frac12\bigl(|u_s|^2+|u_t|^2\bigr)\big|\zeta_x(u(z))\big|^2\\&\ge g\bigl(u_s,\zeta_x(u(z))\bigr)^2+g\bigl(u_t,\zeta_x(u(z))\bigr)^2\,.\end{split} \tag{3.26}$$

Since $\eta'\le0$, (3.24)-(3.26) give

$$\begin{split}&\frac12\eta'\biggl(\frac{d_g(x,u(z))}{\delta}\biggr)\frac{d_g(x,u(z))}{\delta}\bigl(|u_s|^2+|u_t|^2\bigr)+\eta_{u,x,\delta}(z)\bigl(|u_s|^2+|u_t|^2\bigr)\\&\quad\le g\bigl(u_s,\nabla_s\xi_{u,x,\delta}\bigr)+g\bigl(u_t,\nabla_t\xi_{u,x,\delta}\bigr)+C_g(x)\eta_{u,x,\delta}(z)d_g(x,u(z))^2\bigl(|u_s|^2+|u_t|^2\bigr).\end{split} \tag{3.27}$$

Along with (3.20), this implies that

$$-\delta E'_{u,x,\eta}(\delta) + 2E_{u,x,\eta}(\delta) \le \int_\Sigma g\big(\mathrm{d}u \otimes_{\mathfrak{j}} \nabla\xi_{u,x,\delta}\big) + 2C_g(x)\delta^2 E_{u,x,\eta}(\delta) \tag{3.28}$$

for every compact Riemann surface with boundary $(\Sigma, \mathfrak{j})$, J-holomorphic map $u\colon \Sigma \longrightarrow X$, and $\delta \in (0, \delta_g(x))$. Combining this inequality with (3.23), we obtain (3.19).

Suppose $\omega \equiv g(J\cdot,\cdot)$ is a symplectic form on X. We can then run the above argument with $\delta_g(x) = r_g(x)/2$ and $\omega_x = \omega|_{B^g_{r_g(x)}(x)}$. Since J is ω-compatible, $\omega_J = 0$. By Lemma 3.19, (3.28) then becomes

$$-\delta E'_{u,x,\eta}(\delta) + 2E_{u,x,\eta}(\delta) \le 2C_g(x)\delta^2 E_{u,x,\eta}(\delta)\,.$$

The reasoning below (3.19) now yields the second claim of the proposition. If in addition the metric g is flat, $C_g(x)$ vanishes in (3.25), (3.27), (3.28), and above. The reasoning below (3.19) then yields the last claim. □

4 Mean Value Inequality and Applications

We now move to properties of J-holomorphic maps u from Riemann surfaces $(\Sigma, \mathfrak{j})$ into almost complex manifolds (X, J) that are of a more global nature. They generally concern the distribution of the energy of such a map over its domain and are consequences of the Mean Value Inequality for J-holomorphic maps. These fairly technical properties lead to geometric conclusions such as Propositions 4.3 and 5.1.

4.1 Statement and proof

According to Cauchy's Integral Formula, a holomorphic map $u\colon B_R \longrightarrow \mathbb{C}^n$ satisfies

$$u'(0) = \frac{1}{2\pi \mathfrak{i}} \oint_{|z|=r} \frac{u(z)}{z^2}\mathrm{d}z \qquad \forall\, r \in (0, R).$$

This immediately implies that a bounded holomorphic function defined on all of $\mathbb{C}$ is constant. The Mean Value Inequality of Proposition 4.1 bounds the norms of the differentials of J-holomorphic maps of sufficiently small energy away from the boundary of the domain "uniformly" by their L^2-norms. In general, one would not expect the value of a function to be bounded by its integral. The Mean Value Inequality implies that a J-holomorphic map which is defined on all of $\mathbb{C}$ and has sufficiently small energy is in fact constant; see Corollary 4.2.

Proposition 4.1 (Mean Value Inequality). *If (X, J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $\hbar_{J,g}\colon X \times \mathbb{R} \longrightarrow \mathbb{R}^+$ with the following property. If $u\colon B_R \longrightarrow X$ is a J-holomorphic map such that*

$$u(B_R) \subset B^g_r(x) \qquad \text{and} \qquad E_g(u) < \hbar_{J,g}(x, r)$$

for some $x \in X$ and $r \in \mathbb{R}$, then

$$\big|\mathrm{d}_0 u\big|^2_g < \frac{16}{\pi R^2} E_g(u)\,. \tag{4.1}$$

Proof. Let $\phi(z) = \frac{1}{2}|\mathrm{d}_z u|^2_g$. By Lemma 4.7 below, $\Delta\phi \ge -A_{J,g}\phi^2$ with $A_{J,g}\colon X \times \mathbb{R} \longrightarrow \mathbb{R}^+$ determined by (X, J, g). The claim with $\hbar_{J,g} = \pi/8A_{J,g}$ thus follows from Proposition 4.6. □

Corollary 4.2 (Lower Energy Bound)**.** *If (X,J,g) is a compact almost complex Riemannian manifold, then there exists $\hbar_{J,g}\in\mathbb{R}^+$ such that $E_g(u)\ge\hbar_{J,g}$ for every non-constant J-holomorphic map $u\colon\mathbb{C}\longrightarrow X$.*

Proof. By the compactness of X, we can assume that g is compatible with J. Let $\hbar_{J,g}>0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 4.1 on the compact space $X\times[0,\mathrm{diam}_g(X)]$. If $u\colon\mathbb{C}\longrightarrow X$ is J-holomorphic map with $E_g(u)<\hbar_{J,g}$,

$$\big|\mathrm{d}_z u\big|_g^2<\frac{16}{\pi R^2}E_g\big(u;B_R(z)\big)\le\frac{16}{\pi R^2}E_g(u)\qquad\forall\ z\in\mathbb{C},\ R\in\mathbb{R}^+$$

by Proposition 4.1. Thus, $\mathrm{d}_z u=0$ for all $z\in\mathbb{C}$, and so u is constant. □

Since $\mathbb{C}\subset\mathbb{P}^1$, Corollary 4.2 implies that $E_g(u)\ge\hbar_{J,g}$ for every non-constant J-holomorphic map $u\colon\mathbb{P}^1\longrightarrow X$. This lower bound and the compactness of the Deligne-Mumford moduli space of stable marked curves are among the key ingredients in the proof of Theorem 1.5. This theorem in turn implies that for every $a\in\mathbb{Z}^{\ge0}$ there exists $\hbar_{J,g;a}\in\mathbb{R}^+$ such that $E_g(u)\ge\hbar_{J,g;a}$ for every non-constant J-holomorphic map $u\colon\Sigma\longrightarrow X$ from a connected closed surface of arithmetic genus a.

If $\phi\colon U\longrightarrow\mathbb{R}$ is a C^2-function on an open subset of $\mathbb{R}^2$, let

$$\Delta\phi=\frac{\partial^2\phi}{\partial s^2}+\frac{\partial^2\phi}{\partial t^2}\equiv\phi_{ss}+\phi_{tt}$$

denote the Laplacian of ϕ.

Exercise 4.3. Show that in the polar coordinates (r,θ) on $\mathbb{R}^2$,

$$\Delta\phi=\phi_{rr}+r^{-1}\phi_r+r^{-2}\phi_{\theta\theta}\,. \tag{4.2}$$

Lemma 4.4. *If $\phi\colon\overline{B_R}\longrightarrow\mathbb{R}$ is C^2, then*

$$2\pi R\,\phi(0)=-R\int_{(r,\theta)\in B_R}(\ln R-\ln r)\Delta\phi+\int_{\partial B_R}\phi\,. \tag{4.3}$$

Proof. By Stokes' Theorem applied to $\phi\mathrm{d}\theta$ on $\overline{B_R}-B_\delta$,

$$\begin{split}\oint_{\partial B_R}\phi\,\mathrm{d}\theta-\oint_{\partial B_\delta}\phi\,\mathrm{d}\theta&=\int_{\overline{B_R-B_\delta}}\phi_r\,\mathrm{d}r\wedge\mathrm{d}\theta=\int_0^{2\pi}\!\!\int_\delta^R(r\phi_r)r^{-1}\,\mathrm{d}r\mathrm{d}\theta\\&=\int_0^{2\pi}(\ln R-\ln\delta)\delta\,\phi_r(\delta,\theta)\mathrm{d}\theta+\int_0^{2\pi}\!\!\int_\delta^R(\ln R-\ln r)(\phi_{rr}+r^{-1}\phi_r)r\,\mathrm{d}r\mathrm{d}\theta\,;\end{split}$$

the last equality above is obtained by applying integration by parts to the functions $\ln r-\ln R$ and $r\phi_r$. Sending $\delta\longrightarrow0$ and using (4.2), we obtain

$$\frac{1}{R}\int_{\partial B_R}\phi-2\pi\,\phi(0)=0+\int_{(r,\theta)\in B_R}(\ln R-\ln r)\Delta\phi\,,$$

which is equivalent to (4.3). □

Corollary 4.5. *If $\phi\colon\overline{B_R}\longrightarrow\mathbb{R}$ is C^2 and $\Delta\phi\ge-C$ for some $C\in\mathbb{R}^+$, then*

$$\phi(0)\le\frac{1}{8}CR^2+\frac{1}{\pi R^2}\int_{B_R}\phi\,. \tag{4.4}$$

Proof. By (4.3),

$$2\pi r\,\phi(0)\le Cr\int_0^{2\pi}\!\!\int_0^r(\ln r-\ln\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta+\int_{\partial B_r}\phi=Cr\cdot 2\pi\cdot\frac{r^2}{4}+\int_{\partial B_r}\phi\qquad\forall\, r\in(0,R).$$

Integrating the above in $r\in(0,R)$, we obtain

$$2\pi\phi(0)\cdot\frac{R^2}{2}\le 2\pi C\cdot\frac{R^4}{16}+\int_{B_R}\phi.$$

This inequality is equivalent to (4.4). □

Proposition 4.6. *If $\phi\colon B_R\longrightarrow\mathbb{R}^{\ge0}$ is C^2 and $\Delta\phi\ge -A\phi^2$ for some $A\in\mathbb{R}^+$, then*

$$\phi(0)\le\frac{8}{\pi R^2}\int_{B_R}\phi\qquad\textit{or}\qquad\int_{B_R}\phi\ge\frac{\pi}{8A}\,. \tag{4.5}$$

Proof. Replacing A by $\widetilde{A}=R^2A$ and ϕ by

$$\widetilde{\phi}\colon B_1\longrightarrow\mathbb{R},\qquad\widetilde{\phi}(z)=\phi(Rz),$$

we can assume that $R=1$, as well as that ϕ is defined on $\overline{B_1}$.

(1) Define

$$f\colon[0,1)\longrightarrow\mathbb{R}^{\ge0}\qquad\text{by}\qquad f(r)=(1-r)^2\max_{B_r}\phi\,.$$

In particular, $f(0)=\phi(0)$ and $f(1)=0$. Choose $r^*\in[0,1)$ and $z^*\in B_{r^*}$ such that

$$f(r^*)=\sup f\qquad\text{and}\qquad\phi(z^*)=\sup_{B_{r^*}}\phi\equiv c^*\,.$$

Let $\delta=\frac12(1-r^*)>0$; see Figure 6. Thus,

$$f(r^*)=4\delta^2c^*\quad\text{and}\quad\sup_{B_\delta(z^*)}\phi\le\sup_{B_{r^*+\delta}}\phi=\frac{f(r^*+\delta)}{(1-(r^*+\delta))^2}\le\frac{f(r^*)}{\frac14(1-r^*)^2}=4\phi(z^*)=4c^*\,.$$

By the second equation, $\Delta\phi\ge -A\phi^2\ge -16Ac^{*2}$ on $B_\delta(z^*)$.

(2) Using Corollary 4.5, we thus find that

$$c^*=\phi(z^*)\le\frac18\cdot16Ac^{*2}\cdot\rho^2+\frac{1}{\pi\rho^2}\int_{B_\rho(z^*)}\phi\le 2Ac^{*2}\rho^2+\frac{1}{\pi\rho^2}\int_{B_1}\phi\qquad\forall\,\rho\in[0,\delta]\,. \tag{4.6}$$

If $2Ac^*\delta^2\le\frac12$, the $\rho=\delta$ case of the above inequality gives

$$\frac12c^*\le\frac{1}{\pi\delta^2}\int_{B_1}\phi\,,\qquad\phi(0)=f(0)\le f(r^*)=4\delta^2c^*\le\frac{8}{\pi}\int_{B_1}\phi\,.$$

If $2Ac^*\delta^2\ge\frac12$, $\rho\equiv(4Ac^*)^{-\frac12}\le\delta$ and (4.6) gives

$$c^*\le 2Ac^{*2}\cdot\frac{1}{4Ac^*}+\frac{4Ac^*}{\pi}\int_{B_1}\phi\,.$$

Thus, $\displaystyle\int_{B_1}\phi\ge\frac{\pi}{8A}$. □

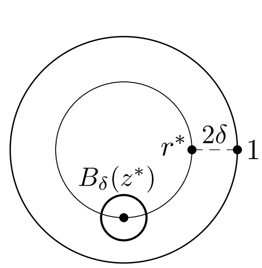

Figure 6: Setup for the proof of Proposition 4.6

Lemma 4.7. *If (X,J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $A_{J,g}\colon X\times\mathbb{R}\longrightarrow\mathbb{R}^+$ with the following property. If $\Omega\subset\mathbb{C}$ is an open subset, $u\colon\Omega\longrightarrow X$ is a J-holomorphic map, and $u(\Omega)\subset B_r^g(x)$ for some $x\in X$ and $r\in\mathbb{R}$, then the function $\phi(z)\equiv\frac{1}{2}|\mathrm{d}_z u|_g^2$ satisfies $\Delta\phi\ge -A_{J,g}(x,r)\phi^2$.*

Proof. Let $z=s+\mathfrak{i}t$ be the standard coordinate on $\mathbb{C}$. Denote by u_s and u_t the s and t-partials of u, respectively. Since u is J-holomorphic, i.e. $u_s=-Ju_t$, and g is J-compatible, i.e. $g(J\cdot,J\cdot)=g(\cdot,\cdot)$, $|u_s|_g^2=|u_t|_g^2$. Since the Levi-Civita connection ∇ of g is g-compatible and torsion-free,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2 t}|u_s|_g^2=|\nabla_t u_s|_g^2+\big\langle\nabla_t\nabla_t u_s,u_s\big\rangle_g=|\nabla_t u_s|_g^2+\big\langle\nabla_t\nabla_s u_t,u_s\big\rangle_g\,. \tag{4.7}$$

Similarly,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2 s}|u_t|_g^2=\big|\nabla_s u_t\big|_g^2+\big\langle\nabla_s\nabla_t u_s,u_t\big\rangle_g. \tag{4.8}$$

Since $u_s=-Ju_t$,

$$\begin{aligned}\langle\nabla_s\nabla_t u_s,u_t\rangle_g&=-\big\langle\nabla_s\nabla_t(Ju_t),u_t\big\rangle_g\\&=-\big\langle J\nabla_s\nabla_t u_t,u_t\big\rangle_g-\big\langle(\nabla_s J)\nabla_t u_t,u_t\big\rangle_g-\big\langle\nabla_s((\nabla_t J)u_t),u_t\big\rangle_g\\&=-\big\langle\nabla_s\nabla_t u_t,u_s\big\rangle_g-\big\langle(\nabla_s J)\nabla_t u_t,u_t\big\rangle_g-\big\langle\nabla_s((\nabla_t J)u_t),u_t\big\rangle_g\,.\end{aligned} \tag{4.9}$$

Putting (4.7)-(4.9), we find that

$$\frac{1}{2}\Delta\phi=\big|\nabla_t u_s\big|_g^2+\big|\nabla_s u_t\big|_g^2+\big\langle R_g(u_t,u_s)u_t,u_s\big\rangle_g-\big\langle(\nabla_s J)\nabla_t u_t,u_t\big\rangle_g-\big\langle\nabla_s((\nabla_t J)u_t),u_t\big\rangle_g\,, \tag{4.10}$$

where R_g is the curvature tensor of the connection ∇. Since $u(\Omega)\subset B_r^g(x)$,

$$\begin{aligned}\big|\langle R_g(u_t,u_s)u_t,u_s\rangle_g\big|&\le C_g(x,r)|u_s|_g^2|u_t|_g^2\,,\\ \big|\langle(\nabla_s J)\nabla_t u_t,u_t\rangle_g\big|&\le C_{J,g}(x,r)|u_s|_g|u_t|_g\big|\nabla_t(Ju_s)\big|_g\le C_{J,g}(x,r)|u_s|_g|u_t|_g\big(|u_s|_g|u_t|_g+|\nabla_t u_s|_g\big)\\&\le C_{J,g}(x,r)|u_s|_g^2|u_t|_g^2+C_{J,g}(x,r)^2|u_s|_g^2|u_t|_g^2+|\nabla_t u_s|_g^2\,,\\ \big|\langle\nabla_s((\nabla_t J)u_t),u_t\rangle_g\big|&\le C_{J,g}(x,r)|u_t|_g^2\big(|u_s|_g|u_t|_g+|\nabla_s u_t|_g\big)\\&\le C_{J,g}(x,r)|u_s|_g|u_t|_g^3+C_{J,g}(x,r)^2|u_t|_g^4+|\nabla_s u_t|_g^2.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we find that

$$\frac{1}{2}\Delta\phi\ge -C(x,r)\big(|u_s|_g^2|u_t|_g^2+|u_s|_g|u_t|_g^3+|u_t|_g^4\big)\ge -3C(x,r)\phi^2,$$

as claimed. $\square$

4.2 Regularity of J-holomorphic maps

By Cauchy's Integral Formula, a continuous extension of a holomorphic map $u\colon B^*_{\mathbb{R}}\longrightarrow\mathbb{C}^n$ over the origin is necessarily holomorphic. By Proposition 4.8 below, the same is the case for a J-holomorphic map $u\colon B^*_{\mathbb{R}}\longrightarrow X$ of bounded energy.

Proposition 4.8. *Let (X,J,g) be an almost complex Riemannian manifold. If $R\in\mathbb{R}^+$ and $u\colon B_R\longrightarrow X$ is a continuous map such that $u|_{B^*_R}$ is a J-holomorphic map and $E_g(u;B^*_R)<\infty$, then u is smooth and J-holomorphic on B_R.*

For a smooth loop $\gamma\colon S^1\longrightarrow X$, define

$$\gamma'(\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}\gamma\big(\mathrm{e}^{\mathrm{i}\theta}\big)\in T_{\gamma(\mathrm{e}^{\mathrm{i}\theta})}X \qquad\text{and}\qquad \ell_g(\gamma)=\int_0^{2\pi}\big|\gamma'(\theta)\big|_g\mathrm{d}\theta\in\mathbb{R}^{\ge0}$$

to be the velocity of γ and the length of γ, respectively.

Lemma 4.9 (Isoperimetric Inequality)**.** *Let (X,J,g), R, and u be as in Proposition 4.8 and*

$$\gamma_r\colon S^1\longrightarrow X,\quad \gamma_r\big(\mathrm{e}^{\mathrm{i}\theta}\big)=u\big(r\mathrm{e}^{\mathrm{i}\theta}\big)\qquad\forall\,r\in(0,R).$$

There exist $\delta\in(0,R)$ and $C\in\mathbb{R}^+$ such that

$$E_g\big(u;B^*_r\big)\le C\ell_g(\gamma_r)^2\qquad\forall\,r\in(0,\delta). \tag{4.12}$$

Proof. Let exp be as above the statement of Lemma 3.16, δ_g and ω_x be as in the first two sentences in the proof of Proposition 3.13,

$$x_0=u(0),\quad \delta_0=\delta_g(x_0),\quad \omega_0=\omega_{x_0},\qquad E\colon(0,R)\longrightarrow\mathbb{R},\quad E(r)=E_g(u;B^*_r).$$

We can assume that the metric g is determined by J and ω_0 on $B^g_{\delta_0}(x_0)$.

For a smooth loop $\gamma\colon S^1\longrightarrow B^g_{\delta_0}(x_0)$, define

$$\begin{aligned}\xi_\gamma\colon S^1\longrightarrow T_{x_0}X\qquad&\text{by}\quad \exp_{\gamma(1)}\xi_\gamma\big(\mathrm{e}^{\mathrm{i}\theta}\big)=\gamma\big(\mathrm{e}^{\mathrm{i}\theta}\big),\quad \big|\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big|<2\delta_0,\\ f_\gamma\colon B_1\longrightarrow X,\qquad& f_\gamma\big(r\mathrm{e}^{\mathrm{i}\theta}\big)=\exp_{\gamma(1)}\big(r\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big).\end{aligned}$$

In particular,

$$\big|\partial_rf_\gamma(\rho\mathrm{e}^{\mathrm{i}\theta})\big|_g=\big|\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big|_g\le\ell_g(\gamma)/2,\quad \big|r^{-1}\partial_\theta f_\gamma(r\mathrm{e}^{\mathrm{i}\theta})\big|_g=\big|\mathrm{d}_{r\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})}(\xi'_\gamma(\theta))\big|_g\le C\big|\gamma'(\theta)\big|_g$$

for some $C\in\mathbb{R}^+$ determined by x_0. Thus,

$$\begin{aligned}\bigg|\int_{B_1}f_\gamma^*\omega_0\bigg|&\le C\int_0^{2\pi}\int_0^1\big|\partial_rf_\gamma(r\mathrm{e}^{\mathrm{i}\theta})\big|_g\big|r^{-1}\partial_\theta f_\gamma(r\mathrm{e}^{\mathrm{i}\theta})\big|_gr\,\mathrm{d}r\mathrm{d}\theta\\ &\le C'\ell_g(\gamma)\int_0^{2\pi}\int_0^r\big|\gamma'(\theta)\big|_gr\,\mathrm{d}r\mathrm{d}\theta=\frac12C'\ell_g(\gamma)^2\end{aligned} \tag{4.13}$$

for some $C,C'\in\mathbb{R}^+$ determined by x_0 and ω_0.

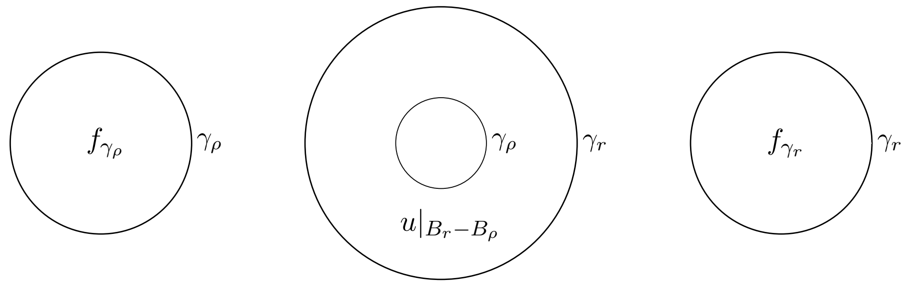

Figure 7: The maps from an annulus and two disks glued together to form the map $F_{\rho;r}\colon S^2\longrightarrow X$ in the proof of Lemma 4.9

By Proposition 4.1 and the finiteness assumption on $E(u;B_R^*)$, there exists $\delta\in(0,R/2)$ such that

$$\big|\gamma_\rho'(\theta)\big|_g^2\equiv\big|\partial_\theta u(\rho\mathrm{e}^{\mathrm{i}\theta})\big|_g^2=\rho^2\big|\partial_\rho u(\rho\mathrm{e}^{\mathrm{i}\theta})\big|_g^2\le\frac{8}{\pi}E(2\rho)\qquad\forall\ \rho\in(0,\delta),\tag{4.14}$$

$$\ell_g(\gamma_\rho)^2\le 32\pi E(2\rho)\qquad\forall\ \rho\in(0,\delta).\tag{4.15}$$

By the continuity of u, we can assume that $u(B_{2\delta})\subset B_{\delta_0}^g(x_0)$. For $r\in(0,\delta)$ and $\rho\in(0,r)$, define

$$F_{\rho;r}\colon S^2\longrightarrow X$$

to be the map obtained from $u|_{B_r-B_\rho}$ by attaching disks to the boundary components ∂B_r and ∂B_ρ and letting $F_{\rho;r}$ be given by f_{γ_r} and f_{γ_ρ} on these two disks, respectively; see Figure 7. Since $F_{\rho;r}$ is homotopic to a constant map and ω_0 is closed,

$$0=\int_{S^2}F_{\rho;r}^*\omega_0=E_g\big(u;B_r-B_\rho\big)+\int_{B_1}f_{\gamma_\rho}^*\omega_0-\int_{B_1}f_{\gamma_r}^*\omega_0\,.$$

Combining this with (4.13) and (4.15), we obtain

$$E_g\big(u;B_r-B_\rho\big)\le C\ell_g(\gamma_r)^2+32\pi E(2\rho)\tag{4.16}$$

for some $C\in\mathbb{R}^+$ independent of r and ρ as above. Since $E_g(u;B_R^*)<0$, $E(2\rho)\longrightarrow0$ as $\rho\longrightarrow0$. Taking the limit of (4.16) as $\rho\longrightarrow0$, we thus obtain (4.12). $\square$

Corollary 4.10. *If (X,J,g), R, and u are as in Proposition 4.8, there exist $\delta\in(0,R)$ and $\mu,C\in\mathbb{R}^+$ such that*

$$\big|\mathrm{d}_{r\mathrm{e}^{\mathrm{i}\theta}}u\big|_g\le Cr^{\mu-1}\qquad\forall\,r\in(0,\delta).\tag{4.17}$$

Proof. Let γ_r, δ, C, and $E(r)$ be as in the statement and proof of Lemma 4.9. Thus,

$$\begin{aligned}E(r)\equiv\frac12\int_0^{2\pi}\!\!\int_0^r\big|\mathrm{d}_{\rho\mathrm{e}^{\mathrm{i}\theta}}u\big|_g^2\rho\mathrm{d}\rho\mathrm{d}\theta\le C\ell_g(\gamma_r)^2&=\frac12Cr^2\bigg(\int_0^{2\pi}\big|\mathrm{d}_{r\mathrm{e}^{\mathrm{i}\theta}}u\big|_g\mathrm{d}\theta\bigg)^2\\&\le C\pi r^2\!\int_0^{2\pi}\big|\mathrm{d}_{r\mathrm{e}^{\mathrm{i}\theta}}u\big|_g^2\mathrm{d}\theta=2C\pi rE'(r)\qquad\forall\,r\in(0,\delta);\end{aligned}$$

the second inequality follows from Hölder's inequality. This implies that

$$\big(r^{-1/2C\pi}E(r)\big)'\ge0,\quad E(r)\le\delta^{-1/2C\pi}E(\delta)\cdot r^{1/2C\pi}\equiv C'r^{2\mu}\qquad\forall\,r\in(0,\delta).$$

Combining this with (4.14), we obtain (4.17) with δ replaced by $\delta/2$. $\square$

Proof of Proposition 4.8. With μ as in Corollary 4.10, let $p\in\mathbb{R}^+$ be such that $p>2$ and $(1-\mu)p<2$. In particular,

$$u|_{B_{R/2}}\in L^p_1\big(B_{R/2};X\big),\qquad \bar\partial_J u|_{B_{R/2}}=0\in L^p\big(B_{R/2};T^*B_{R/2}\otimes_{\mathbb{C}}u^*TX\big).$$

By elliptic regularity, this implies that u is smooth; see [29, Theorem B.4.1]. By the continuity of $\bar\partial_J u$, u is then J-holomorphic on all of B_R. □

4.3 Global structure of J-holomorphic maps

We next combine the local statement of Proposition 3.1 and some of its implications with the regularity statement of Proposition 4.8 to obtain a global description of J-holomorphic maps.

Proposition 4.11. *Let (X,J) be an almost complex manifold, $(\Sigma,\mathfrak{j})$ be a compact Riemann surface, $u\colon\Sigma\longrightarrow X$ be a J-holomorphic map. If u is simple, then u is somewhere injective and all limit points of the set*

$$\big\{z\in\Sigma\colon\,|u^{-1}(u(z))|>1\big\}\tag{4.18}$$

are critical points of u.

Suppose (X,J) is an almost complex manifold, $(\Sigma,\mathfrak{j})$ is a Riemann surface, and $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map. Let

$$\Sigma_u^*=\Sigma-u^{-1}\big(u\big(\{z\in\Sigma:\mathrm{d}_z u=0\}\big)\big)\tag{4.19}$$

be the preimage of the regular values of u and

$$\mathcal{R}_u^*\subset\Sigma_u^*\times\Sigma_u^*$$

be the subset of pairs (z,z') such that there exists a diffeomorphism $\varphi_{z'z}:U_z\longrightarrow U_{z'}$ between neighborhoods of z and z' in Σ satisfying

$$\varphi_{z'z}(z)=z'\qquad\text{and}\qquad u|_{U_z}=u\circ\varphi_{z'z}.\tag{4.20}$$

Denote by $\mathcal{R}_u\subset\Sigma\times\Sigma$ the closure of $\mathcal{R}_u^*$.

It is immediate that $\mathcal{R}_u^*$ is an equivalence relation on Σ and $u(z)=u(z')$ whenever $(z,z')\in\mathcal{R}_u^*$. Thus, $\mathcal{R}_u$ is also a reflexive and symmetric relation and $u(z)=u(z')$ whenever $(z,z')\in\mathcal{R}_u$. By Lemma 4.14 below, $\mathcal{R}_u$ is transitive as well. We denote this equivalence relation by $\sim_u$. Let

$$h_u\colon\Sigma\longrightarrow\Sigma'\equiv\Sigma/\!\sim_u\qquad\text{and}\qquad u'\colon\Sigma'\longrightarrow X\tag{4.21}$$

be the quotient map and the continuous map induced by u, respectively. In particular,

$$u=u'\circ h_u\colon\Sigma\longrightarrow X.$$

In the case Σ is compact, we will show that Σ' inherits a Riemann surface structure $\mathfrak{j}'$ from $\mathfrak{j}$ so that the maps h_u and u' are $\mathfrak{j}'$- and J-holomorphic, respectively. If the degree of h is 1, we will show that all limit points of the set (4.18) are critical points of u.

Lemma 4.12. *Suppose (X,J) is an almost complex manifold, $R\in\mathbb{R}^+$, and $u:B_R\longrightarrow X$ is a non-constant J-holomorphic map such that $\mathrm{d}_z u\neq0$ for all $z\in B_R^*$. Then there exist $m\in\mathbb{Z}^+$ and a neighborhood U_0 of 0 in B_R such that*

$$h_u\colon U_0\cap B_R^*\longrightarrow h_u\big(U_0\cap B_R^*\big)\subset B_R'\tag{4.22}$$

is a covering projection of degree m.

Proof. By the continuity of u, we can assume that $X = \mathbb{C}^n$, $u(0) = 0$, and $J_0 = J_{\mathbb{C}^n}$. As shown in the proof of Corollary 3.11, there exist $\epsilon \in (0, R)$ and $\delta \in (0, \epsilon/2)$ such that

$$U_0 \equiv u^{-1}\big(u(B_\delta)\big) \cap B_\epsilon \subset B_{2\delta}.$$

By Proposition 3.1 and the compactness of $\overline{B_{2\delta}} \subset B_R$, the number

$$m(z) \equiv \big|h_u^{-1}(h_u(z)) \cap U_0\big|$$

is finite for every $z \in U_0 \cap B_R^*$.

Suppose $z_i \in B_\delta^*$ and $z_i' \in U_0$ are sequences such that z_i converges to some $z_0 \in B_\delta^*$ with $z_i \neq z_0$ for all i and $h_u(z_i) = h_u(z_i')$ for all i. Passing to a subsequence, we can assume that z_i' converges to some $z_0' \in \overline{B_{2\delta}}$. By the continuity of u, $u(z_0') = u(z_0)$ and so $z_0' \in U_0$. Corollary 3.10 then implies that $h_u(z_0') = h_u(z_0)$. Since B_δ^* is connected, this implies that the number $m(z)$ is independent of $z \in U_0 \cap B_R^*$; we denote it by m.

Suppose $z \in U_0 \cap B_R^*$ and

$$h_u^{-1}\big(h_u(z)\big) \cap U_0 = \big\{z_1, \ldots, z_m\big\}.$$

Let $\varphi_i : U_1 \longrightarrow U_i$ for $i = 1, \ldots, m$ be diffeomorphisms between neighborhoods of z_1 and z_i in $U_0 \cap B_R^*$ such that

$$\varphi_i(z_1) = z_i, \quad u = u \circ \varphi_i \ \ \forall\, i, \qquad U_i \cap U_j = \emptyset \ \ \forall\, i \neq j,$$

and $u : U_1 \longrightarrow X$ is injective. Then $h_u(U_1) \subset B_R'$ is an open neighborhood of $h_u(z)$,

$$h_u^{-1}\big(h_u(U_1)\big) \cap U_0 = \bigsqcup_{i=1}^{m} U_i\,,$$

and $h_u : U_i \longrightarrow h_u(U_1)$ is a homeomorphism. Thus, (4.22) is a covering projection of degree m. $\square$

Lemma 4.13. *Suppose (X, J), R, and u are as in Lemma 4.12. Then there exists a neighborhood U_0 of 0 in B_R such that*

$$\Psi_0 : h_u(U_0) \longrightarrow \mathbb{C}, \qquad h_u(z) = \prod_{z' \in h_u^{-1}(h_u(z)) \cap U_0} z', \tag{4.23}$$

is a homeomorphism from an open neighborhood of $h_u(0)$ in B_R' to an open neighborhood of 0 in $\mathbb{C}$ and $\Psi_0 \circ h_u|_{U_0}$ is a holomorphic map.

Proof. By Lemma 4.12, there exists a neighborhood U_0 of 0 in B_R so that (4.22) is a covering projection of some degree $m \in \mathbb{Z}^+$. Since the restriction of u to B_R^* is a J-holomorphic immersion, the diffeomorphisms φ_i as in the proof of Lemma 4.12 are holomorphic. Thus, the map

$$\Psi_0 \circ h_u|_{U_0 \cap B_R^*} : U_0 \cap B_R^* \longrightarrow \mathbb{C}, \qquad z \longrightarrow \prod_{z' \in h_u^{-1}(h_u(z)) \cap U_0} z'$$

is holomorphic. Since it is also bounded, it extends to a holomorphic map

$$\widetilde{\Psi}_0 : U_0 \longrightarrow \mathbb{C}.$$

This extension is non-constant and vanishes at 0.

After possibly shrinking U_0, we can assume that there exist $k\in\mathbb{Z}^+$ and $C\in\mathbb{R}^+$ such that

$$C^{-k}|z|^k \le \big|\widetilde{\Psi}_0(z)\big| \le C^k|z|^k \qquad \forall\, z\in U_0. \tag{4.24}$$

Since $\widetilde{\Psi}_0(z')=\widetilde{\Psi}_0(z)$ for all $z'\in h_u^{-1}(h_u(z))\cap U_0$, it follows that

$$\begin{gathered} C^{-2}|z| \le |z'| \le C^2|z| \qquad \forall\; z'\in h_u^{-1}(h_u(z))\cap U_0,\;\; z\in U_0,\\ C^{-2m}|z|^m \le \big|\widetilde{\Psi}_0(z)\big| \le C^{2m}|z|^m \qquad \forall\, z\in U_0. \end{gathered}$$

Along with (4.24), the last estimate implies that $k=m$ and that $\widetilde{\Phi}_0$ has a zero of order precisely m at $z=0$. Thus, shrinking δ in the proof of Lemma 4.12 if necessary, we can assume that $\widetilde{\Phi}_0$ is $m\!:\!1$ over $\overline{U_0}\cap B_R^*$. This implies that the map (4.23) and its extension over the closure of $h_u(U_0)$ in B'_R are continuous and injective. Since the closure of $h_u(U_0)$ is compact and $\mathbb{C}$ is Hausdorff, we conclude that (4.23) is a homeomorphism onto an open subset of $\mathbb{C}$. $\square$

Lemma 4.14. *Suppose (X,J), $(\Sigma,\mathfrak{j})$, and u are as in Proposition 4.11 and $(x,y)\in\mathcal{R}_u$. For every neighborhood U_x of x in Σ, the image of the projection*

$$\mathcal{R}_u\cap(U_x\!\times\! X)\longrightarrow X$$

to the second component contains a neighborhood U_y of y in Σ.

Proof. By Corollary 3.4, the last set in (4.19) is finite. By the same reasoning as in the last part of the proof of Lemma 4.12,

$$h_u\colon \Sigma_u^* \longrightarrow h_u(\Sigma_u^*)\subset\Sigma' \tag{4.25}$$

is a local homeomorphism. Since $u(z)=u(z')$ for all $(z,z')\in\mathcal{R}_u^*$, the definition of Σ_u^* thus implies that (4.25) is a finite-degree covering projection over each topological component of $h_u(\Sigma_u^*)$. Since the complement of finitely many points in a connected Riemann surface is connected, the degree of this covering over a point $h_u(z)$ depends only on the topological component of Σ containing z. For any point $z\in\Sigma$, not necessarily in Σ_u^*, we denote this degree by $d(z)$.

By Corollary 3.4, the set

$$S\equiv u^{-1}\big(u(x)\big)\subset\Sigma$$

is finite. Let $W\subset X$ be a neighborhood of $u(x)$ such that the topological components Σ_s of $u^{-1}(W)$ containing the points $s\in S$ are pairwise disjoint (if U is a union of disjoint balls around the points of S, then

$$W\equiv X-u(\Sigma-U)$$

works). By Lemma 4.12, for each $s\in S$ there exists a neighborhood U'_s of s in Σ_s such that

$$h_u\colon U'_s-\{s\}\longrightarrow h_u\big(U'_s-\{s\}\big)\subset\Sigma'$$

is a covering projection of some degree $m_s\in\mathbb{Z}^+$; we can assume that $U'_x\subset U_x$. Along with the compactness of Σ, the former implies that

$$\begin{aligned} &\big|h_u^{-1}\big(h_u(y')\big)\cap U'_s\big|\in\{0,m_s\} &&\forall\, y'\in U'_{s'}\cap\Sigma_u^*,\;\; s,s'\in S,\\ &\sum_{s\in S}\big|h_u^{-1}\big(h_u(y')\big)\cap U'_s\big| = d(s') &&\forall\; y'\in U'_{s'}\cap\Sigma_u^*,\;\; s'\in S. \end{aligned} \tag{4.26}$$

Define

$$\mathcal{P}_y(S) = \big\{ S' \subset S \colon \sum_{s\in S'} m_s = d(y) \big\}.$$

Let $U''_y \subset U'_y$ be a connected neighborhood of y. For each $S' \in \mathcal{P}_y(S)$, define

$$U''_{y;S'} = \big\{ y' \in U''_y \cap \Sigma^*_u \colon \{ s \in S \colon h_u^{-1}(h_u(y')) \cap U'_s \neq \emptyset \} = S' \big\}.$$

By (4.26), these sets partition $U''_y \cap \Sigma^*_u$. Since each set

$$\big\{ y' \in U''_y \cap \Sigma^*_u \colon h_u^{-1}(h_u(y')) \cap U'_s \neq \emptyset \big\}$$

is open, (4.26) also implies that each set $U''_{y;S'}$ is open. Since the set $U''_y \cap \Sigma^*_u$ is connected, it follows that $U''_y \cap \Sigma^*_u = U''_{y;S_y}$ for some $S_y \in \mathcal{P}_y(S)$. Since $(x,y) \in \mathcal{R}_u$, $x \in S_y$. Thus, the image of the projection

$$\mathcal{R}_u \cap (U'_x \times X) \longrightarrow X$$

to the second component contains the neighborhood U''_y of y in Σ. □

Corollary 4.15. *Suppose (X,J), $(\Sigma,\mathfrak{j})$, and u are as in Proposition 4.11. The quotient map h_u in (4.21) is open and closed.*

Proof. The openness of h_u is immediate from Lemma 4.14. Suppose $A \subset \Sigma$ is a closed subset and $y_i \in h_u^{-1}(h_u(A))$ is a sequence converging to some $y \in \Sigma$. Let $x_i \in A$ be such that $h_u(x_i) = h_u(y_i)$. Passing to a subsequence, we can assume that the sequence x_i converges to some $x \in A$. Since $\Sigma - \Sigma^*_u$ consists of isolated points, we can also assume that $y_i \in \Sigma^*_u$ and so $(x_i, y_i) \in \mathcal{R}^*_u$. Thus, $(x,y) \in \mathcal{R}_u$ and so $y \in h_u^{-1}(h_u(A))$. We conclude that h_u is a closed map. □

Proof of Proposition 4.11. Let Σ', h_u, and u' be as in (4.21). By the second statement in Corollary 4.15 and [33, Lemma 73.3], Σ' is a Hausdorff topological space. Fix a Riemannian metric g on X.

For $(z,z') \in \mathcal{R}^*_u$ with $z \neq z'$, the neighborhoods U_z and $U_{z'}$ as in (4.20) can be chosen so that they are disjoint and $u|_{U_z}$ is an embedding. Since u is J-holomorphic, $\varphi_{z'z}$ is then a biholomorphic map and $h_u|_{U_z}$ is a homeomorphism onto $h_u(U_z) \subset \Sigma'$. Thus, the Riemann surface structure $\mathfrak{j}$ on Σ determines a Riemann surface structure $\mathfrak{j}'$ on $h_u(\Sigma^*_u)$ so that $h_u|_{\Sigma^*_u}$ is a holomorphic covering projection of $h_u(\Sigma^*_u)$ and $u'|_{h_u(\Sigma^*_u)}$ is a J-holomorphic map with

$$E_g\big(u'; h_u(\Sigma^*_u)\big) \le E_g(u). \tag{4.27}$$

By Corollary 3.4, $\Sigma'_u - h_u(\Sigma^*_u)$ consists of finitely many points. By the first statement in Corollary 4.15 and by Lemma 4.13, $\mathfrak{j}'$ extends over these points to a Riemann surface structure on Σ'; we denote the extension also by $\mathfrak{j}'$. Since the continuous map h_u is $\mathfrak{j}'$-holomorphic outside of the finitely many points of $\Sigma - \Sigma^*_u$, it is holomorphic everywhere. Since the continuous map u' is J-holomorphic on $h_u(\Sigma^*_u)$, (4.27) and Proposition 4.8 imply that it is J-holomorphic everywhere.

Suppose $z \in \Sigma$ and $z_i, z'_i \in \Sigma$ with $i \in \mathbb{Z}^+$ are such that

$$\mathrm{d}_z u \neq 0, \qquad z_i \neq z'_i,\ u(z_i) = u(z'_i)\ \ \forall\, i, \qquad \lim_{i\longrightarrow\infty} z_i = z.$$

Passing to a subsequence, we can assume that the sequence z'_i converges to some point $z' \in \Sigma$ with $u(z') = u(z)$. Since the restriction of u to a neighborhood of z is an embedding, $z' \neq z$. By Corollary 3.10, there exists a diffeomorphism $\varphi_{z'z}$ as in (4.20). Thus, $h_u(z) = h_u(z')$, the map h_u is not injective, and u is not simple. □

4.4 Energy bound on long cylinders

Proposition 4.16 and Corollary 4.17 below concern J-holomorphic maps from long cylinders. Their substance is that most of the energy and variation of such maps are concentrated near the ends. These technical statements are used to obtain important geometric conclusions in Sections 5.2 and 5.3.

Proposition 4.16. *If (X,J,g) is an almost complex Riemannian manifold, then there exist continuous functions $\delta_{J,g},\hbar_{J,g},C_{J,g}:X\longrightarrow\mathbb{R}^+$ with the following properties. If $u\colon[-R,R]\times S^1\longrightarrow X$ is a J-holomorphic map such that $\mathrm{Im}\,u\subset B^g_{\delta_{J,g}(u(0,1))}(u(0,1))$, then*

$$E_g\big(u;[-R+T,R-T]\times S^1\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T}E_g(u)\qquad\forall\,T\ge 0\,. \tag{4.28}$$

If in addition $E_g(u)<\hbar_{J,g}\big(u(0,1)\big)$, then

$$\mathrm{diam}_g\big(u([-R+T,R-T]\times S^1)\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T/2}\sqrt{E_g(u)}\quad\forall\,T\ge 1\,. \tag{4.29}$$

Corollary 4.17. *If (X,J,g) is a compact almost complex Riemannian manifold, then there exist constants $\hbar_{J,g},C_{J,g}\in\mathbb{R}^+$ with the following properties. If $R_1,R_2\in\mathbb{R}$ and $u\colon[R_1,R_2]\times S^1\longrightarrow X$ is a J-holomorphic map such that $E_g(u)<\hbar_{J,g}$, then*

$$\begin{aligned}E_g\big(u;[R_1+T,R_2-T]\times S^1\big)&\le C_{J,g}\mathrm{e}^{-T}E_g(u)&&\forall\,T\ge 1,\\ \mathrm{diam}_g\big(u([R_1+T,R_2-T]\times S^1)\big)&\le C_{J,g}\mathrm{e}^{-T/2}\sqrt{E_g(u)}&&\forall\,T\ge 2\,.\end{aligned}$$

As an example, the energy of the injective map

$$[-R,R]\times S^1\longrightarrow\mathbb{C},\qquad(s,\theta)\longrightarrow s\mathrm{e}^{\mathrm{i}\theta}\,,$$

is the area of its image, i.e. $\pi\big(\mathrm{e}^{2R}-\mathrm{e}^{-2R}\big)$. Thus, the exponent e^{-T} in (4.28) can be replaced by e^{-2T} in this case. The proof of Proposition 4.16 shows that in general the exponent can be taken to be $\mathrm{e}^{-\mu T}$ with μ arbitrarily close to 2, but at the cost of increasing $C_{J,g}$ and reducing $\delta_{J,g}$.

Proof of Proposition 4.16. It is sufficient to establish the first statement under the assumption that (X,g) is $\mathbb{C}^n$ with the standard Riemannian metric, J agrees with the standard complex structure $J_{\mathbb{C}^n}$ at $0\in\mathbb{C}^n$, and $u(0,1)=0$. Let

$$\bar\partial u=\frac{1}{2}\big(u_t+J_{\mathbb{C}^n}u_\theta\big)\,.$$

By our assumptions, there exist $\delta',C>0$ (dependent on $u(0,1)$) such that

$$\big|\bar\partial_z u\big|\le C\delta\big|\mathrm{d}_z u\big|\qquad\forall\,z\in u^{-1}\big(B_\delta(0)\big),\ \delta\le\delta'\,. \tag{4.30}$$

Write $u=f+\mathrm{i}g$, with f,g taking values in $\mathbb{R}^n$ and assume that $\mathrm{Im}\,u\subset B_\delta(0)$. By (2.5),

$$|\mathrm{d}u|^2=4\big|\bar\partial u\big|^2+2\mathrm{d}(f\cdot\mathrm{d}g).$$

Combining this with (4.30) and Stokes' Theorem, we obtain

$$\int_{[-t,t]\times S^1}|\mathrm{d}u|^2\le 4C^2\delta^2\int_{[-t,t]\times S^1}|\mathrm{d}u|^2+2\int_{\{t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta-2\int_{\{-t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta\,. \tag{4.31}$$

Let $\widetilde{f} = f - \frac{1}{2\pi}\int_0^{2\pi} f\mathrm{d}\theta$. By Hölder's inequality and Lemma C.5,

$$\begin{aligned}\bigg|\int_{\{\pm t\}\times S^1} f\cdot g_\theta\,\mathrm{d}\theta\bigg| &= \bigg|\int_{\{\pm t\}\times S^1} \widetilde{f}\cdot g_\theta\,\mathrm{d}\theta\bigg| \le \bigg(\int_{\{\pm t\}\times S^1}|\widetilde{f}|^2\mathrm{d}\theta\bigg)^{\frac{1}{2}}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac{1}{2}} \\ &\le \bigg(\int_{\{\pm t\}\times S^1}|\widetilde{f}_\theta|^2\mathrm{d}\theta\bigg)^{\frac{1}{2}}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac{1}{2}} \le \frac{1}{2}\int_{\{\pm t\}\times S^1}|u_\theta|^2\mathrm{d}\theta\,.\end{aligned} \tag{4.32}$$

Since

$$3|u_\theta|^2 = 2|u_\theta|^2 + \big|u_t - 2\bar{\partial}u\big|^2 \le 2|\mathrm{d}u|^2 + 8\big|\bar{\partial}u\big|^2\,,$$

the inequalities (4.30)-(4.32) give

$$\big(1-4C^2\delta^2\big)\int_{[-t,t]\times S^1}|\mathrm{d}u|^2 \le \frac{2}{3}\big(1+4C^2\delta^2\big)\bigg(\int_{\{t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta + \int_{\{-t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\bigg).$$

Thus, the function

$$\varepsilon(T) \equiv E_g\big(u;[-R+T,R-T]\big) \equiv \frac{1}{2}\int_{[-R+T,R-T]\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\mathrm{d}t$$

satisfies $\varepsilon(T) \le -\varepsilon'(T)$ for all $T\in[-R,R]$, if δ is sufficiently small (depending on C). This implies (4.28).

For the purposes of establishing (4.29), we can assume that the metric g on X is compatible with J. Let

$$h_{J,g}\colon X \longrightarrow \mathbb{R}^+, \qquad h_{J,g}(x) = h_{J,g}\big(x,\delta_{J,g}(x)\big),$$

with $h_{J,g}(\cdot,\cdot)$ as in Proposition 4.1 and $\delta_{J,g}(\cdot)$ as provided by the previous paragraph. Suppose u also satisfies the last condition in Proposition 4.16. By Proposition 4.1 and (4.28),

$$|\mathrm{d}_{(t,\theta)}u| \le 3\sqrt{E_g(u;[-|t|-1,|t|+1]\times S^1)} \le 3\sqrt{C_{J,g}(u(0,1))}\mathrm{e}^{(1+|t|-R)/2}\sqrt{E_g(u)}$$

for all $t\in[-R+1,R-1]$ and $\theta\in S^1$. Thus, for all $t_1,t_2\in[-R+T,R-T]$ and $\theta_1,\theta_2\in S^1$ with $T\ge 1$ and $t_1\le t_2$,

$$\begin{aligned}d_g\big(u(t_1,\theta_1),u(t_2,\theta_2)\big) &\le 3\sqrt{C_{J,g}(u(0,1))E_g(u)}\bigg(\pi\mathrm{e}^{(1+|t_1|-R)/2} + \int_{t_1}^{t_2}\mathrm{e}^{(1+|t|-R)/2}\mathrm{d}t\bigg) \\ &\le \big(3\pi+12\big)\sqrt{C_{J,g}(u(0,1))}\,\mathrm{e}^{(1-T)/2}\sqrt{E_g(u)}\,.\end{aligned}$$

This establishes (4.29). $\square$

Lemma 4.18. *If (X,J,g) is a compact almost complex Riemannian manifold, there exists a continuous function $\epsilon_{J,g}\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$ with the following property. If $\delta\in\mathbb{R}^+$ and $u\colon(-R,R)\times S^1\longrightarrow X$ is a J-holomorphic map with $E_g(u)<\epsilon_{J,g}(\delta)$, then*

$$\mathrm{diam}_g\big(u\big([-R+1,R-1]\times S^1\big)\big) \le \delta\,.$$

Proof. We can assume that the metric g is compatible with J. By Proposition 3.13 and the compactness of X, there exists $c_{J,g}\in\mathbb{R}^+$ with the following property. If $(\Sigma,\mathfrak{j})$ is a compact connected Riemann surface with boundary, $u\colon\Sigma\longrightarrow X$ is a non-constant J-holomorphic map, $x\in u(\Sigma)$, and $\delta\in\mathbb{R}^+$ are such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$, then

$$E_g(u)\ge c_{J,g}\delta^2\,. \tag{4.33}$$

Let $\hbar_{J,g}>0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 4.1 on the compact space $X\times[0,\mathrm{diam}_g(X)]$.

Suppose $u\colon(-R,R)\times S^1\longrightarrow X$ is a J-holomorphic map with $E_g(u)<\hbar_{J,g}$ and

$$\delta_u\equiv\mathrm{diam}_g\big(u([-R{+}1,R{-}1]\times S^1)\big)>32\sqrt{E_g(u)}.$$

By the first condition on u,

$$\begin{gathered}\big|\mathrm{d}_zu\big|_g^2\le\frac{16}{\pi}E_g(u)\qquad\forall\,z\in[-R{+}1,R{-}1]\times S^1,\\ \mathrm{diam}_g\big(u(r\times S^1)\big)\le8\sqrt{E_g(u)}\quad\forall\,r\in[-R{+}1,R{-}1].\end{gathered}\tag{4.34}$$

Let $r_-,r_0,r_+\in[-R{+}1,R{-}1]$ and $\theta_-,\theta_0,\theta_+\in S^1$ be such that

$$r_-<r_0<r_+,\quad d_g\big(u(r_0,\theta_0),u(r_\pm,\theta_\pm)\big)\ge\frac12\delta_u\,.$$

By (4.34), we can apply (4.33) with

$$\Sigma=[r_-,r_+]\times S^1,\qquad x=u(r_0,\theta_0),\qquad \delta=\frac14\delta_u,$$

and u replaced by its restriction to Σ. We conclude that

$$E_g(u)\ge\frac{c_{J,g}}{16}\delta_u^2\,.$$

It follows that the function

$$\epsilon_{J,g}\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+,\qquad \epsilon_{J,g}(\delta)=\min\left(\hbar_{J,g},\frac{\delta^2}{32^2},\frac{c_{J,g}}{16}\delta^2\right),$$

has the desired property. □

Proof of Corollary 4.17. Let $\delta\in\mathbb{R}^+$ be the minimum of the function $\delta_{J,g}$ in Proposition 4.16 and $\varepsilon_{J,g}(\cdot)$ be as in Lemma 4.18. Take $C_{J,g}$ to be the maximum of the function $C_{J,g}$ in Proposition 4.16 times e and $\hbar_{J,g}\in\mathbb{R}^+$ to be smaller than the minimum of the function $\hbar_{J,g}$ in Proposition 4.16 and the number $\varepsilon_{J,g}(\delta)$. If u is as in the statement of Corollary 4.17, the required bounds then follow from (4.28) and (4.29) applied with

$$R=(R_2-R_1)/2\qquad\text{and}\qquad \widetilde{u}\colon[-R,R]\times S^1\longrightarrow X,\quad \widetilde{u}(s,\theta)=u\big(s{+}R_1{+}R,\theta\big).$$

The map $\widetilde{u}$ is J-holomorphic and has the same energy as u. □

5 Limiting Behavior of J-Holomorphic Maps

This section studies the limiting behavior of sequences of J-holomorphic maps from Riemann surfaces into a compact almost complex manifold (X,J). The compactness of X plays an essential role in the statements below, in contrast to nearly all statements in Sections 3 and 4.

5.1 Removal of Singularity

By Cauchy's Integral Formula, a bounded holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow\mathbb{C}^n$ extends over the origin. By Proposition 5.1 below, the same is the case for a J-holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow X$ of bounded energy if X is compact.

Proposition 5.1 (Removal of Singularity)**.** *Let (X,J) be a compact almost complex manifold and $u\colon B_R^*\longrightarrow X$ be a J-holomorphic map. If the energy $E_g(u)$ of u, with respect to any metric g on X, is finite, then u extends to a J-holomorphic map $\widetilde{u}\colon B_R\longrightarrow X$.*

A basic example of a holomorphic function $u\colon\mathbb{C}^*\longrightarrow\mathbb{C}$ that does not extend over the origin $0\in\mathbb{C}$ is $z\longrightarrow 1/z$. The energy of $u|_{B_R^*}$ with respect to the standard metric on $\mathbb{C}$ is given by

$$E\big(u;B_R^*\big)=\frac{1}{2}\int_{B_R}|\mathrm{d}u|^2=\int_{B_R}\frac{1}{|z|^4}=\int_0^{2\pi}\!\!\int_0^R r^{-3}\mathrm{d}r\mathrm{d}\theta\not<\infty.$$

The above integral would have been finite if $|\mathrm{d}u|^2$ were replaced by $|\mathrm{d}u|^{1-\epsilon}$ for *any* $\epsilon>0$. This observation illustrates the crucial role played by the energy in the theory of J-holomorphic maps.

By Cauchy's Integral Formula, the conclusion of Proposition 5.1 holds if J is a complex structure and $u(B_\delta^*)$ is contained in a complex coordinate chart for some $\delta\in(0,R)$. We will use the Monotonicity Lemma to show that the latter is the case if the energy of u is finite; the integrability of J turns out to be irrelevant here.

Proof of Proposition 5.1. In light of Proposition 4.8, it is to sufficient to show that u extends continuously over the origin. We can assume that the metric g is compatible with J and $R=1$. Let $c_{J,g},\hbar_{J,g}\in\mathbb{R}^+$ be as in the proof of Lemma 4.18. Define

$$v\colon\mathbb{R}^-\times S^1\longrightarrow X,\qquad v\big(r,\mathrm{e}^{\mathrm{i}\theta}\big)=u\big(\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

This map is J-holomorphic and satisfies $E_g(v)=E_g(u)<\infty$.

Since $E_g(u)<\infty$,

$$\lim_{r\longrightarrow-\infty}E_g\big(v;(-\infty,r)\times S^1\big)=\lim_{r\longrightarrow-\infty}E_g\big(u;B_{e^r}^*\big)=0. \tag{5.1}$$

In particular, there exists $R\in\mathbb{R}^-$ such that

$$E_g\big(v;(-\infty,r)\times S^1\big)<\hbar_{J,g}\qquad\forall\ r<R.$$

By Proposition 4.1 and our choice of $\hbar_{J,g}$, this implies that

$$\begin{aligned}\big|\mathrm{d}_z v\big|_g^2\le\frac{16}{\pi}E_g\big(v;(-\infty,r+1)\times S^1\big)\qquad&\forall\ z\in(-\infty,r)\times S^1,\ r<R-1,\\ \mathrm{diam}_g\big(v(\{r\}\times S^1)\big)\le 4\sqrt{\pi}\sqrt{E_g(v;(-\infty,r+1)\times S^1)}\quad&\forall\ r<R-1.\end{aligned}$$

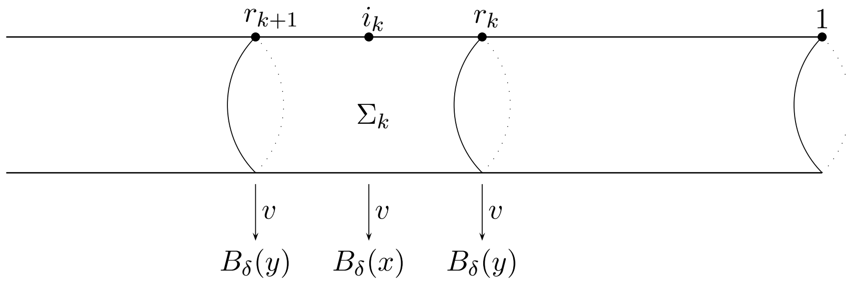

Figure 8: Setup for the proof of Proposition 5.1

Combining the last bound with (5.1), we obtain

$$\lim_{r\longrightarrow-\infty}\mathrm{diam}_g\big(v(\{r\}\times S^1)\big)=0.$$

Thus, it remains to show that $\lim\limits_{r\longrightarrow-\infty}v(r,1)$ exists.

Since X is compact, every sequence in X has a convergent subsequence. Suppose there exist

$$\begin{gathered}\delta\in\mathbb{R}^+,\quad x,y\in X,\quad i_k,r_k\in\mathbb{R}^-\qquad\text{s.t.}\\ d_g(x,y)>3\delta,\quad r_{k+1}<i_k<r_k,\quad v\big(\{i_k\}\times S^1\big)\subset B_\delta(x),\quad v\big(\{r_k\}\times S^1\big)\subset B_\delta(y).\end{gathered}$$

We thus can apply (4.33) with Σ, x, and u replaced by

$$\Sigma_k\equiv[r_{k+1},r_k]\times S^1,\qquad x_k\equiv u(i_k,1),\quad\text{and}\quad v_k\equiv v|_{\Sigma_k}\,,$$

respectively. We conclude that

$$E_g(v)\ge\sum_k E_g\big(v;\Sigma_k\big)=\sum_k E_g(v_k)\ge\sum_k c_{J,g}\delta^2=\infty\,.$$

However, this contradicts the assumption that $E_g(v)\!=\!E_g(u)\!<\!\infty$. □

5.2 Bubbling

The next three statements are used in Section 5.3 to show that no energy is lost under Gromov's convergence procedure, the resulting bubbles connect, and their number is finite.

Lemma 5.2. *Suppose (X,J,g) is an almost complex Riemannian manifold and $u_i\colon B_1\longrightarrow X$ is a sequence of C^1-maps converging to a C^1-map $u\colon B_1^*\longrightarrow X$ C^1-u.c.s. so that $E_g(u)<\infty$ and the limit*

$$\mathfrak{m}\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta)\tag{5.2}$$

exists and is nonzero.

(1) The limit $\mathfrak{m}(\delta)\equiv\lim\limits_{i\longrightarrow\infty}E_g(u_i;B_\delta)$ exists for every $\delta\!\in\!(0,1)$ and is a continuous, non-decreasing function of δ.

(2) For every sequence $z_i\!\in\!B_1$ converging to 0 and $\delta\!\in\!(0,1)$, $\lim\limits_{i\longrightarrow\infty}E_g(u_i;B_\delta(z_i))\!=\!\mathfrak{m}(\delta)$.

(3) For every sequence $z_i \in B_1$ converging to 0, $\mu \in (0, \mathfrak{m})$, and $i \in \mathbb{Z}^+$ sufficiently large, there exists a unique $\delta_i(\mu) \in (0, 1-|z_i|)$ such that $E_g(u_i; B_{\delta_i(\mu)}(z_i)) = \mu$.

Proof. (1) Since $\mathrm{d}u_i$ converges uniformly to $\mathrm{d}u$ on compact subsets of B_1^*,

$$\begin{aligned}\mathfrak{m}(\delta) \equiv \lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta\big) &= \lim_{\delta'\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{\delta'}\big) + \lim_{\delta'\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta - B_{\delta'}\big)\\ &= \mathfrak{m} + \lim_{\delta'\longrightarrow 0} E_g\big(u; B_\delta - B_{\delta'}\big) = \mathfrak{m} + E_g(u; B_\delta).\end{aligned}$$

Since $E_g(u; B_\delta)$ is a continuous, non-decreasing function of δ, so is $\mathfrak{m}(\delta)$.

(2) For all $\delta, \delta' \in \mathbb{R}^+$ and $z_i \in B_{\delta'}$, $B_{\delta-\delta'} \subset B_\delta(z_i) \subset B_{\delta+\delta'}$. Thus,

$$E_g\big(u_i; B_{\delta-\delta'}\big) \le E_g\big(u_i; B_\delta(z_i)\big) \le E_g\big(u_i; B_{\delta+\delta'}\big)$$

for all $i \in \mathbb{Z}^+$ sufficiently large and

$$\lim_{\delta'\longrightarrow 0} \mathfrak{m}(\delta-\delta') \le \lim_{\delta'\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) \le \lim_{\delta'\longrightarrow 0} \mathfrak{m}(\delta+\delta') \qquad \forall\, \delta' \in \mathbb{R}^+ .$$

The claim now follows from (1).

(3) By (2), (1), and (5.2),

$$\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) = \mathfrak{m}(\delta) \ge \mathfrak{m}\,.$$

Thus, there exists $i(\mu) \in \mathbb{Z}^+$ such that

$$E_g\big(u_i; B_\delta(z_i)\big) > \mu \qquad \forall\, i \ge i(\mu).$$

Since $E_g(u_i; B_\delta(z_i))$ is a continuous, increasing function of δ which vanishes at $\delta = 0$, for every $i \ge i(\mu)$ there exists a unique $\delta_i(\mu) \in (0, \delta)$ such that $E_g(u_i; B_{\delta_i(\mu)}(z_i)) = \mu$. □

Corollary 5.3. *If (X, J, g) is a compact Riemannian almost complex manifold, there exists $\hbar_{J,g} \in \mathbb{R}^+$ with the following properties. If $u_i\colon B_1 \longrightarrow X$ is a sequence of J-holomorphic maps converging to a C^1-map $u\colon B_1^* \longrightarrow X$ C^1-u.c.s. so that $E_g(u) < \infty$,*

$$\lim_{i\longrightarrow\infty} \max_{B_{1/2}} \big|\mathrm{d}u_i\big|_g = \infty,$$

and the limit (5.2) exists, then

(1) $\mathfrak{m} \ge \hbar_{J,g}$;

(2) for every sequence $z_i \in B_\delta$ converging to 0, the numbers $\delta_i(\mu) \in (0, 1-|z_i|)$ of Lemma 5.2(3) with $\mu \in (\mathfrak{m}-\hbar_{J,g}, \mathfrak{m})$ satisfy

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{R\delta_i(\mu)}(z_i)\big) = \mathfrak{m}, \tag{5.3}$$

$$\lim_{(R,\delta)\longrightarrow(\infty,0)}\lim_{i\longrightarrow\infty} \mathrm{diam}_g\big(u_i(B_\delta(z_i) - B_{R\delta_i(\mu)}(z_i))\big) = 0. \tag{5.4}$$

Proof. Let $\hbar_{J,g}$ be the smaller of the constants $\hbar_{J,g}$ in Corollaries 4.2 and 4.17. Let u_i, u, and $\mathfrak{m}$ be as in the statement of Corollary 5.3.

(1) For each $i\in\mathbb{Z}^+$, let

$$M_i = \max_{\overline{B_{1/2}}} \big|\mathrm{d}_z u_i\big|_g \in \mathbb{R}^+$$

and $z_i \in \overline{B_{1/2}}$ be such that $|\mathrm{d}_{z_i} u_i|_g = M_i$. Since $M_i \longrightarrow \infty$ as $i \longrightarrow \infty$ and u_i converges to u u.c.s., $z_i \longrightarrow 0$. For $i\in\mathbb{Z}^+$ such that $|z_i|+1/\sqrt{M_i}<1/2$, define

$$v_i\colon B_{\sqrt{M_i}} \longrightarrow X, \qquad v_i(w) = u_i\big(z_i+w/M_i\big).$$

Thus, v_i is a J-holomorphic map with

$$\sup \big|\mathrm{d}v_i\big|_g = \big|\mathrm{d}_0 v_i\big|_g = 1, \qquad E_g(v_i) = E_g\big(u_i; B_{1/\sqrt{M_i}}(z_i)\big) \le E_g\big(u_i; B_{|z_i|+1/\sqrt{M_i}}\big)\,. \tag{5.5}$$

By the first statement in (5.5) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of v_i converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}$ to a non-constant J-holomorphic map $v\colon \mathbb{C}\longrightarrow X$. By the second statement in (5.5) and Lemma 5.2(1),

$$E_g(v) \le \limsup_{i\longrightarrow\infty} E_g\big(u_i; B_{1/\sqrt{M_i}}(z_i)\big) \le \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta\big) = \mathfrak{m}. \tag{5.6}$$

By Proposition 5.1, v thus extends to a J-holomorphic map $\widetilde{v}\colon \mathbb{P}^1\longrightarrow X$. By Corollary 4.2,

$$E_g(v) = E_g(\widetilde{v}) \ge \hbar_{J,g}\,.$$

Combining this with (5.6), we obtain the first claim.

(2) By the first two statements in Lemma 5.2 and (5.2),

$$\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) = \lim_{\delta\longrightarrow 0}\mathfrak{m}(\delta) = \mathfrak{m}. \tag{5.7}$$

After passing to a subsequence of u_i, we can thus assume that there exists a sequence $\delta_i \longrightarrow 0$ such that

$$\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{\delta_i}(z_i)\big) = \mathfrak{m}. \tag{5.8}$$

Since $\delta_i \longrightarrow 0$, (5.7) and (5.8) imply that

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{R\delta_i}(z_i)\big) = \mathfrak{m}. \tag{5.9}$$

Suppose $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$. By (5.9) and the definition of $\delta_i(\mu)$,

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{R\delta_i}(z_i)-B_{\delta_i(\mu)}(z_i)\big) = \mathfrak{m}-\mu < \hbar_{J,g}\,.$$

Thus, Corollary 4.17 applies with (R_1,R_2,T) replaced by $(\ln\delta_i(\mu), \ln\delta_i+\ln R, \ln R)$ and u replaced by the J-holomorphic map

$$v_i\colon (R_1,R_2)\times S^1 \longrightarrow X, \qquad v_i\big(r,\mathrm{e}^{\mathrm{i}\theta}\big) = u_i\big(z_i+\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

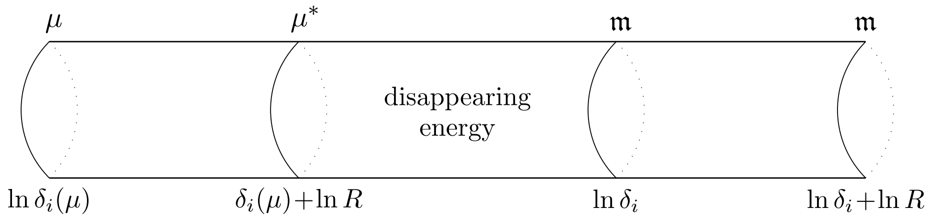

Figure 9: Illustration for the proof of (5.3)

By the first statement of this corollary,

$$E_g\big(u;B_{\delta_i}(z_i)\big)-E_g\big(u;B_{R\delta_i(\mu)}(z_i)\big)=E_g\big(u;B_{\delta_i}(z_i)-B_{R\delta_i(\mu)}(z_i)\big)\le\frac{C_{J,g}}{R}E_g(u)$$

for all i sufficiently large (depending on R); see Figure 9. Combining this with (5.8), we obtain (5.3).

It remains to establish (5.4). By Lemma 5.2(2), (5.2), and the definition of $\delta_i(\mu)$.

$$\begin{aligned}&\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{R\delta}(z_i)-B_{\delta_i(\mu)}(z_i)\big)\\&\qquad=\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_\delta(z_i)\big)-\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{\delta_i(\mu)}(z_i)\big)=\mathfrak{m}-\mu.\end{aligned}$$

Thus, for all $R>0$ and $\delta\in(0,\delta(R))$ there exists $i(R,\delta)\in\mathbb{Z}^+$ such that

$$E_g\big(u_i;B_{R\delta}(z_i)-B_{\delta_i(\mu)}(z_i)\big)<\hbar_{J,g}\qquad\forall\ i>i(R,\delta).$$

Thus, Corollary 4.17 applies with (R_1,R_2,T) replaced by $(\ln\delta_i(\mu),\ln\delta+\ln R,\ln R)$ and u replaced by the J-holomorphic map

$$v_i\colon(R_1,R_2)\times S^1\longrightarrow X,\qquad v_i\big(r,\mathrm{e}^{\mathrm{i}\theta}\big)=u_i\big(z_i+\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

By the second statement of this corollary,

$$\mathrm{diam}_g\big(u_i(B_\delta(z_i)-B_{R\delta_i(\mu)}(z_i))\big)\le\frac{C_{J,g}}{\sqrt{R}}\hbar_{J,g}\qquad\forall\ i>i(R,\delta),\ \delta\in(0,\delta(R)).$$

Since increasing R does not increase the left-hand side above, we obtain (5.4). $\square$

Lemma 5.4. *If (X,J,g) is a compact almost complex Riemannian manifold, then there exists a function $N\colon\mathbb{R}\longrightarrow\mathbb{Z}$ with the following property. If $(\Sigma,\mathfrak{j})$ is compact Riemann surface, $S_0\subset\Sigma$ is a finite subset, and $u_i\colon U_i\longrightarrow X$ is a sequence of J-holomorphic maps from open subsets of Σ with*

$$U_i\subset U_{i+1},\qquad\Sigma-S_0=\bigcup_{i=1}^{\infty}U_i,\quad\text{and}\quad E\equiv\liminf_{i\longrightarrow\infty}E_g(u_i)<\infty,\tag{5.10}$$

then there exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, so that the set

$$S\equiv\Big\{z\in\Sigma-S_0\colon\lim_{\delta\longrightarrow0}\sup_{B_\delta(z)}\big|\mathrm{d}u_i\big|_g=\infty\Big\}$$

is of cardinality at most $N(E)$ and the sequence u_i converges to a J-holomorphic map $u\colon\Sigma\longrightarrow X$ C^1-u.c.s. on $\Sigma-S_0\cup S$.

Proof. Let $\hbar_{J,g}$ be the minimal value of the function provided by Proposition 4.1. For $E\in\mathbb{R}^+$, let $N(E)\in\mathbb{Z}^{\ge 0}$ be the smallest integer such that $E\le N(E)\hbar_{J,g}$.

Let Σ, S_0, u_i, and E be as in the statement of the lemma and $N=N(E)+|S_0|$. Fix a Riemannian metric g_Σ on Σ. For $z\in\Sigma$ and $\delta\in\Sigma$, let $B_\delta(z)\subset\Sigma$ denote the ball of radius δ around z. By Proposition 4.1, there exists $C\in\mathbb{R}^+$ with the following property. If $u\colon B_\delta(z)\longrightarrow X$ is a J-holomorphic map, $z\in\Sigma$, and $\delta\in\mathbb{R}^+$, then

$$E_g\big(u;B_\delta(z)\big)<\hbar_{J,g}\qquad\Longrightarrow\qquad \big|\mathrm{d}_z u\big|_g\le C/\delta^2\,. \tag{5.11}$$

For every pair $i,j\in\mathbb{Z}^+$, let $\{z_{ij}^k\}_{k=1}^N$ be a subset of points of Σ containing S_0 such that

$$z\in\Sigma_{ij}^*\equiv\Sigma-\bigcup_{k=1}^N B_{2/j}\big(z_{ij}^k\big)\qquad\Longrightarrow\qquad E_g\big(u_i;B_{1/j}(z)\cap U_i\big)<\hbar_{J,g}\,. \tag{5.12}$$

By (5.11) and (5.12),

$$\big|\mathrm{d}_z u_i\big|_g\le Cj^2\qquad\forall\,z\in\Sigma_{ij}^*\text{ s.t. }B_{1/j}(z)\subset U_i\,. \tag{5.13}$$

After passing to a subsequence of $\{u_i\}$, we can assume that the sequence $E_g(u_i)$ converges to E and that the sequence $\{z_{ij}^k\}_{i\in\mathbb{Z}^+}$ converges to some $z_j^k\in\Sigma$ for every $k=1,\ldots,N$ and $j\in\mathbb{Z}^+$. Along with (5.13) and the first two assumptions in (5.10), this implies that

$$\limsup_{i\longrightarrow\infty}\big|\mathrm{d}_z u_i\big|_g\le Cj^2\qquad\forall\,z\in\Sigma_{ij}^*\,. \tag{5.14}$$

After passing to another subsequence of $\{u_i\}$, we can assume that the sequence $\{z_j^k\}_{j\in\mathbb{Z}^+}$ converges to some $z^k\in\Sigma$ for every $k=1,\ldots,N$.

For each $j\in\mathbb{Z}^+$, let

$$\Sigma_j^*=\Sigma-\bigcup_{k=1}^N B_{2/j}\big(z_j^k\big).$$

By (5.14) and the ellipticity of the $\bar\partial$-operator, a subsequence of u_i converges uniformly in the C^∞-topology on compact subsets of Σ_1^* to a J-holomorphic map $v_1\colon\Sigma_1^*\longrightarrow X$. By (5.14) and the ellipticity of the $\bar\partial$-operator, a subsequence of this subsequence in turn converges uniformly in the C^∞-topology on compact subsets of Σ_2^* to a J-holomorphic map $v_2\colon\Sigma_2^*\longrightarrow X$. Continuing in this way, we obtain a subsequence of u_i converging uniformly in the C^∞-topology on compact subsets of Σ_j^* to a J-holomorphic map $v_j\colon\Sigma_j^*\longrightarrow X$ for every $j\in\mathbb{Z}^+$. The limiting maps satisfy

$$v_j|_{\Sigma_j\cap\Sigma_{j'}^*}=v_{j'}|_{\Sigma_j^*\cap\Sigma_{j'}^*}\qquad\forall\,j,j'\in\mathbb{Z}^+\,.$$

Thus, the map

$$u\colon\Sigma^*\equiv\Sigma-\big\{z^1,\ldots,z^N\big\}\longrightarrow X,\qquad u(z)=v_j(z)\ \ \forall\,z\in\Sigma_j^*,$$

is well-defined and J-holomorphic.

Since the convergence is C^∞ on compact subsets of Σ^*, $S\subset\{z^1,\ldots,z^N\}$ and

$$E_g(u)\le\liminf_{i\longrightarrow\infty}E_g(u_i)=E\,.$$

By Proposition 5.1, u thus extends to a J-holomorphic map $\Sigma \longrightarrow X$, which we denote by u as well. If $z^k \notin S_0 \cup S$, a subsequence of u_i converges uniformly in the C^∞-topology on $B_\delta(z^k)$ to a J-holomorphic map $v\colon B_\delta(z^k) \longrightarrow X$ for some $\delta \in \mathbb{R}^+$. Thus, a subsequence of u_i converges uniformly in the C^∞-topology on compact subsets of $\Sigma - S_0 \cup S$ to the J-holomorphic map u above. □

5.3 Gromov's convergence

We next show that a sequence of maps as in Corollary 5.3 gives rise to a continuous map from a tree of spheres attached at $0 \in B_1$, i.e. a connected union of spheres that have a distinguished base component and no loops; the distinguished component will be attached at $\infty \in \mathbb{P}^1$ to $0 \in B_1$. The combinatorial structure of such a tree is described by a finite rooted linearly ordered set (or rooted tree), i.e. a partially ordered set $(\mathscr{V}, \prec)$ such that

(RS1) there is a unique smallest element (root) $v_0 \in \mathscr{V}$, i.e. $v_0 \prec v$ for every $v \in I - \{v_0\}$, and

(RS2) for all $v, v_1, v_2 \in \mathscr{V}$ with $v_1, v_2 \prec v$, either $v_1 = v_2$, or $v_1 \prec v_2$, or $v_2 \prec v_1$.

For each $v \in \mathscr{V} - \{v_0\}$, let $p(v) \in \mathscr{V}$ denote the immediate predecessor of v, i.e. $p(v) \in \mathscr{V}$ such that $p(v) \prec v$ and $v' \prec p(v)$ for all $v' \in \mathscr{V} - \{p(v)\}$ such that $v' \prec v$. Such $p(v) \in \mathscr{V}$ exists by (RS1) and is unique by (RS2). In the first diagram in Figure 10, the vertices (dots) represent the elements of a rooted linearly ordered set $(\mathscr{V}, \prec)$ and the edges run from $v \in \mathscr{V} - \{v_0\}$ down to $p(v)$. For $v \in \mathscr{V}$, let

$$S_v(\mathscr{V}) \equiv \big\{v' \in \mathscr{V} - \{v_0\} \colon p(v') = v\big\}$$

be the set of immediate successors of v.

Given a finite tree $(\mathscr{V}, \prec)$ with root v_0 and a function

$$\mu\colon \mathscr{V} - \{v_0\} \longrightarrow \mathbb{C} \quad \text{s.t.} \quad \big(p(v_1), \mu(v_1)\big) \neq \big(p(v_2), \mu(v_2)\big) \ \ \forall\, v_1, v_2 \in \mathscr{V} - \{v_0\},\ v_1 \neq v_2, \tag{5.15}$$

let

$$\Sigma_{\mathscr{V},\mu} = \bigg(\bigsqcup_{v \in \mathscr{V}} \{v\} \times \mathbb{P}^1\bigg)\Big/\!\sim, \qquad (v, \infty) \sim \big(p(v), \mu(v)\big) \quad \forall\, v \in \mathscr{V} - \{v_0\};$$

see the second diagram in Figure 10. Thus, the tree $\Sigma_{\mathscr{V},\mu}$ of spheres is obtained by attaching ∞ in the sphere

$$\mathbb{P}^1_v \equiv \{v\} \times \mathbb{P}^1$$

to $\mu(v) \in \mathbb{P}^1_{p(v)}$. By (5.15), $\Sigma_{\mathscr{V},\mu}$ is a nodal Riemann surface, i.e. each non-smooth point (node) of $\Sigma_{\mathscr{V},\mu}$ has only two local branches (pieces homeomorphic to $\mathbb{C}$). We call a function μ as in (5.15) an attaching map for $(\mathscr{V}, \prec)$.

Proposition 5.5. *Let (X, J, g) be a compact almost complex Riemannian manifold and $u_i\colon B_1 \longrightarrow X$ be a sequence of J-holomorphic maps converging to a J-holomorphic map $u\colon B_1 \longrightarrow X$ C^1-u.c.s. on B_1^*. If the limit (5.2) exists and is nonzero, then there exist*

(1) a J-holomorphic map $u_{v_0}\colon \mathbb{P}^1 \longrightarrow X$ with $u_{v_0}(\infty) = u(0)$ and a finite subset $S_{v_0} \subset \mathbb{C}$,

(2) a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, and

(3) for every $i \in \mathbb{Z}^+$, a biholomorphic map $\psi_i\colon U_i \longrightarrow B_{1/2}$, where $U_i \subset \mathbb{C}$ is an open subset,

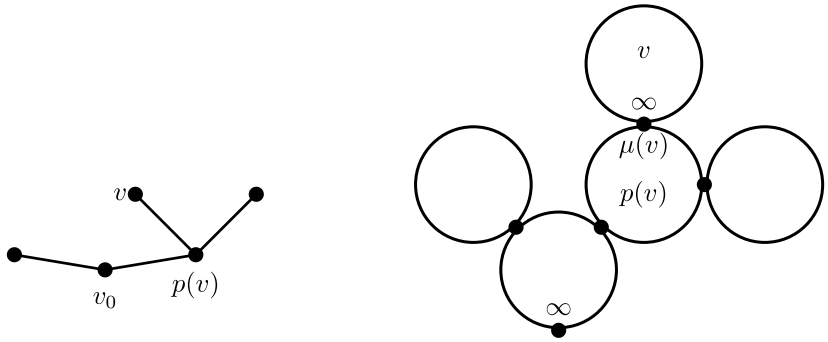

Figure 10: A tree $(\mathscr{V}, \prec)$ with root v_0 and the associated tree $\Sigma_{\mathscr{V},\mu}$ of spheres

such that

(a) $U_i \subset U_{i+1}$ *for all* $i \in \mathbb{Z}^+$, $\mathbb{C} = \bigcup\limits_{i=1}^{\infty} U_i$, *and the sequence* $u_i \circ \psi_i$ *converges to* u_{v_0} C^1*-u.c.s. on* $\mathbb{C} - S_{v_0}$,

(b) *the limit*

$$\mathfrak{m}_w \equiv \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i \circ \psi_i; B_\delta(w)\big), \tag{5.16}$$

exists and is nonzero for every $w \in S_{v_0}$ *and* $\mathfrak{m} = E_g\big(u_{v_0}\big) + \sum\limits_{w \in S_{v_0}} \mathfrak{m}_w$,

(c) *if* u_{v_0} *is constant, then* $|S_{v_0}| \geq 2$.

Proof. Let $\hbar \equiv \hbar_{J,g}$ be as in Corollary 5.3. For each $i \in \mathbb{Z}^+$ sufficiently large, choose $z_i \in \overline{B_{1/2}}$ so that

$$\max_{z \in \overline{B_{1/2}}} \big|\mathrm{d}u_i\big|_g = \big|\mathrm{d}_{z_i} u_i\big|_g. \tag{5.17}$$

Since u_i converges uniformly in the C^∞-topology on compact subsets of B_1^* to u, $z_i \longrightarrow 0$ as $i \longrightarrow \infty$. Thus, $B_{1/2}(z_i) \subset B_1$ for all $i \in \mathbb{Z}^+$ sufficiently large. By Lemma 5.2(3), for all $i \in \mathbb{Z}^+$ sufficiently large there exists $\delta_i \in (0, 1/2)$ such that

$$E_g\big(u_i; B_{\delta_i}(z_i)\big) = \mathfrak{m} - \frac{\hbar}{2}\,. \tag{5.18}$$

Define

$$\psi_i\colon U_i \equiv \big\{w \in \mathbb{C}\colon z_i + \delta_i w \in B_{1/2}\big\} \longrightarrow B_{1/2} \qquad \text{by} \qquad \psi_i(w) = z_i + \delta_i w\,.$$

Since $\delta_i \longrightarrow 0$, the second property in (b) holds. By taking a subsequence of $\{u_i\}$, we can assume that the first property in (b) holds as well.

For each $i \in \mathbb{Z}^+$ sufficiently large, let

$$u_{v_0;i} = u_i \circ \psi_i \colon U_i \longrightarrow X. \tag{5.19}$$

Since u_i is J-holomorphic and ψ_i is biholomorphic onto its image, $u_{v_0;i}$ is a J-holomorphic map with $E_g(u_{v_0;i}) = E_g(u_i; B_{1/2})$. Along with Lemma 5.2(2), this implies that

$$\lim_{i \longrightarrow \infty} E_g(u_{v_0;i}) = \mathfrak{m}(1/2) < \infty\,.$$

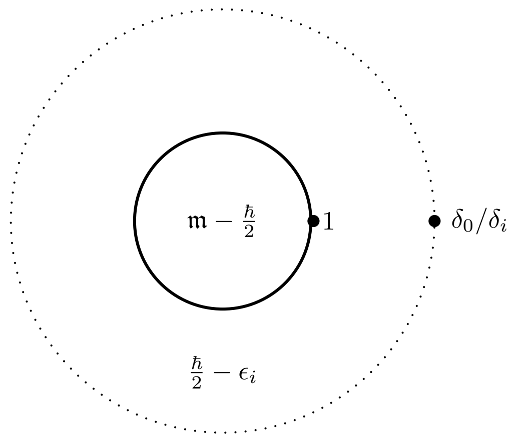

Figure 11: The energy distribution of the rescaled map $u_{v0;i}$ in the proof of Proposition 5.5

By Lemma 5.4, there thus exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, so that the set

$$S_{v_0} \equiv \big\{w\in\mathbb{C}\colon \lim_{\delta\longrightarrow 0}\sup_{B_\delta(w)}\big|\mathrm{d}u_{v_0;i}\big|_g=\infty\big\} \tag{5.20}$$

is finite and the sequence $u_{v_0;i}$ converges to a J-holomorphic map $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$ C^1-u.c.s. on $\mathbb{C}-S_{v_0}$. In particular, the last property in (a) holds and $|\mathrm{d}u_{v_0;i}|_g$ is uniformly bounded on compact subsets of $\mathbb{C}-S_{v_0}$.

For all $R>0$ so that $S_{v_0}\subset B_R$, $\delta>0$ so that $B_\delta(w)\subset B_R$ for every $w\in S_{v_0}$ and $B_\delta(w)\cap B_\delta(w')=\emptyset$ for all $w,w'\in S_{v_0}$ distinct, and $i\in\mathbb{Z}^+$ so that $B_R\subset U_i$,

$$E_g\Big(u_{v_0;i},B_R-\bigcup_{w\in S_{v_0}}B_\delta(w)\Big)+\sum_{w\in S_{v_0}}E_g\big(u_{v_0;i};B_\delta(w)\big)=E_g\big(u_{v_0;i},B_R\big). \tag{5.21}$$

We can pass to a further subsequence of $\{u_i\}$ so that the limit (5.16) exists for every $w\in S_{v_0}$. In light of Corollary 5.3(1), $\mathfrak{m}_w\ge\hbar$ for all $w\in S_{v_0}$. By (a) and (5.21),

$$\begin{aligned}
E_g(u_{v_0})+\sum_{w\in S_{v_0}}\mathfrak{m}_w &= \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\Big(u_{v_0;i},B_R-\bigcup_{w\in S_{v_0}}B_\delta(w)\Big)\\
&\qquad\qquad+\sum_{w\in S_{v_0}}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i};B_\delta(w)\big)\\
&=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i},B_R\big)=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i,B_{R\delta_i}(z_i)\big)=\mathfrak{m}\,;
\end{aligned} \tag{5.22}$$

the last equality holds by (5.3).

We next show that $u(0)=u_{v_0}(\infty)$. Note that

$$\begin{aligned}
d_g\big(u(0),u_{v_0}(\infty)\big) &= \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}d_g\big(u(\delta),u_{v_0}(R)\big)=\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}d_g\big(u_i(z_i+\delta),u_{v_0;i}(R)\big)\\
&=\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}d_g\big(u_i(z_i+\delta),u_i(z_i+R\delta_i)\big)\\
&\le\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}\mathrm{diam}_g\big(u_i(B_\delta(z_i)-B_{R\delta_i}(z_i))\big).
\end{aligned} \tag{5.23}$$

Along with (5.4), this implies that $u(0)=u_{v_0}(\infty)$.

Suppose $u_{v_0}:\mathbb{P}^1\longrightarrow X$ is a constant map. By (5.22), $S_{v_0}\neq\emptyset$. Since

$$E_g\big(u_{v_0;i};B_1\big)=E_g\big(u_i;B_{\delta_i}(z_i)\big)=\mathfrak{m}-\frac{\hbar}{2}\geq\frac{\hbar}{2} \tag{5.24}$$

by (5.18), $S_{v_0}\cap\overline{B_1}\neq\emptyset$. By (5.17) and the definition of ψ_i, $|\mathrm{d}_0u_{v_0;i}|\geq|\mathrm{d}_wu_{v_0;i}|$ for all $w\in U_i$. Thus, $0\in S_{v_0}$. By (5.24),

$$\mathfrak{m}_0\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g(u_{v_0;i};B_\delta)\leq\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i};B_1\big)=\mathfrak{m}-\frac{\hbar}{2}<\mathfrak{m},$$

and so $|S_{v_0}|\geq2$, as claimed in (c). Since the amount of energy of $u_{v_0;i}$ contained in $\mathbb{C}-B_1$ approaches $\hbar/2$, as illustrated in Figure 11, there must be in particular a bubble point $w\in S_{v_0}$ with $|w|=1$, though this is not material. $\square$

Corollary 5.6. *Let (X,J,g), u_i, u, and $\mathfrak{m}>0$ be as in Proposition 5.5. There exist*

(0) a finite tree $(\mathscr{V},\prec)$ with root v_0 and an attaching map μ for $(\mathscr{V},\prec)$,

(1) a J-holomorphic map $u_\infty:\Sigma_{\mathscr{V},\mu}\longrightarrow X$ with $u_\infty|_{\mathbb{P}^1_{v_0}}(\infty)=u(0)$,

(2) a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, and

(3) for every $v\in\mathscr{V}$ and $i\in\mathbb{Z}^+$, $\delta_v\in\mathbb{R}^+$ and a biholomorphic map $\psi_{v;i}:U_{v;i}\longrightarrow B_{\delta_v}(\mu(v))$, where $U_{v;i}\subset\mathbb{C}$ is an open subset and $\mu(v_0)\equiv0$,

such that

(a) for every $v\in\mathscr{V}$, $U_{v;i}\subset U_{v;i+1}$ for all $i\in\mathbb{Z}^+$, $\mathbb{C}=\bigcup_{i=1}^{\infty}U_{v;i}$, and the sequence $u_{v;i}$, where $u_{p(v_0);i}\equiv u_i$ and $u_{v;i}\equiv u_{p(v);i}\circ\psi_{v;i}$ if $v\in\mathscr{V}-\{v_0\}$, converges to $u_\infty|_{\mathbb{P}^1_v}$ C^1-u.c.s. on $\mathbb{C}-\mu(S_v(\mathscr{V}))$,

(b) for every $v\in\mathscr{V}$,

$$\mathfrak{m}_v\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_{p(v);i};B_\delta(\mu(v))\big)=E_g\big(u_\infty|_{\mathbb{P}^1_v}\big)+\sum_{v'\in S_v(\mathscr{V})}\mathfrak{m}_{v'}, \tag{5.25}$$

(c) if $v\in\mathscr{V}$ and $u_\infty|_{\mathbb{P}^1_v}$ is constant, then $|S_v(\mathscr{V})|\geq2$.

If $v\in\mathscr{V}$ is a maximal element, $\mathfrak{m}_v=E_g(u_\infty|_{\mathbb{P}^1_v})$ by (5.25). Since the set $\mathscr{V}$ is finite, it follows from (5.25) that

$$\mathfrak{m}\equiv\mathfrak{m}_{v_0}=E_g\big(u_\infty\big).$$

Proof of Corollary 5.6. Let $\hbar$ be the smaller of the numbers $\hbar_{J,g}$ in Corollaries 4.2 and 5.3. In particular, $\mathfrak{m}\geq\hbar$. Let $N\in\mathbb{Z}^+$ be the largest integer so that $N\hbar\leq\mathfrak{m}$.

Let u_{v_0}, S_{v_0}, $\{u_i\}$, $\psi_{v_0;i}\equiv\psi_i$, and $U_{v_0;i}\equiv U_i$ be as in (1)-(3) of Proposition 5.5 and $\delta_{v_0}=1/2$. If $N=1$, $S_{v_0}=\emptyset$ by (b) and (c) in Proposition 5.5. We then take $\mathscr{V}\equiv\{v_0\}$ and $u_\infty=u_{v_0}$, establishing the conclusion of Corollary 5.6 in this case.

Suppose $N \geq 2$ and the conclusion Corollary 5.6 holds for all smaller values of N. Let $w \in S_{v_0}$ and $\delta_w \in \mathbb{R}^+$ be such that $B_{2\delta_w}(w) \subset \mathbb{C}$ is disjoint from $S_{v_0} - \{w\}$. By a translation and rescaling, we identify $B_{2\delta_w}(w)$ with B_1. By (b) and (c) in Proposition 5.5, $\mathfrak{m}_w < N\hbar$. By the inductive assumption, there exist

(0) a finite tree $(\mathscr{V}_w, \prec_w)$ with root w and an attaching map μ_w for $(\mathscr{V}_w, \prec_w)$,

(1) a J-holomorphic map $u_{w;\infty} : \Sigma_{\mathscr{V}_w,\mu_w} \longrightarrow X$ with $u_{w;\infty}|_{\mathbb{P}^1_w}(\infty) = u_{v_0}(w)$,

(2) a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, and

(3) for every $v \in \mathscr{V}_w$ and $i \in \mathbb{Z}^+$, $\delta_v \in \mathbb{R}^+$ and a biholomorphic map $\psi_{v;i} : U_{v;i} \longrightarrow B_{\delta_v}(\mu_w(v))$, where $U_{v;i} \subset \mathbb{C}$ is an open subset and $\mu_w(w) \equiv \{v_0\}$,

such that

(a) for every $v \in \mathscr{V}_w$, $U_{v;i} \subset U_{v;i+1}$ for all $i \in \mathbb{Z}^+$, $\mathbb{C} = \bigcup_{i=1}^{\infty} U_{v;i}$, and

$$\mathfrak{m}_v \equiv \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_{p(v);i}; B_\delta(\mu_w(v))\big) = E_g\big(u_{w;\infty}|_{\mathbb{P}^1_v}\big) + \sum_{v' \in S_v(\mathscr{V}_w)} \mathfrak{m}_{v'},$$

where $u_{p(w);i} \equiv u_{v_0;i}$ and $u_{v;i} \equiv u_{p(v);i} \circ \psi_{v;i}$ if $v \in \mathscr{V}_w - \{w\}$,

(b) the sequence $u_{v;i}$ converges to $u_{w;\infty}|_{\mathbb{P}^1_v}$ uniformly in the C^∞-topology on compact subsets of $\mathbb{C} - \mu_w(S_v(\mathscr{V}_w))$,

(c) if $v \in \mathscr{V}_w$ and $u_{w;\infty}|_{\mathbb{P}^1_v}$ is constant, then $|S_v(\mathscr{V}_w)| \geq 2$.

We take

$$\mathscr{V} = \{v_0\} \sqcup \bigsqcup_{w \in S_{v_0}} \mathscr{V}_w, \quad v_0 \prec v \quad \forall\, v \in \mathscr{V}_w,\, w \in S_{v_0}, \quad v \prec v' \quad \forall\, v, v' \in \mathscr{V}_w,\, w \in S_{v_0} \text{ s.t. } v \prec_w v',$$

$$\mu : \mathscr{V} - \{v_0\} \longrightarrow \mathbb{C}, \qquad \mu(v) = \begin{cases} w, & \text{if } w \in S_{v_0}; \\ \mu_w(v), & \text{if } v \in \mathscr{V}_w - \{w\},\ w \in S_{v_0}; \end{cases}$$

$$u_\infty : \Sigma_{\mathscr{V},\mu} \longrightarrow X, \qquad u_\infty(z) = \begin{cases} u_{v_0}(z), & \text{if } z \in \mathbb{P}^1_{v_0}; \\ u_{w;\infty}(z), & \text{if } z \in \mathbb{P}^1_v,\ v \in \mathscr{V}_w,\ w \in S_{v_0}. \end{cases}$$

By (1) in Proposition 5.5 and (1) above, the map u_∞ is well-defined and satisfies the conditions in (1) in Corollary 5.6. By (a)-(c) in Proposition 5.5 and (b)-(c) above, the requirements (a)-(c) of Corollary 5.6 are satisfied as well. □

Proof of Theorem 1.3. Fix a Riemannian metric g_Σ on Σ. For $z \in \Sigma$ and $\delta \in \Sigma$, let $B_\delta(z) \subset \Sigma$ denote the ball of radius δ around z.

By Lemma 5.4, there exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, so that the set

$$S \equiv \Big\{ z \in \Sigma : \lim_{\delta \longrightarrow 0} \sup_{B_\delta(z)} \big|\mathrm{d}u_i\big|_g = \infty \Big\}$$

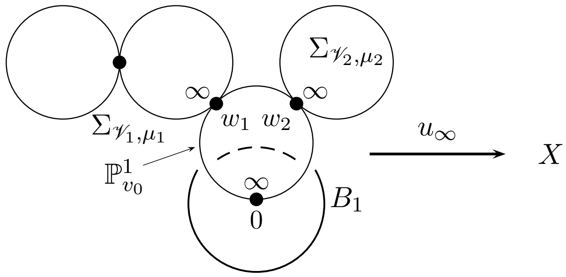

Figure 12: Gromov's limit of a sequence of J-holomorphic maps $u_i\colon B_1 \longrightarrow X$

is finite and the sequence u_i converges to a J-holomorphic map $u\colon \Sigma \longrightarrow X$ C^1-u.c.s. on $\Sigma - S$. In particular, $|\mathrm{d}u_i|_g$ is uniformly bounded on compact subsets of $\Sigma - S$. We can also assume that the limit

$$\mathfrak{m}_z \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z)\big)$$

exists for every $z\in S$. We note that

$$\begin{aligned} E_g(u) + \sum_{z\in S}\mathfrak{m}_z &= \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u;\Sigma - \bigcup_{z\in S}^{\ell} B_\delta(z)\big) + \sum_{z\in S}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z)\big) \\ &= \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g(u_i) = \lim_{i\longrightarrow\infty} E_g(u_i). \end{aligned} \tag{5.26}$$

For each $z \in S$, Corollary 5.6 provides a tree $\Sigma_{\mathscr{V}_z,\mu_z}$ of Riemann spheres $\mathbb{P}^1$ with a distinguished smooth point ∞ and a J-holomorphic map

$$u_{z;\infty}\colon \Sigma_{\mathscr{V}_z,\mu_z} \longrightarrow X \qquad \text{s.t.}\quad u_{z;\infty}(\infty) = u(z) \;\text{ and }\; E_g(u_{z;\infty}) = \mathfrak{m}_z.$$

Combining the last equality with (5.26), we obtain

$$E_g(u) + \sum_{z\in S} E_g(u_{z;\infty}) = \lim_{i\longrightarrow\infty} E_g(u_i)\,.$$

Identifying the distinguished point ∞ of each $\Sigma_{\mathscr{V}_z,\mu_z}$ with $z \in \Sigma$, we obtain a Riemann surface $(\Sigma_\infty, \mathfrak{j}_\infty)$ and a J-holomorphic map $u_\infty\colon \Sigma_\infty \longrightarrow X$ with the desired properties.

By Corollary 5.6(c), the limiting map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ constructed above is stable unless u is a constant map, $\Sigma = \mathbb{P}^1$, and $|S| \le 2$. If u is a constant, then $S \neq \emptyset$ by (5.26). Suppose $\Sigma = \mathbb{P}^1$. Let $h\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a holomorphic automorphism so that the set $h^{-1}(S)\cap B_1$ consists of $0 \in \mathbb{C}$ only. By replacing each u_i with $u_i\circ h$, we can assume that $S\cap B_1(0) = \{0\}$. For each $i\in\mathbb{Z}^+$, let $z_i \in \overline{B_{1/2}}$ be such that

$$M_i \equiv \sup_{\overline{B_{1/2}}}\big|\mathrm{d}u_i\big| = \big|\mathrm{d}_{z_i}u_i\big|, \tag{5.27}$$

with the norms taken with respect to the standard Euclidean metric on $\overline{B_{1/2}} \subset \mathbb{C}$. Define

$$h_i\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1,\;\; h_i(z) = z_i + z/M_i, \qquad \text{and} \qquad v_i = u_i\circ h_i\colon \mathbb{P}^1 \longrightarrow X.$$

In particular, $E_g(v_i) = E_g(u_i)$. Since $S\cap B_1(0) = \{0\}$, $z_i \longrightarrow 0$ and $M_i \longrightarrow \infty$. Along with (5.27), this gives

$$\sup_{\overline{B_1}}\big|\mathrm{d}v_i\big| = \big|\mathrm{d}_0 v_i\big| = 1 \tag{5.28}$$

for all $i\in\mathbb{Z}^+$ sufficiently large. By Lemma 5.4, there exists a subsequence of $\{v_i\}$, still denoted by $\{v_i\}$, so that the set

$$S'\equiv\big\{z\in\mathbb{P}^1\colon \lim_{\delta\longrightarrow 0}\sup_{B_\delta(z)}\big|\mathrm{d}v_i\big|_g=\infty\big\}$$

is finite and the sequence v_i converges to a J-holomorphic map $u\colon\mathbb{P}^1\longrightarrow X$ C^1-u.c.s. on $\mathbb{P}^1-S'$. By (5.28), $S'\cap B_1=\emptyset$. Thus, $|\mathrm{d}_0u|_g=1$ and the map u is not constant. Along with Corollary 5.6(c), this implies that the limiting map $(\Sigma_\infty,\mathfrak{j}_\infty,u_\infty)$ obtained by applying the above construction to the sequence $\{v_i\}$ is a stable J-holomorphic map. □

5.4 An example

We now give an example illustrating Gromov's convergence in a classical setting.

Let $n\in\mathbb{Z}^+$, with $n\geq 2$, and $\mathbb{P}^{n-1}=\mathbb{CP}^{n-1}$. Denote by ℓ the positive generator of $H_2(\mathbb{P}^{n-1};\mathbb{Z})\approx\mathbb{Z}$, i.e. the homology class represented by the standard $\mathbb{P}^1\subset\mathbb{P}^{n-1}$. A degree d map $f\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a continuous map such that $f_*[\mathbb{P}^1]=d\ell$. A holomorphic degree d map $f\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is given by

$$[u,v]\longrightarrow\big[R_1(u,v),\ldots,R_n(u,v)\big]$$

for some degree d homogeneous polynomials $R_1,\ldots,R_d$ on $\mathbb{C}^2$ without a common linear factor. Since the tuple $(\lambda R_1,\ldots,\lambda R_n)$ determines the same map as $(R_1,\ldots,R_n)$ for any $\lambda\in\mathbb{C}^*$, the space of degree d holomorphic maps $f\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a dense open subset of

$$\mathfrak{X}_{n,d}\equiv\big((\mathrm{Sym}^d\mathbb{C}^2)^n-\{0\}\big)/\mathbb{C}^*\approx\mathbb{P}^{(d+1)n-1}\,.$$

Suppose $f_k\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a sequence of holomorphic degree $d\geq 1$ maps and

$$\mathbf{R}_k=\big[R_{k;1},\ldots,R_{k;n}\big]\in\mathfrak{X}_{n,d}$$

are the associated equivalence classes of n-tuples of homogeneous polynomials without a common linear factor. Passing to a subsequence, we can assume that $[\mathbf{R}_k]$ converges to some

$$\mathbf{R}\equiv\big[(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_1,\ldots,(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_n\big]\in\mathfrak{X}_{n,d}\,,\tag{5.29}$$

with $d_1,\ldots,d_m\in\mathbb{Z}^+$ and homogeneous polynomials

$$\mathbf{S}\equiv[S_1,\ldots,S_n]\in\mathfrak{X}_{n,d_0}$$

without a common linear factor and with $d_0\in\mathbb{Z}^{\geq 0}$. By (5.29),

$$d_0+d_1+\ldots+d_m=d.$$

Rescaling $(R_{k;1},\ldots,R_{k;n})$, we can assume that

$$\lim_{k\longrightarrow\infty}R_{k;i}=(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_i\qquad\forall\,i=1,\ldots,n.\tag{5.30}$$

Suppose $z_0\in\mathbb{C}-\{u_1/v_1,\ldots,u_m/v_m\}$. Since the polynomials $S_1,\ldots,S_n$ do not have a common linear factor, $S_{i_0}(z_0,1)\neq 0$ for some $i_0=1,\ldots,n$. This implies that $R_{k;i_0}(z_0,1)\neq 0$ for all k large enough and so

$$\lim_{k\longrightarrow\infty}\frac{R_{k;i}(z,1)}{R_{k;i_0}(z,1)}=\frac{\lim\limits_{k\longrightarrow\infty}R_{k;i}(z,1)}{\lim\limits_{k\longrightarrow\infty}R_{k;i_0}(z,1)}=\frac{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_i(z,1)}{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_{i_0}(z,1)}=\frac{S_i(z,1)}{S_{i_0}(z,1)}$$

for all $i=1,\ldots,n$ and z close to z_0. Furthermore, the convergence is uniform on a neighborhood of z_0. Thus, the sequence f_k C^∞-converges on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$ to the holomorphic degree d_0 map $g\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ determined by $\mathbf{S}$.

Let ω be the Fubini-Study symplectic form on $\mathbb{P}^{n-1}$ normalized so that $\langle\omega,\ell\rangle=1$ and $E(\cdot)$ be the energy of maps into $\mathbb{P}^{n-1}$ with respect to the associated Riemannian metric. For each $\delta>0$ and $j=1,\ldots,m$, denote by $B_\delta([u_j,v_j])$ the ball of radius δ around $[u_j,v_j]$ in $\mathbb{P}^1$ and let

$$\mathbb{P}^1_\delta = \mathbb{P}^1 - \bigcup_{j=1}^m B_\delta([u_j,v_j])\,.$$

For each $j=1,\ldots,m$, let

$$\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big) = \lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty} E\big(f_k;B_\delta([u_j,v_j])\big)\in\mathbb{R}^{\ge0}$$

be the energy sinking into the bubble point $[u_j,v_j]$. By Theorem 1.3, the number $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})$ is the value of ω on some element of $H_2(\mathbb{P}^{n-1};\mathbb{Z})$, i.e. an integer. Below we show that $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$.

Since the sequence f_k C^∞-converges to the degree d_0 map $g\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$,

$$d_0=\langle\omega,d_0\ell\rangle=E(g)=\lim_{\delta\longrightarrow0}E_g\big(g;\mathbb{P}^1_\delta\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;\mathbb{P}^1_\delta\big).$$

Thus,

$$\begin{aligned}\sum_{j=1}^m\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)&=\sum_{j=1}^m\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;B_\delta([u_j,v_j])\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;\bigcup_{j=1}^m B_\delta([u_j,v_j])\big)\\&=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}\big(E_g(f_k)-E_g\big(f_k;\mathbb{P}^1_\delta\big)\big)=d-d_0=d_1+\ldots+d_m\,.\end{aligned}$$

In particular, $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$ if $m=1$, no matter what the "residual" tuple of polynomials $\mathbf{S}$ is. We use this below to establish this energy identity for $m>1$ as well.

By (5.30), for all $k\in\mathbb{Z}^+$ sufficiently large there exist $\lambda_{k;i;j;p}\in\mathbb{C}$ with $i=1,\ldots,n$, $j=1,\ldots,m$, and $p=1,\ldots,d_j$ and tuples

$$\mathbf{S}_k\equiv\big[S_{k;1},\ldots,S_{k;n}\big]\in\mathfrak{X}_{n;d_0}$$

of polynomials without a common linear factor such that

$$\lim_{k\longrightarrow\infty}\mathbf{S}_k=\mathbf{S},\qquad \lim_{k\longrightarrow\infty}\lambda_{k;i;j;p}=1\quad\forall\,i,j,p,$$

$$R_{k;i}(u,v)=\prod_{j=1}^m\prod_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v)\quad\forall\,k,i\,.$$

For each $j_0=1,\ldots,m$, let

$$\mathbf{T}_{j_0}\equiv\big[T_{j_0;1},\ldots,T_{j_0;n}\big]\in\mathfrak{X}_{n;d-d_{j_0}}$$

be a tuple of polynomials without a common linear factor. If in addition, $i=1,\ldots,n$, $\epsilon\in\mathbb{R}$, and $k\in\mathbb{Z}^+$, let

$$\begin{aligned}
S_{i;j_0;\epsilon}(u,v) &\equiv \prod_{j\neq j_0}^{m}(v_ju-u_jv)^{d_j}\cdot S_i(u,v)+\epsilon T_{j_0;i}(u,v)\,, && i=1,\ldots,n,\\
R_{k;i;j_0;\epsilon}(u,v) &\equiv R_{k;i}(u,v)+\epsilon\prod_{p=1}^{d_{j_0}}(v_{j_0}u-\lambda_{k;i;j_0;p}u_{j_0}v)\cdot T_{j_0;i}(u,v), && i=1,\ldots,n.
\end{aligned}$$

The polynomials within each tuple $(S_{i;j_0;\epsilon})_{i=1,\ldots,n}$ and $(R_{k;i;j_0;\epsilon})_{i=1,\ldots,n}$ have no common linear factor for all $\epsilon\in\mathbb{R}^+$ sufficiently small and k sufficiently large (with the conditions on ϵ and k mutually independent). We denote by

$$f_{k;j_0;\epsilon}\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$$

the holomorphic degree d map determined by the tuple

$$\mathbf{R}_{k;j_0;\epsilon}\equiv\big[R_{k;1;j_0;\epsilon},\ldots,R_{k;n;j_0;\epsilon}\big]\,.$$

Since

$$\lim_{k\longrightarrow\infty}\mathbf{R}_{k;j_0;\epsilon}=\big[(v_1u-u_1v)^{d_{j_0}}S_{1;j_0;\epsilon},\ldots,(v_1u-u_1v)^{d_{j_0}}S_{n;j_0;\epsilon}\big]\in\mathfrak{X}_{n;d}$$

and the polynomials $S_{1;j_0;\epsilon},\ldots,S_{n;j_0;\epsilon}$ have no linear factor in common,

$$\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_{k;j_0;\epsilon};B_\delta([u_{j_0},v_{j_0}])\big)\equiv\mathfrak{m}_{[u_{j_0},v_{j_0}]}\big(\{f_{k;j_0;\epsilon}\}\big)=d_{j_0}\tag{5.31}$$

by the $m=1$ case established above.

For $\delta\in\mathbb{R}^+$ sufficiently small, $\epsilon\in\mathbb{R}^+$ sufficiently small, and k sufficiently large,

$$\prod_{j\neq j_0}^{m}\prod_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v)\neq0\qquad\forall\;[u,v]\in B_{2\delta}\big([u_{j_0},v_{j_0}]\big).$$

Thus, the ratios

$$\frac{R_{k;i;j_0;\epsilon}(u,v)}{R_{k;i}(u,v)}=1+\epsilon\frac{T_{j_0;i}(u,v)}{\prod\limits_{j\neq j_0}^{m}\prod\limits_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v)}$$

converge uniformly to 1 on $B_\delta([u_{j_0},v_{j_0}])$ as $\epsilon\longrightarrow0$. Thus, there exists $k^*\in\mathbb{Z}^+$ such that

$$\lim_{\epsilon\longrightarrow0}\sup_{k\ge k^*}\sup_{z\in B_\delta([u_{j_0},v_{j_0}])}\left|\frac{|\mathrm{d}_zf_{k;j_0;\epsilon}|}{|\mathrm{d}_zf_k|}-1\right|=0.$$

It follows that

$$\begin{aligned}
\mathfrak{m}_{[u_{j_0},v_{j_0}]}\big(\{f_k\}\big)&\equiv\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;B_\delta([u_{j_0},v_{j_0}])\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}\lim_{\epsilon\longrightarrow0}E\big(f_{k;j_0;\epsilon};B_\delta([u_{j_0},v_{j_0}])\big)\\
&=\lim_{\epsilon\longrightarrow0}\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_{k;j_0;\epsilon};B_\delta([u_{j_0},v_{j_0}])\big)=\lim_{\epsilon\longrightarrow0}d_{j_0}=d_{j_0};
\end{aligned}$$

the second-to-last equality above holds by (5.31).

Suppose that either $d_0 \geq 1$ or $m \geq 3$. Otherwise, the maps f_k can be reparametrized so that $d_0 \neq 0$; see the last paragraph of the proof of Theorem 1.3 at the end of Section 5.3. By Theorem 1.3 and the above, a subsequence of $\{f_k\}$ converges to the equivalence class of a holomorphic degree d_0 map $f : \Sigma \longrightarrow \mathbb{P}^{n-1}$, where Σ is a nodal Riemann surface consisting of the component $\Sigma_0 = \mathbb{P}^1$ corresponding to the original $\mathbb{P}^1$ and finitely many trees of $\mathbb{P}^1$'s coming off from Σ_0. The maps on the components in the trees are defined only up reparametrization of the domain. By the above, $f|_{\Sigma_0}$ is the map g determined by the "relatively prime part" $\mathbf{S}$ of the limit $\mathbf{R}$ of the tuples of polynomials. The trees are attached at the roots $[u_j, v_j]$ of the common linear factors $v_j u - u_j v$ of the polynomials in $\mathbf{R}$; the degree of the restriction of f to each tree is the power of the multiplicity d_j of the corresponding common linear factor.

The same reasoning as above applies to the sequence of maps

$$\big(\mathrm{id}_{\mathbb{P}^1}, f_k\big) : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^{n-1},$$

but the condition that either $d_0 \geq 1$ or $m \geq 3$ is no longer necessary for the analogue of the conclusion in the previous paragraph. This implies that the map

$$\mathfrak{M}_{0,0}\big(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d)\big) \longrightarrow \mathfrak{X}_{n,d}, \qquad [f,g] \longrightarrow \big[g \circ f^{-1}\big],$$

from the subspace of $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d))$ corresponding to maps from $\mathbb{P}^1$ extends to a continuous surjective map

$$\overline{\mathfrak{M}}_{0,0}\big(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d)\big) \longrightarrow \mathfrak{X}_{n,d}\,. \tag{5.32}$$

In particular, Gromov's moduli spaces refine classical compactifications of spaces of holomorphic maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}$. On the other hand, the former are defined for arbitrary almost Kahler manifolds, which makes them naturally suited for applying topological methods. The right-hand side of (5.32) is known as the linear sigma model in the Mirror Symmetry literature. The morphism (5.32) plays a prominent role in the proof of mirror symmetry for the genus 0 Gromov-Witten invariants in [14] and [24]; see [20, Section 30.2].

5.5 Convergent sequences and topologies

We now discuss topologies induced by collections of "convergent sequences", in the spirit of [29, Section 5.6]. Such a sequence in a set $\mathfrak{M}$ can be identified with a tuple

$$\big(x, (x_k)_{k \in \mathbb{Z}^+}\big) \in \mathfrak{M} \times \mathfrak{M}^{\mathbb{Z}^+}.$$

If a subset A of a topological space $(\mathfrak{M}, \mathcal{T})$ is closed, i.e. is the complement of an element of $\mathcal{T}$, then a limit of every sequence in A convergent in $\mathfrak{M}$ is contained in A. The converse holds if the topological space $(\mathfrak{M}, \mathcal{T})$ is $T1$, i.e. the one-point subsets of $\mathfrak{M}$ are closed, and first countable. This underpins the discussion below.

For a topology $\mathcal{T}$ on a set $\mathfrak{M}$, let

$$\mathscr{C}(\mathcal{T}) = \big\{\big(x, (x_k)_{k \in \mathbb{Z}^+}\big) \in \mathfrak{M} \times \mathfrak{M}^{\mathbb{Z}^+} \colon \forall \mathcal{U} \in \mathcal{T} \text{ with } x \in U,\ \exists\, N \in \mathbb{Z}^+ \text{ s.t. } x_k \in U\ \forall\, k \geq N\big\}.$$

It is immediate that the subset $\mathscr{C} \equiv \mathscr{C}(\mathcal{T})$ of $\mathfrak{M} \times \mathfrak{M}^{\mathbb{Z}^+}$ satisfies

($\mathscr{C}1$) if $x \in X$ and $x_k = x$ for all $k \in \mathbb{Z}^+$, then $(x, (x_k)_{k \in \mathbb{Z}^+}) \in \mathscr{C}$;

($\mathscr{C}2$) if $(x,(x_k)_{k\in\mathbb{Z}^+})\in\mathscr{C}$ and $\iota\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ is a strictly increasing function, then $(x,(x_{\iota(k)})_{k\in\mathbb{Z}^+})\in\mathscr{C}$;

($\mathscr{C}3$) if $(x,(x_k)_{k\in\mathbb{Z}^+})\in\mathfrak{M}\times\mathfrak{M}^{\mathbb{Z}^+}$ and for every strictly increasing function $\iota:\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ there exists a strictly increasing function $\iota':\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ such that $(x,(x_{\iota(\iota'(k))})_{k\in\mathbb{Z}^+})\in\mathscr{C}$, then $(x,(x_k)_{k\in\mathbb{Z}^+})\in\mathscr{C}$.

If $(\mathfrak{M},\mathcal{T})$ is a first countable topological space, then $\mathscr{C}\equiv\mathscr{C}(\mathcal{T})$ satisfies

($\mathscr{C}4$) if $(x,(x_k)_{k\in\mathbb{Z}^+})\in\mathscr{C}$ and $(x,(x_{k,n})_{k\in\mathbb{Z}^+})\in\mathscr{C}$ for every $k\in\mathbb{Z}^+$, then there exist functions $\iota_1,\iota_2\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ so that $(x_0,(x_{\iota_1(k),\iota_2(k)})_{k\in\mathbb{Z}^+})\in\mathscr{C}$.

If $(\mathfrak{M},\mathcal{T})$ is a Hausdorff topological space, then $\mathscr{C}\equiv\mathscr{C}(\mathcal{T})$ satisfies

($\mathscr{C}5$) if $(x,(x_k)_{k\in\mathbb{Z}^+})\in\mathscr{C}$ and $(x',(x_k)_{k\in\mathbb{Z}^+})\in\mathscr{C}$, then $x=x'$.

For a subset $\mathscr{C}$ of $\mathfrak{M}\times\mathfrak{M}^{\mathbb{Z}^+}$, let

$$\begin{aligned}\mathcal{T}_{\rm op}(\mathscr{C})&=\big\{\mathcal{U}\subset\mathfrak{M}\colon\forall\big(x,(x_k)_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}\text{ with }x\in U,\ \exists\,N\in\mathbb{Z}^+\text{ s.t. }x_k\in U\ \forall\,k\ge N\big\},\\ \mathcal{T}_{\rm cl}(\mathscr{C})&=\big\{\mathcal{U}\subset\mathfrak{M}\colon\forall\big(x,(x_k)_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}\text{ with }x_k\notin U\ \forall\,k\in\mathbb{Z}^+,\ x\notin U\big\}\\ &=\big\{\mathcal{U}\subset\mathfrak{M}\colon\forall\big(x,(x_k)_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}\text{ with }x\in U,\ \exists\,k\in\mathbb{Z}^+\text{ s.t. }x_k\in U\big\}\supset\mathcal{T}_{\rm op}(\mathscr{C}).\end{aligned}$$

It is immediate that $\mathcal{T}_{\rm op}(\mathscr{C})$ is a topology on $\mathfrak{M}$ so that

$$\mathscr{C}\subset\mathscr{C}\big(\mathcal{T}_{\rm op}(\mathscr{C})\big)\qquad\text{and}\qquad\mathcal{T}_{\rm op}(\mathscr{C})=\mathcal{T}_{\rm op}\big(\mathscr{C}(\mathcal{T}_{\rm op}(\mathscr{C}))\big).\tag{5.33}$$

If $\mathscr{C}$ satisfies ($\mathscr{C}1$), then $\mathcal{T}_{\rm op}(\mathscr{C})$ is a $T1$ topology. The collection $\mathcal{T}_{\rm cl}(\mathscr{C})$ may not be a topology on $\mathfrak{M}$.

Exercise 5.7. Let $\mathscr{C}\subset\mathfrak{M}\times\mathfrak{M}^{\mathbb{Z}^+}$. Show that

(1) if $\mathscr{C}$ satisfies ($\mathscr{C}2$), then $\mathcal{T}_{\rm op}(\mathscr{C})=\mathcal{T}_{\rm cl}(\mathscr{C})$;

(2) if $\mathscr{C}$ satisfies ($\mathscr{C}1$), ($\mathscr{C}2$), and ($\mathscr{C}4$), then the closure of a subset $A\subset\mathfrak{M}$ with respect to $\mathcal{T}_{\rm op}(\mathscr{C})$ is given by
$$\overline{A}=\big\{x\in\mathfrak{M}\colon\exists(x_k)_{k\in\mathbb{Z}^+}\in A^{\mathbb{Z}^+}\text{ s.t. }\big(x,(x_k)_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}\big\}.$$

Lemma 5.8 ([29, Lemma 5.6.4])**.** *Let $\mathscr{C}\subset\mathfrak{M}\times\mathfrak{M}^{\mathbb{Z}^+}$. If $\mathscr{C}$ satisfies ($\mathscr{C}1$)-($\mathscr{C}5$), then $\mathscr{C}=\mathscr{C}(\mathcal{T}_{\rm op}(\mathscr{C}))$.*

Proof. In light of the first statement in (5.33), it remains to show that $\mathscr{C}\subset\mathscr{C}(\mathcal{T}(\mathscr{C}))$. Let

$$\big(x,(x_k)_{k\in\mathbb{Z}^+}\big)\in\mathfrak{M}\times\mathfrak{M}^{\mathbb{Z}^+}-\mathscr{C}.$$

By ($\mathscr{C}3$), there exists a strictly increasing function $\iota\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ so that

$$\big(x,(x_{\iota(\iota'(k))})_{k\in\mathbb{Z}^+}\big)\notin\mathscr{C}$$

for every strictly increasing function $\iota'\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$. Along with ($\mathscr{C}1$), this implies that there exists $N\in\mathbb{Z}^+$ so that $x_{\iota(k)}\neq x$ for all $k\ge N$. Let

$$A=\big\{x_{\iota(k)}\colon k\ge N\big\}\subset\mathfrak{M}$$

and $\overline{A}\subset A$ be the closure of A with respect to $\mathcal{T}_{\mathrm{op}}(\mathscr{C})$. We show below that $x\notin\overline{A}$. Thus,

$$U\equiv\mathfrak{M}-\overline{A}\in\mathcal{T}_{\mathrm{op}}(A), \qquad x\in U, \qquad \text{and} \qquad x_{\iota(k)}\notin U \quad \forall\, k\geq N.$$

This means that $(x,(x_k)_{k\in\mathbb{Z}^+})\notin\mathscr{C}(\mathcal{T}_{\mathrm{op}}(\mathscr{C}))$.

Suppose $x\in\overline{A}$. By Exercise 5.7(2), there exists a function $\iota'\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ so that

$$\iota'(k)\geq N \quad \forall\, k\in\mathbb{Z}^+ \qquad \text{and} \qquad \big(x,(x_{\iota(\iota'(k))})_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}.$$

If ι' were unbounded, there would exist a strictly increasing function $\iota''\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ so that the composition $\iota'\circ\iota''\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ is also strictly increasing. By ($\mathscr{C}$2),

$$\big(x,(x_{\iota(\iota'(\iota''(k)))})_{k\in\mathbb{Z}^+}\big)\in\mathscr{C}\,;$$

this would contradict the assumption on ι. Thus, ι' is bounded and there exists a strictly increasing function $\iota''\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ so that the composition $\iota'\circ\iota''\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ is a constant function; let $m\geq N$ be its value. By ($\mathscr{C}$2) and ($\mathscr{C}$1),

$$\big(x,(x_{\iota(\iota'(\iota''(k)))})_{k\in\mathbb{Z}^+}\big)\in\mathscr{C} \qquad \text{and} \qquad \big(x_{\iota(m)},(x_{\iota(\iota'(\iota''(k)))})_{k\in\mathbb{Z}^+}\big)\in\mathscr{C},$$

respectively. Along with ($\mathscr{C}$5), this implies that $x=x_{\iota(m)}$. However, this contradicts the assumptions on ι and N. Thus, $x\notin\overline{A}$. $\square$

The conclusion of Lemma 5.8 and ($\mathscr{C}$5) imply the uniqueness of the limits of convergent sequences in $(\mathfrak{M},\mathcal{T}_{\mathrm{op}}(\mathscr{C}))$. The uniqueness of the limits of convergent sequences in a first countable topological space implies Hausdorffness. However, the topological space $(\mathfrak{M},\mathcal{T}_{\mathrm{op}}(\mathscr{C}))$ need not be first countable even if $\mathscr{C}$ satisfies ($\mathscr{C}$1)-($\mathscr{C}$5).

Exercise 5.9. Let $(\mathfrak{M},\mathcal{T})$ be the topological space obtained by identifying the origins of countably many one-dimensional vector spaces. Show that

(1) the topological space $(\mathfrak{M},\mathcal{T})$ is Hausdorff, but not first countable;

(2) the subset $\mathscr{C}\equiv\mathscr{C}(\mathcal{T})$ of $\mathfrak{M}\times\mathfrak{M}^+$ satisfies ($\mathscr{C}$1)-($\mathscr{C}$5) and $\mathcal{T}_{\mathrm{op}}(\mathscr{C}(\mathcal{T}))=\mathcal{T}$.

6 Proof of Theorem 1.5

6.1 Convergence for marked maps

We first modify Proposition 5.5 and Corollary 5.6 by treating $0\in B_1$ as a marked point. A sequence of maps as in Corollary 5.3 then gives rise to a continuous map from a tree of spheres $\Sigma_{\mathscr{V},\mu}$ attached at $0\in B_1$ with an additional marked point

$$z_1=0_{v^*}\in\mathbb{C}-\mu\big(S_{v^*}(\mathscr{V})\big)\subset\mathbb{P}^1_{v^*} \tag{6.1}$$

for some $v^*\in\mathscr{V}$.

Proposition 6.1. *Let (X,J,g), u_i, u, and $\mathfrak{m}>0$ be as in Proposition 5.5. There exist $(\mathscr{V},\prec,v_0,\mu)$, u_∞, a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, δ_v, $U_{v;i}$, and $\psi_{v;i}$ as in (0)-(3) of Corollary 5.6 and z_1 as in (6.1) satisfying (b) and (a) of Corollary 5.6 along with* *(c′) if $v\in\mathscr{V}$ and $u_\infty|_{\mathbb{P}^1_v}$ is constant, then either $|S_v(\mathscr{V})|\ge 2$ or $S_v(\mathscr{V})\neq\emptyset$ and $z_1\in\mathbb{P}^1_v$;*

(d′) if $v\preceq v^*$, then $\mu(v)=\psi_{v;i}(0)=0$ for all $i\in\mathbb{Z}^+$.

Proof. We begin by modifying the conclusion of Proposition 5.5. We show that there exist u_{v_0}, S_{v_0}, a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, U_i, and ψ_i as in (1)-(3) in the statement of this proposition satisfying (a) and (b) there, $\psi_i(0)=0$, and

(d′) if u_{v_0} is constant, then $|S_{v_0}\cup\{0\}|\ge 2$.

Let $z_i\in B_1$ and $\delta_i\in\mathbb{R}^+$ be as in the proof of Proposition 5.5. By passing to a subsequence of $\{u_i\}$, we can assume that z_i/δ_i converges to some $z_{v_0}\in\mathbb{P}^1$.

Case 1: $z_{v_0}\in\mathbb{C}$. We can then assume that δ_i is decreasing. Define

$$\psi_i\colon U_i\equiv B_{1/2\delta_i}\longrightarrow B_{1/2}\qquad\text{by}\qquad \psi_i(w)=\delta_i w\,.$$

Since $\delta_i\longrightarrow 0$, (b) in Proposition 5.5 holds. As in its proof, the rescaled maps $u_{v_0;i}$ in (5.19) are J-holomorphic. After passing to a subsequence of $\{u_i\}$, we can again assume that the set (5.20) is finite, the sequence $u_{v_0;i}$ converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}-S_{v_0}$ to a J-holomorphic map $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$, the limit (5.16) exists for every $w\in S_{v_0}$, and (5.22) holds. Since $z_i/\delta_i\longrightarrow z_{v_0}$, u_{v_0} differs from the map u_{v_0} in the proof of Proposition 5.5 by the composition with the shift by $-z_{v_0}$. Thus, the properties (a)-(c) in the statement of the proposition continue to hold.

Case 2: $z_{v_0}=\infty$. We can then assume that $z_i\neq 0$ for all $i\in\mathbb{Z}^+$ and $|z_i|$ is decreasing. Define

$$\psi_i\colon U_i\equiv B_{1/2|z_i|}\longrightarrow B_{1/2}\qquad\text{by}\qquad \psi_i(w)=z_i w\,.$$

Since $z_i\longrightarrow 0$, (b) in Proposition 5.5 holds. As in its proof, the rescaled maps $u_{v_0;i}$ in (5.19) are J-holomorphic. After passing to a subsequence of $\{u_i\}$, we can again assume that the set (5.20) is finite, the sequence $u_{v_0;i}$ converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}-S_{v_0}$ to a J-holomorphic map $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$, the limit (5.16) exists for every $w\in S_{v_0}$, and (5.22) holds. Since $z_i/\delta_i\longrightarrow\infty$,

$$\begin{aligned}\mathfrak{m}_1\equiv\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i};B_\delta(1)\big)&=\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{\delta|z_i|}(z_i)\big)\\&\ge\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{R\delta_i}(z_i)\big)=\mathfrak{m};\end{aligned}$$

the last equality holds by (5.3). Along with (5.22), this implies that $S_{v_0}=\{1\}$ and u_{v_0} is a constant map. By the same reasoning as in the proof of Proposition 5.5, we obtain $u(0)=u_{v_0}(\infty)$.

We now take $\hbar$ and N as in the proof of Corollary 5.6 and u_{v_0}, S_{v_0}, $\psi_{v_0;i}=\psi_i$ and $U_{v_0;i}\equiv U_i$ as constructed just above. We then proceed again by induction on N, assuming Corollary 5.6. For each $w\in S_{v_0}-\{0\}$, we take $(\mathscr{V}_w,\prec_w,\mu_w)$, $u_{w;\infty}$, a subsequence of $\{u_i\}$, δ_v, $\psi_{v;i}$, and $U_{v;i}$ exactly as in the proof of Corollary 5.6. If $0\in S_{v_0}$, we take $(\mathscr{V}_0,\prec_0,\mu_0)$, $u_{0;\infty}$, a subsequence of $\{u_i\}$, δ_v, $\psi_{v;i}$, and $U_{v;i}$ as provided by Proposition 6.1 and the inductive assumption. We then combine these collections as at the end of the proof of Corollary 5.6 to conclude the inductive step of the proof. □

We also include a modification of Proposition 5.5 and Corollary 5.6 for S^*-marked maps. A sequence of maps as in Corollary 5.3 then gives rise to a continuous map from a tree of spheres $\Sigma_{\mathscr{V},\mu}$ attached at $0\in B_1$ with marked points

$$z_s \in \mathbb{C}-\mu\big(S_{v_s}(\mathscr{V})\big) \subset \mathbb{P}^1_{v_s} \tag{6.2}$$

for some $v_z\in\mathscr{V}$.

Proposition 6.2. *Let (X,J,g), u_i, u, and $\mathfrak{m}$ be as in Proposition 5.5, S^* be a finite nonempty set so that either $|S^*|\geq 2$ or $\mathfrak{m}>0$, and $z_{i;s}\in B_1$ be a sequence of points for each $s\in S^*$ converging to 0 so that $z_{i;s}\neq z_{i;s'}$ for all $s\neq s'$. There exist $(\mathscr{V},\prec,v_0,\mu)$, u_∞, a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, δ_v, $U_{v;i}$, and $\psi_{v;i}$ as in (0)-(3) of Corollary 5.6 and $(v_s,z_s)\in\mathscr{V}\times\mathbb{C}$ for each $s\in S^*$ with $z_s\notin\mu(S_v(\mathscr{V}))$ satisfying (b) and (a) of Corollary 5.6, $(v_s,z_s)\neq(v_{s'},z_{s'})$ for all $s\neq s'$, and*

- *(c′) if $v\in\mathscr{V}$ and $u_\infty|_{\mathbb{P}^1_v}$ is constant, then $|S_v(\mathscr{V})|+|\{s\in S^*: v_s=v\}|\geq 2$;*
- *(d′) $z_{p(v);i;s}\in B_{\delta_v}(\mu(v))$ for all $v\preceq v_s$, $i\in\mathbb{Z}^+$, and $s\in S^*$, where $z_{v_0;i;s}=z_{i;s}$ and $z_{v';i;s}=\psi^{-1}_{v';i}(z_{p(v');i;s})$ if $v_0\prec v'\leq v_s$;*
- *(e′) for every $s\in S^*$, the sequence $z_{v_s;i;s}$ converges to z_s.*

Proof. We first show that there exist u_{v_0}, S_{v_0}, a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, U_i, and ψ_i as in (1)-(3) in the statement of Proposition 5.5 and $z_{v_0;s}\in\mathbb{C}$ for each $s\in S$ satisfying (a) and (b) there and

- (d′) if u_{v_0} is constant, then $|S_{v_0}\cup\{z_{v_0;s}: s\in S^*\}|\geq 2$;
- (e′) $z_{i;s}\in B_{1/2}$ for all $i\in\mathbb{Z}^+$ and $s\in S^*$;
- (f′) for every $s\in S^*$, the sequence $z_{v_0;i;s}\equiv\psi_i^{-1}(z_{i;s})$ converges to $z_{v_0;s}$.

If $\mathfrak{m}\neq 0$, let $z_i\in B_1$ and $\delta_i\in\mathbb{R}^+$ be as in the proof of Proposition 5.5; otherwise, let $z_i=0$ and $\delta_i=1$. By passing to a subsequence of $\{u_i\}$, we can assume that (e′) above holds and the sequence $(z_{i;s}-z_i)/\delta_i$ converges to some $z'_{v_0;s}\in\mathbb{P}^1$ for every $s\in S^*$.

Case 1: $\mathfrak{m}\neq 0$ and $z'_{v_0;s}\in\mathbb{C}$ for all $s\in S$. We then take U_i, ψ_i, a subsequence of $\{u_i\}$, u_{v_0} and S_{v_0} as in the proof of Proposition 5.5 and $z_{v_0;s}=z'_{v_0;s}$.

Case 2: $\mathfrak{m}\neq 0$ and $z'_{v_0;s}=\infty$ for some $s\in S$. By passing to a subsequence of $\{u_i\}$, we can assume that the sequence $(z_i-z_{i;s})/(z_i-z_{i;s^*})$ converges to some $z_{v_0;s,s^*}\in\mathbb{P}^1$ for all $s\neq s^*$. Let $s^*\in S$ be such that $z_{v_0;s,s^*}\in\mathbb{C}$ for every $s\in S$. Thus, $z'_{v_0;s^*}=\infty$. We can then assume that $z_{i;s^*}\neq z_i$ for all $i\in\mathbb{Z}^+$. Define

$$\psi_i\colon U_i\equiv\big\{w\in\mathbb{C}: z_{i;s^*}+(z_i-z_{i;s^*})w\in B_{1/2}\big\}\longrightarrow B_{1/2} \quad\text{by}\quad \psi_i(w)=z_{i;s^*}+(z_i-z_{i;s^*})w\,.$$

By passing to a subsequence of $\{u_i\}$, we can assume that (b) in Proposition 5.5 holds. Furthermore,

$$z_{v_0;i;s}\equiv\psi_i^{-1}(z_{i;s})=1-\frac{z_i-z_{i;s}}{z_i-z_{i;s^*}}\longrightarrow z_{v_0;s}\equiv 1-z_{v_0;s,s^*}\in\mathbb{C}$$

for all $s\in S^*$. As in the proof of Proposition 5.5, the rescaled maps $u_{v_0;i}$ in (5.19) are J-holomorphic. After passing to a subsequence of $\{u_i\}$, we can again assume that the set (5.20) is finite, the sequence

$u_{v0;i}$ converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}-S_{v_0}$ to a J-holomorphic map $u_{v_0}\colon \mathbb{P}^1 \longrightarrow X$, the limit (5.16) exists for every $w\in S_{v_0}$, and (5.22) holds. Since $(z_i-z_{i;s^*})/\delta_i \longrightarrow \infty$,

$$\begin{aligned}\mathfrak{m}_1 \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_{v0;i};B_\delta(1)\big) &= \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i;B_{\delta|z_i-z_{i;s^*}|}(z_i)\big)\\ &\ge \lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u_i;B_{R\delta_i}(z_i)\big)=\mathfrak{m};\end{aligned}$$

the last equality holds by (5.3). Along with (5.22), this implies that $S_{v_0}=\{1\}$ and u_{v_0} is a constant map. By the same reasoning as in the proof of Proposition 5.5, we obtain $u(0)=u_{v_0}(\infty)$.

Case 3: $\mathfrak{m}=0$ and thus $|S|\ge 2$. Fix $s_1\in S$. By passing to a subsequence of $\{u_i\}$, we can assume that the sequences $(z_{i;s}-z_{i;s_1})/(z_{i;s'}-z_{i;s_1})$ converge to some $z_{v0;s,s'}\in\mathbb{P}^1$ for all $s,s'\neq s_1$. Let $s_2\in S-\{s_1\}$ be such that $z_{v0;s,s_2}\in\mathbb{C}$ for all $s\in S$. Define

$$\psi_i\colon U_i\equiv\big\{w\in\mathbb{C}\colon z_{i;s_1}+(z_{i;s_2}-z_{i;s_1})w\in B_{1/2}\big\}\longrightarrow B_{1/2}\quad\text{by}\quad \psi_i(w)=z_{i;s_1}+(z_{i;s_2}-z_{i;s_1})w\,.$$

By passing to a subsequence of $\{u_i\}$, we can assume that (b) in Proposition 5.5 holds. Furthermore,

$$z_{v0;i;s}\equiv\psi_i^{-1}(z_{i;s})=\frac{z_{i;s}-z_{i;s_1}}{z_{i;s_2}-z_{i;s_1}}\longrightarrow z_{v0;s}\equiv z_{v0;s,s_2}\in\mathbb{C}$$

for all $s\in S^*$. By the reasoning in Case 2 and the assumption $\mathfrak{m}=0$, the set (5.20) is empty in this case and the sequence $u_{v0;i}$ converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}$ to a constant map $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$ with value $u(0)$. The set $\{z_{v0;s}\colon s\in S^*\}$ contains $z_{v0;s_1}=0$ and $z_{v0;s_2}=1$.

We now take $\hbar$ and N as in the proof of Corollary 5.6, u_{v_0}, S_{v_0}, $\psi_{v0;i}=\psi_i$, $U_{v0;i}\equiv U_i$, and $z_{v0;s}$ as constructed just above, and

$$S'_{v_0}=S_{v_0}\cup\big\{w\in\mathbb{C}\colon \big|\{s\in S\colon z_{v0;s}=w\}\big|\ge 2\big\}.$$

We then proceed by induction on $N+|S^*|$, assuming Corollary 5.6. For each $w\in S'_{v_0}$, let

$$S^*_w=\big\{s\in S\colon z_{v0;s}=w\big\}.$$

We take $(\mathscr{V}_w,\prec_w,\mu_w)$, $u_{w;\infty}$, a subsequence of $\{u_i\}$, δ_v, $\psi_{v;i}$, $U_{v;i}$, and $z_{w;s}$ with $s\in S^*_w$ as in the proof of Corollary 5.6 if $S^*_w=\emptyset$ and as provided by Proposition 6.2 and the inductive assumption with $S^*=S^*_w$ if $S^*_w\neq\emptyset$. We then combine these collections as at the end of the proof of Corollary 5.6 to conclude the inductive step of the proof. □

6.2 Bubbling on thin necks

For $\delta_x,\delta_y\in\mathbb{R}^+$ and $\lambda\in\mathbb{C}$, let

$$A_{\delta_x,\delta_y}(\lambda)=\big\{(x,y)\in\mathbb{C}^2\colon xy=\lambda,\ |x|<\delta_x,|y|<\delta_y\big\}.$$

The space $A_{\delta_x,\delta_y}(0)$ is the wedge of the disks

$$B^x_{\delta_x}\equiv\big\{(x,0)\in\mathbb{C}^2\colon |x|<\delta_x\big\}\qquad\text{and}\qquad B^y_{\delta_y}\equiv\big\{(0,y)\in\mathbb{C}^2\colon |y|<\delta_y\big\}$$

identified at $0_x\in B^x_{\delta_x}$ and $0_y\in B^y_{\delta_y}$. Let $A^*_{\delta_x,\delta_y}(0)\subset A_{\delta_x,\delta_y}(0)$ be the complement of the resulting node. If $\lambda\neq 0$ and $\delta_x\delta_y>|\lambda|$, $A_{\delta_x,\delta_y}(\lambda)$ is the annulus with the ratio of the radii equal $\delta_x\delta_y/|\lambda|$. In such a case, we identify $A_{\delta_x,\delta_y}(\lambda)$ with the annuli $B^x_{\delta_x}-\overline{B}^x_{|\lambda_i|/\delta_y}$ and $B^y_{\delta_y}-\overline{B}^y_{|\lambda_i|/\delta_x}$ via the projections

$$\pi_{\lambda;x},\pi_{\lambda;y}\colon A_{\delta_x,\delta_y}(\lambda)\longrightarrow B^x_{\delta_x},B^y_{\delta_y},\qquad \pi_x(x,y)=(x,0)\quad \pi_y(x,y)=(0,y).$$

For maps $u_x\colon B^*_{\delta_x}\longrightarrow X$ and $u_y\colon B^*_{\delta_y}\longrightarrow X$, define

$$u_x\cup u_y\colon A^*_{\delta_x,\delta_y}(0)\longrightarrow X, \qquad u_x\cup u_y(z)=\begin{cases} u_x(x), & \text{if } z=(x,0);\\ u_y(y), & \text{if } z=(0,y).\end{cases}$$

We say that a sequence $u_i\colon A_{\delta_x,\delta_y}(\lambda_i)\longrightarrow X$ of C^ℓ-maps into a smooth manifold X converges to a C^ℓ-map $u\colon A^*_{\delta_x,\delta_y}(0)\longrightarrow X$ C^ℓ-u.c.s. if the sequence λ_i converges to 0 in $\mathbb{C}$ and the sequences

$$u_i\circ\pi^{-1}_{\lambda_i;x}\colon B^x_{\delta_x}-\overline{B}^x_{|\lambda_i|/\delta_y}\longrightarrow X \qquad\text{and}\qquad u_i\circ\pi^{-1}_{\lambda_i;y}\colon B^y_{\delta_y}-\overline{B}^y_{|\lambda_i|/\delta_x}\longrightarrow X$$

of smooth maps converge to smooth maps $u|_{B^x_{\delta_x}-\{0_x\}}$ and $u|_{B^y_{\delta_y}-\{0_y\}}$, respectively, C^ℓ-u.c.s.

We now obtain analogues of the statements of Section 5.2 for sequences of maps from the thin necks $A_{\delta_x,\delta_y}(\lambda)$. Except as noted, they follow by the same reasoning as the corresponding statements (if any) in Section 5.2.

Lemma 6.3. *Let (X,J,g) be an almost complex Riemannian manifold. Suppose $\delta_0\in\mathbb{R}^+$, and $u_i\colon A_{\delta_0,\delta_0}(\lambda_i)\longrightarrow X$ is a sequence of C^1-maps converging to a C^1-map $u\colon A^*_{\delta_0,\delta_0}(0)\longrightarrow X$ C^1-u.c.s. so that $E_g(u)<\infty$ and the limit*

$$\mathfrak{m}\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{\delta,\delta}(\lambda_i)\big) \tag{6.3}$$

exists.

(1) The limit $\mathfrak{m}(\delta_x,\delta_y)\equiv\lim_{i\longrightarrow\infty}E_g(u_i;A_{\delta_x,\delta_y}(\lambda_i))$ exists for all $\delta_x,\delta_y\in(0,\delta_0)$ and is a continuous, non-decreasing function of δ_x,δ_y.

(3) For every $\mu\in(0,\mathfrak{m})$ and $i\in\mathbb{Z}^+$ sufficiently large, there exists a unique $\delta_i(\mu)\in(0,\delta_0)$ such that

$$E_g\big(u_i;A_{\delta_i(\mu),\delta_i(\mu)}(\lambda_i)\big)=\mu.$$

For all $\mu\in(0,\mathfrak{m})$, $\mu'\in(0,\mu)$, and $i\in\mathbb{Z}^+$ sufficiently large, there exists a unique $\delta_i(\mu,\mu')\in(0,\delta_0)$ such that

$$E_g\big(u_i;A_{\delta_i(\mu,\mu'),\delta_i(\mu)}(\lambda_i)\big)=\mu'.$$

(4) If X is a compact, $\mathfrak{m}=0$, and the maps u_i are J-holomorphic, then u extends to a J-holomorphic map $A_{\delta_0,\delta_0}(0)\longrightarrow X$.

Proof. If the maps u_i are J-holomorphic, so are the maps $u|_{B^x_1-\{0\}}$ and $u|_{B^y_1-\{0\}}$. Since $E_g(u)<\infty$, they extend to J-holomorphic maps

$$\widetilde{u}_x\colon B^x_1\longrightarrow X \qquad\text{and}\qquad \widetilde{u}_y\colon B^y_1\longrightarrow X$$

by Proposition 5.1. The substance of (4) is that $\widetilde{u}_x(0_x)=\widetilde{u}_y(0_y)$ if $\mathfrak{m}=0$.

Suppose $\mathfrak{m}=0$. Let $\hbar_{J,g}$ be as in Corollary 4.17 and $\delta\in(0,\delta_0)$ be such that $\mathfrak{m}(\delta,\delta)<\hbar_{J,g}$. Since $E_g(u_i;A_{\delta,\delta}(\lambda_i))<\hbar_{J,g}$ for all i sufficiently large, Corollary 4.17 applies with (R_1,R_2,R) replaced by $(\ln|\lambda_i|-\ln\delta,\ln\delta,\ln\delta-\ln r)$ and u replaced by the J-holomorphic map

$$v_i\colon(R_1,R_2)\times S^1\longrightarrow X, \qquad v_i\big(s,\mathrm{e}^{\mathrm{i}\theta}\big)=u_i\big(\mathrm{e}^{s+\mathrm{i}\theta},\lambda_i\mathrm{e}^{-s-\mathrm{i}\theta}\big).$$

By the second statement of this corollary,

$$\mathrm{diam}_g\big(u_i(A_{r,r}(\lambda_i))\big) \le C_{J,g}\sqrt{r}\hbar_{J,g} \qquad \forall\, r\in(0,\delta/9),\ i>i(\delta).$$

Since the maps u_i converge to u C^0-u.c.s. and the maps $\widetilde{u}_x$ and $\widetilde{u}_y$ are continuous,

$$\begin{aligned} d_g\big(\widetilde{u}_x(0_x),\widetilde{u}_y(0_y)\big) = \lim_{r\longrightarrow 0} d_g\big(u(r,0),u(0,r)\big) &= \lim_{r\longrightarrow 0}\lim_{i\longrightarrow\infty} d_g\big(u_i(r,\lambda_i/r),u_i(\lambda_i/r,r)\big)\\ &\le \lim_{r\longrightarrow 0}\lim_{i\longrightarrow\infty}\mathrm{diam}_g\big(u_i(A_{r,r}(\lambda_i))\big). \end{aligned}$$

Combining the last two equations, we obtain $\widetilde{u}_x(0_x)=\widetilde{u}_y(0_y)$. $\square$

Corollary 6.4. *If (X,J,g) is a compact Riemannian almost complex manifold, there exists $\hbar_{J,g}\in\mathbb{R}^+$ with the following properties. If $\delta_0\in\mathbb{R}^+$ and $u_i\colon A_{\delta_0,\delta_0}(\lambda_i)\longrightarrow X$ is a sequence of J-holomorphic maps converging to a C^1-map $u\colon A^*_{\delta_0,\delta_0}(0)\longrightarrow X$ C^1-u.c.s. so that $E_g(u)<\infty$ and the limit (6.3) exists and is not zero, then*

(1) $\mathfrak{m}\ge\hbar_{J,g}$*;*

(2) for all $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$ and sequences $\delta_{x;i},\delta_{y;i}\in(0,\delta_0)$ converging to 0 with $E_g(u_i;A_{\delta_{x;i},\delta_{y;i}}(\lambda_i))=\mu$,

$$\lim_{(R,\delta)\longrightarrow(\infty,0)}\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{R\delta_{x;i},\delta}(\lambda_i)\big),\ \lim_{R_x,R_y\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{R_x\delta_{x;i},R_y\delta_{y;i}}(\lambda_i)\big)=\mathfrak{m}, \tag{6.4}$$

$$\lim_{(R,\delta)\longrightarrow(\infty,0)}\lim_{i\longrightarrow\infty}\mathrm{diam}_g\big(u_i(A_{\delta,|\lambda_i|/R\delta_{x;i}}(\lambda_i))\big)=0\,; \tag{6.5}$$

(3) for all $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$ and $\mu'\in(\mathfrak{m}-\hbar_{J,g},\mu)$,

$$\mu-\mu'\le\liminf_{R\longrightarrow\infty}\liminf_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i(\mu,\mu'),\delta_i(\mu,\mu')}(\lambda_i)\big)\le\mathfrak{m}-\mu'. \tag{6.6}$$

Proof. (2) For $r,R\in\mathbb{R}^+$ and $\lambda\in\mathbb{C}$, define

$$A^x_{R,r}(\lambda)=\big\{(x,y)\in A_{\delta_0,\delta_0}(\lambda)\colon r<|x|<R\big\}\quad\text{and}\quad A^y_{R,r}(\lambda)=\big\{(x,y)\in A_{\delta_0,\delta_0}(\lambda)\colon r<|y|<R\big\}. \tag{6.7}$$

Let $\hbar_{J,g}$ be the smaller of the constants $\hbar_{J,g}$ in Corollaries 4.2 and 4.17. Let u_i, u, and $\mathfrak{m}$ be as in the statement of Corollary 6.4 and $\delta_{x;i},\delta_{y;i}\in(0,\delta_0)$ be as in (2). After passing to a subsequence of $\{u_i\}$, we can choose $\delta'_{x;i}\in(\delta_{x;i},\delta_0)$ and $\delta'_{y;i}\in(\delta_{y;i},\delta_0)$ so that

$$\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{\delta'_{x;i},\delta'_{y;i}}(\lambda_i)\big)=\mathfrak{m}.$$

Applying the reasoning in the proof of (5.3) to the annuli $A^x_{R_x\delta'_{x;i},\delta_{x;i}}(\lambda_i)$ and $A^y_{R_y\delta'_{y;i},\delta_{y;i}}(\lambda_i)$, we then obtain (6.4) and (6.5) whenever $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$ and $\mu>0$.

(1) Suppose $0<\mathfrak{m}<\hbar_{J,g}$. Let $\delta_i=\delta_i(\mathfrak{m}/2)$ for $i\in\mathbb{Z}^+$ sufficiently large,

$$\mathfrak{m}_x=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i,\delta_i}(\lambda_i)\big),\quad\text{and}\quad \mathfrak{m}_y=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A^y_{R\delta_i,\delta_i}(\lambda_i)\big).$$

By the second equation in (6.4) and the definitions of $\mathfrak{m}$ and δ_i, $\mathfrak{m}_x+\mathfrak{m}_y=\mathfrak{m}/2$. By symmetry, we can assume that $\mathfrak{m}_x>0$.

For $i\in\mathbb{Z}^+$ sufficiently large (so that δ_i is defined and thus $|\lambda_i|<\delta_i^2$), define

$$v_i\colon U_i\equiv\big\{w\in\mathbb{C}\colon 2|\lambda_i|/\delta_0\delta_i<|w|<\delta_0/2\delta_i\big\}\longrightarrow X, \qquad v_i(w)=u_i\big(\delta_i w,\lambda_i/(\delta_i w)\big).$$

Thus, $\mathbb{C}^*=\bigcup_{i=1}^{\infty}U_i$ and v_i is a J-holomorphic map with

$$\lim_{i\longrightarrow\infty}E_g(v_i)=\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{\delta_0/2,\delta_0/2}(\lambda_i)\big)=\mathfrak{m}\big(\delta_0/2,\delta_0/2\big)<\infty. \tag{6.8}$$

After passing to a subsequence of $\{u_i\}$, we can assume that the sequences δ_i and $|\lambda_i|/\delta_i$ are decreasing and thus $U_i\subset U_{i+1}$.

By (6.8) and Lemma 5.4, there exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, so that the set

$$S\equiv\Big\{w\in\mathbb{C}^*\colon \lim_{\delta\longrightarrow 0}\sup_{B_\delta(w)}\big|\mathrm{d}v_i\big|_g=\infty\Big\}$$

is finite and the sequence u_i converges to a J-holomorphic map $v\colon\mathbb{P}^1\longrightarrow X$ C^1-u.c.s. on $\mathbb{C}^*-S$. For $w\in S$, let

$$\mathfrak{m}_w=\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\big(v_i;B_\delta(w)\big).$$

Similarly to (5.22),

$$\begin{aligned}E_g(v)+\sum_{w\in S}\mathfrak{m}_w&=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(v_i;B_R-B_{1/R}\big)=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i,\delta_i/R}(\lambda_i)\big)\\&\le\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g\big(u_i;A_{\delta,\delta}(\lambda_i)\big)=\mathfrak{m}<\hbar_{J,g}.\end{aligned}$$

Along with Corollary 5.3(2), this implies that $S=\emptyset$ and

$$\hbar_{J,g}>E_g(v)\ge\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i,\delta_i}(\lambda_i)\big)=\mathfrak{m}_x>0.$$

Thus, $v\colon\mathbb{P}^1\longrightarrow X$ is a non-constant J-holomorphic map with $E_g(v)<\hbar_{J,g}$. Since this is a contradiction, we must have $\mathfrak{m}\ge\hbar_{J,g}$.

(3) By the definitions of $\delta_i(\mu)$ and $\delta_i(\mu,\mu')$, $\delta_i(\mu,\mu')<\delta_i(\mu)$ and

$$E_g\big(u_i;A^x_{\delta_i(\mu),\delta_i(\mu,\mu')}(\lambda_i)\big)=\mu-\mu'.$$

By the second equation in (6.4),

$$\begin{aligned}&\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i(\mu),R\delta_i(\mu,\mu')}(\lambda_i)\big)\\&\quad=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}\big(E_g\big(u_i;A_{R\delta_i(\mu),R\delta_i(\mu)}(\lambda_i)\big)-\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}\big(E_g\big(u_i;A_{R\delta_i(\mu,\mu'),R\delta_i(\mu)}(\lambda_i)\big)=0.\end{aligned}$$

By the last two equations,

$$\begin{aligned}&\liminf_{R\longrightarrow\infty}\liminf_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i(\mu,\mu'),\delta_i(\mu,\mu')}(\lambda_i)\big)\\&\qquad\ge\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}\Big(E_g\big(u_i;A^x_{\delta_i(\mu),\delta_i(\mu,\mu')}(\lambda_i)\big)-E_g\big(u_i;A^x_{R\delta_i(\mu),R\delta_i(\mu,\mu')}(\lambda_i)\big)\Big)=\mu-\mu'.\end{aligned}$$

This establishes the left inequality in (6.6). The right one follows from (6.4) with μ replaced by μ' and the definition of $\delta_i(\mu,\mu')$. $\square$

Proposition 6.5. *Let (X,J,g) be a compact almost complex Riemannian manifold. Suppose $\delta_0 \in \mathbb{Z}^+$ and $u_i\colon A_{2\delta_0,2\delta_0}(\lambda_i)\longrightarrow X$ is a sequence of J-holomorphic maps converging to a C^1-map $u\colon A^*_{2\delta_0,2\delta_0}(0)\longrightarrow X$ C^1-u.c.s.. If $E_g(u)<\infty$ and the limit (6.3) exists and is nonzero, then $u|_{B^x_{2\delta_0}-\{0\}}$ extends to a J-holomorphic map $u_x\colon B^x_{2\delta_0}\longrightarrow X$ and there exist*

(1) a J-holomorphic map $u_{v_0}\colon \mathbb{P}^1\longrightarrow X$ with $u_{v_0}(\infty)=u_x(0_x)$ and a finite subset $S_{v_0}\subset\mathbb{C}^*$,

(2) a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, and

(3) for every $i\in\mathbb{Z}^+$, a biholomorphic map $\psi_i\colon B_{R_i}-\overline{B}_{r_i}\longrightarrow A_{\delta_0,\delta_0}(\lambda_i)$,

(4) $\delta_0'\in(0,\delta_0)$ and a sequence $\lambda_i'\in\mathbb{C}^*$ converging to 0,

such that

(a) $R_i\le R_{i+1}$ and $r_i\ge r_{i+1}$ for all $i\in\mathbb{Z}^+$, $(R_i,r_i)\longrightarrow(\infty,0)$ as $i\longrightarrow\infty$, and the sequences

$$u_i\circ\psi_i\colon B_{R_i}-\overline{B}_{r_i}\longrightarrow X \quad\textit{and}\quad u_i\circ\psi_i\circ\pi_{\lambda_i';x}\colon A_{\delta_0',\delta_0'}(\lambda_i')\longrightarrow X$$

converge to u_{v_0} C^1-u.c.s. on $\mathbb{C}^*-S_{v_0}$ and to $u_{v_0}|_{B_{\delta_0'}}\cup u|_{B^y_{\delta_0'}}$ C^1-u.c.s., respectively,

(b) the limits

$$\begin{aligned}\mathfrak{m}_0&\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i\circ\psi_i;B_\delta-B_{r_i/\delta}\big)\quad\textit{and}\\ \mathfrak{m}_w&\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i\circ\psi_i;B_\delta(w)\big)\quad\textit{with}\ \ w\in S_{v_0}\end{aligned}\tag{6.9}$$

exist, $\mathfrak{m}_w\neq0$ for $w\in S_{v_0}$, and $\mathfrak{m}=E_g\big(u_{v_0}\big)+\mathfrak{m}_0+\sum\limits_{w\in S_{v_0}}\mathfrak{m}_w$,

(c) if u_{v_0} is constant, then $S_{v_0}\neq\emptyset$.

Proof. Let $\hbar\equiv\hbar_{J,g}$ be the smaller of the numbers $\hbar_{J,g}$ in Corollaries 5.3 and 6.4 and $\mu=\mathfrak{m}-\hbar/4$. For every $i\in\mathbb{Z}^+$ sufficiently large, let

$$\delta_i=\delta_i\big(\mu,\mathfrak{m}-\hbar/2\big)\in(0,\delta_0).$$

After passing to a subsequence of $\{u_i\}$, we can assume that the limit

$$\mathfrak{m}_x\equiv\lim_{i\longrightarrow\infty}E_g\big(u_i;A^x_{R\delta_i,\delta_i}(\lambda_i)\big)$$

exists and lies in $[\hbar/4,\hbar/2]$; see Corollary 6.4(3). Define

$$\begin{gathered}R_i=\delta_0/\delta_i,\qquad r_i=|\lambda_i|/\delta_0\delta_i,\qquad \lambda_i'=\lambda_i/\delta_i,\\ \psi_i\colon B_{R_i}-\overline{B}_{r_i}\longrightarrow A_{\delta_0,\delta_0}(\lambda_i),\quad \psi_i(w)=\big(\delta_iw,\lambda_i/(\delta_iw)\big).\end{gathered}$$

Since $|\lambda_i|<\delta_i(\mu)\delta_i$, $\delta_i<\delta_i(\mu)$, and $\delta_i(\mu)\longrightarrow0$, $(R_i,r_i)\longrightarrow(\infty,0)$ as $i\longrightarrow\infty$. After passing to a subsequence of $\{u_i\}$, we can assume that R_i (resp. r_i) is an increasing (resp. decreasing) sequence. Thus, the first two properties in (a) are satisfied.

For each $i\in\mathbb{Z}^+$ sufficiently large, let

$$u_{v_0;i}=u_i\circ\psi_i\colon B_{R_i}-\overline{B}_{r_i}\longrightarrow X.$$

Since u_i is J-holomorphic and ψ_i is biholomorphic onto its image, $u_{v_0;i}$ is a J-holomorphic map with $E_g(u_{v_0;i})=E_g(u_i;A_{\delta_0,\delta_0}(\lambda_i))$. Thus,

$$\lim_{i\longrightarrow\infty}E_g(u_{v_0;i})=\mathfrak{m}+E_g\big(u;A_{\delta_0,\delta_0}(\lambda_i)\big)<\infty\,.$$

By Lemma 5.4, there thus exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, so that the set

$$S_{v_0}\equiv\big\{w\in\mathbb{C}^*\colon\lim_{\delta\longrightarrow0}\sup_{B_\delta(w)}\big|\mathrm{d}u_{v_0;i}\big|_g=\infty\big\}$$

is finite and the sequence $u_{v_0;i}$ converges to a J-holomorphic map $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$ C^1-u.c.s. on $\mathbb{C}^*-S_{v_0}$. In particular, the third property in (a) is satisfied and $|\mathrm{d}u_{v_0;i}|_g$ is uniformly bounded on compact subsets of $\mathbb{C}^*-S_{v_0}$.

For all $R>0$ so that $S_{v_0}\subset B_R$, $\delta>0$ so that $B_\delta(w)\subset B_R$ for every $w\in\{0\}\sqcup S_{v_0}$ and $B_\delta(w)\cap B_\delta(w')=\emptyset$ for all $w,w'\in\{0\}\sqcup S_{v_0}$ distinct, and $i\in\mathbb{Z}^+$ so that $R_i>R$ and $r_i<\delta^2$,

$$\begin{split}E_g\Big(u_{v_0;i},B_R-B_\delta-\bigcup_{w\in S_{v_0}}B_\delta(w)\Big)+E_g\big(u_{v_0;i};B_\delta-B_{r_i/\delta}\big)\qquad\\+\sum_{w\in S_{v_0}}E_g\big(u_{v_0;i};B_\delta(w)\big)=E_g\big(u_{v_0;i},B_R-B_{r_i/\delta}\big).\end{split}\tag{6.10}$$

We can pass to a further subsequence of $\{u_i\}$ so that all limits in (6.9) exist. In light of Corollary 5.3(1), $\mathfrak{m}_w\ge\hbar$ for all $w\in S_{v_0}$. By the conclusion of the previous paragraph and (6.10),

$$\begin{split}E_g(u_{v_0})+\mathfrak{m}_0+\sum_{w\in S_{v_0}}\mathfrak{m}_w&=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i},B_R-B_{r_i/\delta}\big)\\&=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i,A_{R\delta_i,\delta_0\delta}(\lambda_i)\big)=\mathfrak{m}\,;\end{split}\tag{6.11}$$

the last equality holds by the first statement in (6.4).

Let $\delta_0'\in(0,\delta_0)$ be such that $B_{2\delta_0'}\cap S_{v_0}=\emptyset$. Since

$$\begin{aligned}&\big\{u_i\circ\psi_i\circ\pi_{\lambda_i';x}\big\}\circ\pi_{\lambda_i';x}^{-1}=u_i\circ\psi_i\colon B_{\delta_0'}^x-B_{|\lambda_i'|/\delta_0'}^x\longrightarrow X\qquad\text{and}\\&\big\{u_i\circ\psi_i\circ\pi_{\lambda_i';x}\big\}\circ\pi_{\lambda_i';y}^{-1}=u_i\circ\pi_{\lambda_i;y}^{-1}\colon B_{\delta_0'}^y-B_{|\lambda_i'|/\delta_0'}^y\longrightarrow X,\end{aligned}$$

the last property in (a) is satisfied. By (6.5) and the reasoning in (5.23), $u_x(0_x)=u_{v_0}(\infty)$. By the reasoning in (6.11) and by (6.6),

$$E_g(u_{v_0})+\sum_{w\in S_{v_0}}\mathfrak{m}_w\ge\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_{v_0;i},B_R-B_1\big)=\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty}E_g\big(u_i,A^x_{R\delta_i,\delta_i}(\lambda_i)\big)\ge\hbar/4.$$

If $u_{v_0}\colon\mathbb{P}^1\longrightarrow X$ is a constant map, this implies that $S_{v_0}\neq\emptyset$. $\square$

If $(\mathscr{V},\prec)$ is a finite tree with root v_0, we call a subset $\mathscr{V}_\bullet\subset\mathscr{V}$ a stem of $(\mathscr{V},\prec)$ if $v_0\in\mathscr{V}$ and there is an ordering $v_1,\ldots,v_m$ of the elements of $\mathscr{V}_\bullet-\{v_0\}$ so that $p(v_i)=v_{i-1}$ for every $i=1,\ldots,m$. In such a case, we define $\max\mathscr{V}_\bullet=v_m$. If $v\in\mathscr{V}-\mathscr{V}_\bullet$, then $S_v(\mathscr{V})\cap\mathscr{V}_\bullet=\emptyset$.

Corollary 6.6. *Let (X,J,g), u_i, and u be as in Proposition 6.5. If $E_g(u)<\infty$ and the limit (6.3) exists and is nonzero, then $u|_{B^x_{2\delta_0}-\{0\}}$ and $u|_{B^y_{2\delta_0}-\{0\}}$ extend to J-holomorphic maps $u_x\colon B^x_{2\delta_0}\longrightarrow X$ and $u_y\colon B^y_{2\delta_0}\longrightarrow X$, respectively, and there exist*

(0) a finite tree $(\mathscr{V},\prec)$ with root v_0, a stem $\mathscr{V}_\bullet\subset\mathscr{V}$, and an attaching map μ for $(\mathscr{V},\prec)$ so that $\mu(\mathscr{V}_\bullet)=\{0\}$,

(1) a J-holomorphic map $u_\infty\colon\Sigma_{\mathscr{V},\mu}\longrightarrow X$ with $u_\infty|_{\mathbb{P}^1_{v_0}}(\infty)=u_x(0_x)$ and $u_\infty|_{\mathbb{P}^1_{\max\mathscr{V}_\bullet}}(0)=u_y(0_y)$,

(2) a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, and

(3) for every $v\in\mathscr{V}_\bullet$ and $i\in\mathbb{Z}^+$, $\delta_v\in(0,\delta_{p(v)})$, $\lambda_{v;i}\in\mathbb{C}^*$, and a biholomorphic map

$$\psi_{v;i}\colon B_{R_{v;i}}-\overline{B}_{r_{v;i}}\longrightarrow A_{\delta_{p(v)},\delta_{p(v)}}(\lambda_{p(v);i}),$$

where $\delta_{p(v_0)}\equiv\delta_0$ and $\lambda_{p(v_0);i}\equiv\lambda_i$,

(4) for every $v\in\mathscr{V}-\mathscr{V}_\bullet$ and $i\in\mathbb{Z}^+$, $\delta_v\in\mathbb{R}^+$ and a biholomorphic map $\psi_{v;i}\colon U_{v;i}\longrightarrow B_{\delta_v}(\mu(v))$, where $U_{v;i}\subset\mathbb{C}$ is an open subset,

such that

(a) for every $v\in\mathscr{V}_\bullet$, $R_{v;i}\le R_{v;i+1}$ and $r_{v;i}\ge r_{v;i+1}$ for all $i\in\mathbb{Z}^+$, $(R_{v;i},r_{v;i})\longrightarrow(\infty,0)$ as $i\longrightarrow\infty$, and the sequences

$$u_{v;i}\colon B_{R_{v;i}}-\overline{B}_{r_{v;i}}\longrightarrow X\quad\text{and}\quad u_{v;i}\circ\pi_{\lambda_{v;i};x}\colon A_{2\delta_v,2\delta_v}(\lambda_{v;i})\longrightarrow X,$$

where $u_{p(v_0);i}\equiv u_i$ and $u_{v;i}\equiv u_{p(v);i}\circ\psi_{v;i}$ if $v\in\mathscr{V}_\bullet-\{v_0\}$, converge to $u_\infty|_{\mathbb{P}^1_v}$ C^1-u.c.s. on $\mathbb{C}^*-\mu(S_v(\mathscr{V}))$, and to $u_v|_{B_{2\delta_v}}\cup u|_{B^y_{2\delta_v}}$ C^1-u.c.s., respectively,

(b) for every $v\in\mathscr{V}-\mathscr{V}_\bullet$, $U_{v;i}\subset U_{v;i+1}$ for all $i\in\mathbb{Z}^+$, $\mathbb{C}=\bigcup\limits_{i=1}^{\infty}U_{v;i}$ and the sequence $u_{v;i}\colon U_{v;i}\longrightarrow X$, where $u_{v;i}\equiv u_{p(v);i}\circ\psi_{v;i}$, converges to $u_\infty|_{\mathbb{P}^1_v}$ u.c.s. on $\mathbb{C}-\mu(S_v(\mathscr{V}))$,

(c) the limits

$$\begin{aligned}\mathfrak{m}_{v;0}&\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_{v;i};B_\delta-B_{r_{v;i}/\delta}\big)\quad\text{with } v\in\mathscr{V}_\bullet\qquad\text{and}\\ \mathfrak{m}_v&\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_{p(v);i};B_\delta(\mu(v))\big)\quad\text{with } v\in\mathscr{V}-\mathscr{V}_\bullet,\end{aligned}$$

exist, $\mathfrak{m}_{\max\mathscr{V}_\bullet}=0$, and

$$\begin{aligned}\mathfrak{m}_{p(v);0}&=E_g\big(u_\infty|_{\mathbb{P}^1_v}\big)+\mathfrak{m}_{v;0}+\sum_{v'\in S_v(\mathscr{V})}\mathfrak{m}_{v'}\qquad\forall\ v\in\mathscr{V}_\bullet,\\ \mathfrak{m}_v&=E_g\big(u_\infty|_{\mathbb{P}^1_v}\big)+\sum_{v'\in S_v(\mathscr{V})}\mathfrak{m}_{v'}\qquad\forall\ v\in\mathscr{V}-\mathscr{V}_\bullet,\end{aligned}\tag{6.12}$$

where $\mathfrak{m}_{p(v_0);0}\equiv\mathfrak{m}$, *(d) if $v\in\mathscr{V}_\bullet$ (resp. $v\in\mathscr{V}-\mathscr{V}_\bullet$) and $u_\infty|_{\mathbb{P}^1_v}$ is constant, then $S_v(\mathscr{V})\neq\emptyset$ (resp. $|S_v(\mathscr{V})|\geq 2$).*

If $v\in\mathscr{V}$ is a maximal element, $\mathfrak{m}_v=E_g(u_\infty|_{\mathbb{P}^1_v})$ by (6.12). Since the set $\mathscr{V}$ is finite, it follows from (6.12) that

$$\mathfrak{m}\equiv\mathfrak{m}_{p(v_0);0}=E_g\big(u_\infty\big).$$

Proof of Corollary 6.6. Let $\hbar$ be the smallest of the numbers $\hbar_{J,g}$ in Corollaries 4.2, 5.3, and 6.4. In particular, $\mathfrak{m}\geq\hbar$. Let $N\in\mathbb{Z}^+$ be the largest integer so that $N\hbar\leq\mathfrak{m}$. We proceed by induction on the admissible values of N.

Let u_{v_0}, S_{v_0}, $\{u_i\}$, $\delta_{v_0}\equiv\delta_0'/2$, $\lambda_{v_0;i}\equiv\lambda_i'$, $R_{v_0;i}\equiv R_i$, $r_{v_0;i}\equiv r_i$, and $\psi_{v_0;i}\equiv\psi_i$ be as in (1)-(4) of Proposition 6.5. By (b) and (c) in Proposition 6.5 and Corollaries 4.2 and 5.3(1),

$$\mathfrak{m}_{v_0;0}\equiv\mathfrak{m}_0\leq\mathfrak{m}-\hbar.$$

For each $w\in S_{v_0}$, we take $(\mathscr{V}_w,\prec_w,\mu_w)$, $u_{w;\infty}$, a subsequence of $\{u_i\}$, δ_v, $\psi_{v;i}$, and $U_{v;i}$ exactly as in the proof of Corollary 5.6.

Case 1: $\mathfrak{m}_{v_0;0}=0$. We then combine the above collections as at the end of the proof of Corollary 5.6 to obtain a finite tree $(\mathscr{V},\prec)$ with root v_0, an attaching map μ for $(\mathscr{V},\prec)$, a J-holomorphic map $u_\infty:\Sigma_{\mathscr{V},\mu}\longrightarrow X$ with $u_\infty|_{\mathbb{P}^1_{v_0}}(\infty)=u_x(0_x)$, and δ_v, $\lambda_{v_0;i}$, $R_{v_0;i}$, $r_{v_0;i}$, $U_{v;i}$, and $\psi_{v;i}$ as in (3) and (4) satisfying (a)-(d) with $\mathscr{V}_\bullet=\{v_0\}$. By Lemma 6.3(4) applied to the sequence $u_{v_0;i}\circ\pi_{\lambda_{v_0;i};x}$, $u_\infty|_{\mathbb{P}^1_{v_0}}(0)=u_y(0_y)$.

Case 2: $\mathfrak{m}_{v_0;0}>0$. We take $(\mathscr{V}_0,\prec_0,\mathscr{V}_{0;\bullet},\mu_0)$, $u_{0;\infty}$, a subsequence of $\{u_i\}$, δ_v, $\lambda_{v;i}$, $R_{v;i}$, $r_{v;i}$, $U_{v;i}$, and $\psi_{v;i}$ as provided by Corollary 6.6 and the inductive assumption for the sequence

$$u_{v_0;i}\circ\pi_{\lambda_{v_0;i};x}:A_{2\delta_{v_0},2\delta_{v_0}}(\lambda_{v_0;i})\longrightarrow X.$$

We then combine this collection with the collections above *Case 1* as at the end of the proof of Corollary 5.6 to obtain $(\mathscr{V},\prec,\mu)$, $\mathscr{V}_\bullet\equiv\{v_0\}\sqcup\mathscr{V}_{0;\bullet}$, u_∞, δ_v, $\lambda_{v;i}$, $R_{v;i}$, $r_{v;i}$, $U_{v;i}$, and $\psi_{v;i}$ in (0)-(4) satisfying (a)-(d). $\square$

A Connections in real vector bundles

A.1 Connections and splittings

Suppose X is a smooth manifold and $\pi_E:E\longrightarrow X$ is a vector bundle. We identify X with the zero section of E. Denote by

$$\mathfrak{a}:E\oplus E\longrightarrow E\qquad\text{and}\qquad\pi_{E\oplus E}:E\oplus E\longrightarrow X$$

the associated addition map and the induced projection map, respectively. For $f\in C^\infty(X;\mathbb{R})$, define

$$m_f:E\longrightarrow E\qquad\text{by}\qquad m_f(v)=f\big(\pi_E(v)\big)\cdot v\quad\forall\,v\in E.\tag{A.1}$$

In particular,

$$\pi_{E\oplus E}=\pi_E\circ\mathfrak{a},\qquad\pi_E=\pi_E\circ m_f\quad\forall\,f\in C^\infty(X;\mathbb{R}).$$

The total spaces of the vector bundles

$$\pi_{E\oplus E}\colon E\oplus E \longrightarrow X \qquad \text{and} \qquad \pi_E^*E \longrightarrow E$$

consist of the pairs (v,w) in $E\times E$ such that $\pi_E(v)=\pi_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E\colon \pi_E^*E \longrightarrow TE, \qquad \iota_E(v,w)=\frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0}. \tag{A.2}$$

Since the restriction of ι_E to the fiber over $v\in E$ is the composition of the isomorphism

$$E_{\pi_E(v)} \longrightarrow T_vE_{\pi_E(v)}, \qquad w \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0},$$

with the differential of the embedding of the fiber $E_{\pi_E(v)}$ into E, ι_E is an injective bundle homomorphism. Furthermore,

$$\mathrm{d}\pi_E\circ\iota_E=0, \quad m_f^*\iota_E\circ\pi_E^*m_f=\mathrm{d}m_f\circ\iota_E, \quad \mathfrak{a}^*\iota_E\circ\pi_{E\oplus E}^*\mathfrak{a}=\mathrm{d}\mathfrak{a}\circ\iota_{E\oplus E}, \tag{A.3}$$

$$TE|_X \approx TX\oplus \mathrm{Im}\,\iota_E\,. \tag{A.4}$$

By the first statement in (A.3), the injectivity of ι_E, and surjectivity of $\mathrm{d}\pi_E$,

$$0\longrightarrow \pi_E^*E \xrightarrow{\iota_E} TE \xrightarrow{\mathrm{d}\pi_E} \pi_E^*TX \longrightarrow 0 \tag{A.5}$$

is an exact sequence of vector bundles over E. By the second statement in (A.3), the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TX \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}"] & 0 \\
0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TX \arrow[r] & 0
\end{tikzcd} \tag{A.6}$$

of vector bundle homomorphisms over E commutes. By the third statement in (A.3), the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{E\oplus E}^*(E\oplus E) \arrow[r, "\iota_{E\oplus E}"] \arrow[d, "\pi_{E\oplus E}^*\mathfrak{a}"] & T(E\oplus E) \arrow[r, "\mathrm{d}\pi_{E\oplus E}"] \arrow[d, "\mathrm{d}\mathfrak{a}"] & \pi_{E\oplus E}^*TX \arrow[r] \arrow[d, "\pi_{E\oplus E}^*\mathrm{id}"] & 0 \\
0 \arrow[r] & \pi_{E\oplus E}^*E \arrow[r, "\mathfrak{a}^*\iota_E"] & \mathfrak{a}^*TE \arrow[r, "\mathfrak{a}^*\mathrm{d}\pi_E"] & \pi_{E\oplus E}^*TX \arrow[r] & 0
\end{tikzcd} \tag{A.7}$$

of vector bundle homomorphisms over $E\oplus E$ commutes.

A connection in E is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(X;E)\longrightarrow \Gamma(X;T^*X\otimes_{\mathbb{R}}E) \qquad \text{s.t.}$$
$$\nabla(f\xi)=\mathrm{d}f\otimes\xi+f\nabla\xi \quad \forall\ f\in C^\infty(X),\ \xi\in\Gamma(X;E). \tag{A.8}$$

The Leibnitz property (A.8) implies that any two connections in E differ by a 1-form on X. In other words, if ∇ and $\widetilde{\nabla}$ are connections in E there exists

$$\begin{gathered}\theta \in \Gamma\big(X; T^*X \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(E,E)\big) \qquad \text{s.t.} \\ \widetilde{\nabla}_v \xi = \nabla_v \xi + \big\{\theta(v)\big\}\xi \qquad \forall\, \xi \in \Gamma(X;E),\ v \in T_xX,\ x \in X.\end{gathered} \tag{A.9}$$

If U is a neighborhood of $x \in X$ and f is a smooth function on X supported in U such that $f(x)=1$, then

$$\nabla\xi\big|_x = \nabla\big(f\xi\big)\big|_x - \mathrm{d}_x f \otimes \xi(x) \tag{A.10}$$

by (A.8). The right-hand side of (A.10) depends only on $\xi|_U$. Thus, a connection ∇ in E is a local operator, i.e. the value of $\nabla\xi$ at a point $x \in X$ depends only on the restriction of ξ to any neighborhood U of x.

Suppose U is an open subset of X and $\xi_1, \ldots, \xi_n \in \Gamma(U;E)$ is a frame for E on U, i.e.

$$\xi_1(x), \ldots, \xi_n(x) \in E_x$$

is a basis for E_x for all $x \in U$. By definition of ∇, there exist

$$\theta_l^k \in \Gamma(U; T^*U) \qquad \text{s.t.} \qquad \nabla\xi_l = \sum_{k=1}^{k=n} \xi_k \theta_l^k \equiv \sum_{k=1}^{k=n} \theta_l^k \otimes \xi_k \quad \forall\, l=1,\ldots,n.$$

We call

$$\theta \equiv \big(\theta_l^k\big)_{k,l=1,\ldots,n} \in \Gamma\big(U; T^*U \otimes_{\mathbb{R}} \mathrm{Mat}_n\mathbb{R}\big)$$

the connection 1-form of ∇ with respect to the frame $(\xi_k)_k$.

For an arbitrary section

$$\xi = \sum_{l=1}^{l=n} f^l \xi_l \in \Gamma(U;E),$$

by (A.8) we have

$$\nabla\xi = \sum_{k=1}^{k=n} \xi_k \Big(\mathrm{d}f^k + \sum_{l=1}^{l=n} \theta_l^k f^l\Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{\xi} \cdot \underline{f}^t\big) = \underline{\xi} \cdot \big\{\mathrm{d} + \theta\big\}\underline{f}^t, \tag{A.11}$$

$$\text{where} \qquad \underline{\xi} = (\xi_1, \ldots, \xi_n), \quad \underline{f} = (f^1, \ldots, f^n). \tag{A.12}$$

This implies that

$$\nabla\xi\big|_x = \pi_2|_x \circ \mathrm{d}_x\xi \colon T_xX \longrightarrow E_x \qquad \forall\, \xi \in \Gamma(U;E) \text{ s.t. } \xi(x)=0, \tag{A.13}$$

where $\pi_2|_x \colon T_xE \longrightarrow E_x$ is the projection to the second component in (A.4).

By (A.11), ∇ is a first-order differential operator. By (A.8), its symbol is given by

$$\sigma_\nabla \colon T^*X \longrightarrow \mathrm{Hom}\big(E, T^*X \otimes_{\mathbb{R}} E\big), \qquad \big\{\sigma_\nabla(\eta)\big\}(f) = \eta \otimes f.$$

Lemma A.1. *Suppose X is a smooth manifold and $\pi_E\colon E\longrightarrow X$ is a vector bundle. A connection ∇ in E induces a splitting*

$$TE \approx \pi_E^*TX \oplus \pi_E^*E \tag{A.14}$$

of the exact sequence (A.5) extending the splitting (A.4) such that

$$\nabla\xi\big|_x = \pi_2|_x\circ \mathrm{d}_x\xi\colon T_xX \longrightarrow E_x \qquad \forall\ \xi\in\Gamma(X;E),\ x\in X, \tag{A.15}$$

where $\pi_2|_x\colon T_xE\longrightarrow E_x$ is the projection onto the second component in (A.14). Furthermore,

$$\mathrm{d}m_t \approx \pi_E^*\mathrm{id}\oplus\pi_E^*m_t \ \ \forall\ t\in\mathbb{R} \qquad \textit{and} \qquad \mathfrak{a}\approx\pi_{E\oplus E}^*\mathrm{id}\oplus\pi_{E\oplus E}^*\mathfrak{a}, \tag{A.16}$$

with respect to the splitting (A.14), i.e. it is consistent with the commutative diagrams (A.6) and (A.7).

Proof. Given $x\in X$ and $v\in E_x$, choose $\xi\in\Gamma(X;E)$ such that $\xi(x)=v$ and let

$$T_vE^{\mathrm{h}} = \mathrm{Im}\,\{\mathrm{d}\xi-\nabla\xi\}\big|_x \subset T_vE.$$

Since $\pi_E\circ\xi=\mathrm{id}_X$,

$$\mathrm{d}_v\pi_E\circ\big\{\mathrm{d}\xi-\nabla\xi\big\}\big|_x = \mathrm{id}_{T_xX} \qquad \Longrightarrow \qquad T_vE\approx T_vE^{\mathrm{h}}\oplus E_x\approx T_xX\oplus E_x.$$

This splitting of T_vE satisfies (A.15) at v.

With the notation as in (A.11),

$$\{\mathrm{d}\xi-\nabla\xi\}\big|_x = \Big(\mathrm{d}_x\mathrm{id}_X, -\sum_{l=1}^{l=n} f^l(x)\theta_l^1|_x,\ldots-\sum_{l=1}^{l=n} f^l(x)\theta_l^n|_x\Big)\colon T_xX\longrightarrow T_xX\oplus\mathbb{R}^n$$

with respect to the identification $E|_U\approx U\times\mathbb{R}^k$ determined by the frame $(\xi_k)_k$. Thus, T_vE^{h} is independent of the choice of ξ. Furthermore, the resulting splitting (A.14) of (A.5) extends (A.4) and satisfies (A.16). $\square$

A.2 Metric-compatible connections

Suppose $E\longrightarrow X$ is a smooth vector bundle. Let g be a metric on E, i.e.

$$g\in\Gamma(X;E^*\otimes_{\mathbb{R}}E^*) \qquad \text{s.t.} \qquad g(v,w)=g(w,v),\ \ g(v,v)>0 \quad \forall\ v,w\in E_x,\ v\neq0,\ x\in X.$$

A connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(\xi,\zeta)\big) = g(\nabla\xi,\zeta)+g(\xi,\nabla\zeta)\in\Gamma(X;T^*X) \qquad \forall\ \xi,\zeta\in\Gamma(X;E).$$

Suppose U is an open subset of X and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a frame for E on U. For $i,j=1,\ldots,n$, let

$$g_{ij} = g(\xi_i,\xi_j)\in C^\infty(\mathrm{U}).$$

If ∇ is a connection in E and θ_{kl} is the connection 1-form for ∇ with respect to the frame $\{\xi_k\}_k$, then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta_j^k + g_{jk}\theta_i^k\big) = \mathrm{d}g_{ij} \qquad \forall\ i,j=1,2,\ldots,n. \tag{A.17}$$

A.3 Torsion-free connections

If X is a smooth manifold, a connection ∇ in TX is torsion-free if

$$\nabla_\xi \zeta - \nabla_\zeta \xi = [\xi, \zeta] \qquad \forall\, \xi, \zeta \in \Gamma(X; TX).$$

If $(x_1, \ldots, x_n)\colon \mathrm{U} \longrightarrow \mathbb{R}^n$ is a coordinate chart on X, let

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \Gamma(\mathrm{U}; TX)$$

be the corresponding frame for TX on U. If ∇ is a connection in TX, the corresponding connection 1-form θ can be written as

$$\theta^k_j = \sum_{i=1}^{i=n} \Gamma^k_{ij} \mathrm{d}x^i, \qquad \text{where} \qquad \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{k=1}^{k=n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$

The connection ∇ is torsion-free on $TX|_{\mathrm{U}}$ if and only if

$$\Gamma^k_{ij} = \Gamma^k_{ji} \qquad \forall\, i, j, k = 1, \ldots, n. \tag{A.18}$$

Lemma A.2. *If (X, g) is a Riemannian manifold, there exists a unique torsion-free g-compatible connection ∇ in TX.*

Proof. (1) Suppose ∇ and $\widetilde{\nabla}$ are torsion-free g-compatible connections in TX. By (A.9), there exists

$$\begin{gathered} \theta \in \Gamma\big(X; T^*X \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(TX, TX)\big) \qquad \text{s.t.} \\ \widetilde{\nabla}_v \xi - \nabla_v \xi = \big\{\theta(X)\big\} Y \qquad \forall\, \xi \in \Gamma(X; TX),\ v \in T_x X,\ x \in X. \end{gathered}$$

Since ∇ and $\widetilde{\nabla}$ are torsion-free,

$$\big\{\theta(v)\big\} w = \big\{\theta(v)\big\} w \qquad \forall\, v, w \in T_x X,\ x \in X. \tag{A.19}$$

Since ∇ and $\widetilde{\nabla}$ are g-compatible,

$$\begin{cases} g\big(\{\theta(v)\}w, w'\big) + g\big(w, \{\theta(v)\}w'\big) = 0 \\ g\big(\{\theta(w)\}v, w'\big) + g\big(v, \{\theta(w)\}w'\big) = 0 \\ g\big(\{\theta(w')\}v, w\big) + g\big(v, \{\theta(w')\}w\big) = 0 \end{cases} \qquad \forall\, v, w, w' \in T_x X,\ x \in X. \tag{A.20}$$

Adding the first two equations in (A.20), subtracting the third, and using (A.19) and the symmetry of g, we obtain

$$2g\big(\{\theta(v)\}w', w\big) = 0 \quad \forall\, v, w, w' \in T_x X,\ x \in X \qquad \Longrightarrow \qquad \theta \equiv 0.$$

Thus, $\widetilde{\nabla} = \nabla$.

(2) Let $(x_1,\ldots,x_n)\colon \mathrm{U}\longrightarrow\mathbb{R}^n$ be a coordinate chart on X. With notation as in the paragraph preceding Lemma A.2, ∇ is g-compatible on $TX|_{\mathrm{U}}$ if and only if

$$\sum_{l=1}^{l=n}\big(g_{il}\Gamma^l_{kj}+g_{jl}\Gamma^l_{ki}\big)=\partial_{x_k}g_{ij};\tag{A.21}$$

see (A.17). Define a connection ∇ in $TX|_{\mathrm{U}}$ by

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{l=1}^{l=n}g^{kl}\big(\partial_{x_i}g_{jl}+\partial_{x_j}g_{il}-\partial_{x_l}g_{ij}\big)\qquad\forall\, i,j,k=1,\ldots,n,$$

where g^{ij} is the (i,j)-entry of the inverse of the matrix $(g_{ij})_{i,j=1,\ldots,n}$. Since $g_{ij}=g_{ji}$, Γ^k_{ij} satisfies (A.18); a direct computation shows that Γ^k_{ij} also satisfies (A.21). Therefore, ∇ is a torsion-free g-compatible connection on $TX|_{\mathrm{U}}$. In this way, we can define a torsion-free g-compatible connection on every coordinate chart. By the uniqueness property, these connections agree on the overlaps. $\square$

B Complex structures

B.1 Complex linear connections

Suppose X is a smooth manifold and $\pi:(E,\mathfrak{i})\longrightarrow X$ is a complex vector bundle. Similarly to Section A.1, there is an exact sequence

$$0\longrightarrow\pi_E^*E\xrightarrow{\iota_E}TE\xrightarrow{\mathrm{d}\pi_E}\pi_E^*TX\longrightarrow 0\tag{B.1}$$

of vector bundles over E. The homomorphism ι_E is now $\mathbb{C}$-linear. If $f\in C^\infty(X;\mathbb{C})$ and $m_f\colon E\longrightarrow E$ is defined as in (A.1), there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TX \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}"] & 0\\
0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TX \arrow[r] & 0
\end{tikzcd}\tag{B.2}$$

of bundle maps over E.

Suppose

$$\nabla\colon\Gamma(X;E)\longrightarrow\Gamma(X;T^*X\otimes_{\mathbb{R}}E)$$

is a $\mathbb{C}$-linear connection, i.e.

$$\nabla_v(\mathfrak{i}\xi)=\mathfrak{i}(\nabla_v\xi)\qquad\forall\,\xi\in\Gamma(X;E),\ v\in TX.$$

If U is an open subset of X and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta^k_l\in\Gamma(X;T^*X)\qquad\text{s.t.}\qquad\nabla\xi_l=\sum_{k=1}^{k=n}\xi_k\theta^k_l\equiv\sum_{k=1}^{k=n}\theta^k_l\otimes\xi_k\quad\forall\ l=1,\ldots,n.$$

We will call

$$\theta\equiv\big(\theta^k_l\big)_{k,l=1,\ldots,n}\in\Gamma\big(\Sigma;T^*X\otimes_{\mathbb{R}}\mathrm{Mat}_n\mathbb{C}\big)$$

the complex connection 1-form of ∇ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi = \sum_{l=1}^{l=n} f^l \xi_l \in \Gamma(\mathrm{U}; E),$$

by (A.8) and $\mathbb{C}$-linearity of ∇ we have

$$\nabla\xi = \sum_{k=1}^{k=n} \xi_k \Big(\mathrm{d}f^k + \sum_{l=1}^{l=n} \theta_l^k f^l \Big), \qquad \text{i.e.} \qquad \nabla\big(\underline{\xi}\cdot\underline{f}^t\big) = \underline{\xi}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t, \tag{B.3}$$

where $\underline{\xi}$ and $\underline{f}$ are as (A.12).

Let g be a hermitian metric on E, i.e.

$$g \in \Gamma\big(X; \mathrm{Hom}_{\mathbb{C}}(\bar{E}\otimes_{\mathbb{C}} E, \mathbb{C})\big) \quad \text{s.t.} \quad g(v,w) = \overline{g(w,v)}, \quad g(v,v) > 0 \quad \forall\, v,w \in E_x,\ v\neq 0,\ x\in X.$$

A $\mathbb{C}$-linear connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(\xi,\zeta)\big) = g(\nabla\xi,\zeta) + g(\xi,\nabla\zeta) \in \Gamma(X; T^*X\otimes_{\mathbb{R}}\mathbb{C}) \qquad \forall\, \xi,\zeta \in \Gamma(X;E).$$

With notation as in the previous paragraph, let

$$g_{ij} = g(\xi_i,\xi_j) \in C^\infty(\mathrm{U};\mathbb{C}) \qquad \forall\, i,j=1,\ldots,n.$$

Then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n} \big(g_{ik}\theta_j^k + \bar{g}_{jk}\bar{\theta}_i^k\big) = \mathrm{d}g_{ij} \qquad \forall\, i,j = 1,2,\ldots,n. \tag{B.4}$$

B.2 Generalized $\bar\partial$-operators

If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, let

$$T^*\Sigma^{1,0} \equiv \big\{\eta\in T^*\Sigma\otimes_{\mathbb{R}}\mathbb{C} \colon \eta\circ\mathfrak{j} = \mathfrak{i}\,\eta\big\} \qquad \text{and} \qquad T^*\Sigma^{0,1} \equiv \big\{\eta\in T^*\Sigma\otimes_{\mathbb{R}}\mathbb{C} \colon \eta\circ\mathfrak{j} = -\mathfrak{i}\,\eta\big\}$$

be the bundles of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on Σ. If $(\Sigma,\mathfrak{j})$ and (X,J) are smooth almost complex manifolds and $u\colon\Sigma\longrightarrow X$ is a smooth function, define

$$\bar\partial_{J,\mathfrak{j}}u \in \Gamma\big(\Sigma; T^*\Sigma^{0,1}\otimes_{\mathbb{C}} u^*TX\big) \qquad \text{by} \qquad \bar\partial_{J,\mathfrak{j}}u = \frac{1}{2}\big(\mathrm{d}u + J\circ\mathrm{d}u\circ\mathfrak{j}\big). \tag{B.5}$$

A smooth map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ will be called $(J,\mathfrak{j})$-holomorphic if $\bar\partial_{J,\mathfrak{j}}u=0$.

Definition B.1. Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold and $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle. A $\bar\partial$-operator on $(E,\mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar\partial\colon \Gamma(\Sigma;E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1}\otimes_{\mathbb{C}} E)$$

such that

$$\bar\partial\big(f\xi\big) = (\bar\partial f)\otimes\xi + f(\bar\partial\xi) \qquad \forall\, f\in C^\infty(\Sigma),\ \xi\in\Gamma(\Sigma;E), \tag{B.6}$$

where $\bar\partial f = \bar\partial_{\mathfrak{i},\mathfrak{j}}f$ is the usual $\bar\partial$-operator on complex-valued functions.

Similarly to Section A.1, a $\bar\partial$-operator on $(E,\mathfrak{i})$ is a first-order differential operator. If U is an open subset of X and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta_l^k\in\Gamma(U;T^*U^{0,1})\qquad\text{s.t.}\qquad \bar\partial\xi_l=\sum_{k=1}^{k=n}\xi_k\theta_l^k\equiv\sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k\quad\forall\, l=1,\ldots,n.$$

We call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(U;T^*U^{0,1}\otimes_{\mathbb{C}}\mathrm{Mat}_n\mathbb{C}\big)$$

the connection 1-form of $\bar\partial$ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathrm{U};E),$$

by (B.6) we have

$$\bar\partial\xi=\sum_{k=1}^{k=n}\xi_k\Big(\bar\partial f^k+\sum_{l=1}^{l=n}\theta_l^kf^l\Big),\qquad\text{i.e.}\qquad \bar\partial\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\{\bar\partial+\theta\}\underline{f}^t,\tag{B.7}$$

where $\underline{\xi}$ and $\underline{f}$ are as in (A.12). It is immediate from (B.6) that the symbol of $\bar\partial$ is given by

$$\sigma_{\bar\partial}\colon T^*\Sigma\longrightarrow\mathrm{Hom}_{\mathbb{C}}\big(E,T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big),\qquad \big\{\sigma_{\bar\partial}(\eta)\big\}(f)=\big(\eta+\mathfrak{i}\,\eta\circ\mathfrak{j}\big)\otimes f.$$

In particular, $\bar\partial$ is an elliptic operator (i.e. $\sigma_{\bar\partial}(\eta)$ is an isomorphism for $\eta\neq0$) if $(\Sigma,\mathfrak{j})$ is a Riemann surface.

Lemma B.2. *Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold and $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle. If*

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$, there exists a unique almost complex structure $J=J_{\bar\partial}$ on (the total space of) E such that π is a $(\mathfrak{j},J)$-holomorphic map, the restriction of J to the vertical tangent bundle $TE^{\mathrm{v}}\approx\pi^*E$ agrees with $\mathfrak{i}$, and

$$\bar\partial_{J,\mathfrak{j}}\xi=0\in\Gamma(\mathrm{U};T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\xi^*TE)\qquad\Longleftrightarrow\qquad\bar\partial\xi=0\in\Gamma(\mathrm{U};T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\tag{B.8}$$

for every open subset U of Σ and $\xi\in\Gamma(\mathrm{U};E)$.

Proof. (1) With notation as above, define

$$\varphi\colon\mathrm{U}\times\mathbb{C}^n\longrightarrow E|_{\mathrm{U}}\qquad\text{by}\qquad\varphi(x,c^1,\ldots,c^n)=\underline{\xi}(x)\cdot\underline{c}^t\equiv\sum_{k=1}^{k=n}c^k\xi_k(x)\in E_x.$$

The map φ is a trivialization of E over U. If $J\equiv J_{\bar\partial}$ is an almost complex structure on E, let $\widetilde{J}$ be the almost complex structure on $\mathrm{U}\times\mathbb{C}^n$ given by

$$\widetilde{J}_{(x,\underline{c})}=\big\{\mathrm{d}_{(x,\underline{c})}\varphi\big\}^{-1}\circ J_{\varphi(x,\underline{c})}\circ\mathrm{d}_{(x,\underline{c})}\varphi\qquad\forall\,(x,\underline{c})\in\mathrm{U}\times\mathbb{C}^n.\tag{B.9}$$

The almost complex structure J restricts to $\mathfrak{i}$ on TE^{v} if and only if

$$\widetilde{J}_{(x,\underline{c})}w=\mathfrak{i}w\in T_{\underline{c}}\mathbb{C}^n\subset T_{(x,\underline{c})}(\mathrm{U}\times\mathbb{C}^n)\qquad\forall\,w\in T_{\underline{c}}\mathbb{C}^n.\tag{B.10}$$

If J restricts to $\mathfrak{i}$ on TE^{v}, the projection π is $(\mathfrak{j},J)$-holomorphic on $E|_{\mathrm{U}}$ if and only if there exists

$$\begin{aligned}
&\widetilde{J}^{\mathrm{vh}} \in \Gamma\big(\mathrm{U}\times\mathbb{C}^n; \mathrm{Hom}_{\mathbb{R}}(\pi_{\mathrm{U}}^*T\mathrm{U}, \pi_{\mathbb{C}^n}^*T\mathbb{C}^n)\big) \qquad \text{s.t.}\\
&\widetilde{J}_{(x,\underline{c})}w = \mathfrak{j}_x w + \widetilde{J}^{\mathrm{vh}}_{(x,\underline{c})}w \qquad \forall\ w\in T_x\mathrm{U}\subset T_{(x,\underline{c})}(\mathrm{U}\times\mathbb{C}^n).
\end{aligned} \tag{B.11}$$

If $\xi\in\Gamma(\mathrm{U};E)$, let

$$\widetilde{\xi}\equiv\varphi^{-1}\circ\xi\equiv\big(\mathrm{id}_{\mathrm{U}},\underline{f}\big), \qquad \text{where} \qquad \underline{f}\in C^\infty(\mathrm{U};\mathbb{C}^n).$$

By (B.9)-(B.11),

$$\begin{aligned}
2\,\bar\partial_{J,\mathfrak{j}}\xi\big|_x = \mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ 2\bar\partial_{\widetilde{J},\mathfrak{j}}\widetilde{\xi}\big|_x &= \mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ\big\{\big(\mathrm{Id}_{T_x\mathrm{U}},\mathrm{d}_x\underline{f}\big)+\widetilde{J}_{\widetilde{\xi}(x)}\circ\big(\mathrm{Id}_{T_x\mathrm{U}},\mathrm{d}_x\underline{f}\big)\circ\mathfrak{j}_x\big\}\\
&= \mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ\big(0, 2\,\bar\partial f|_x+\widetilde{J}^{\mathrm{vh}}_{\widetilde{\xi}(x)}\circ\mathfrak{j}_x\big).
\end{aligned} \tag{B.12}$$

On the other hand, by (B.7),

$$\begin{aligned}
\bar\partial\xi|_x = \bar\partial\big(\underline{\xi}\cdot f^t\big)\big|_x &= \underline{\xi}(x)\cdot\big\{\bar\partial+\theta\big\}f^t\big|_x\\
&= \varphi\big(\bar\partial f|_x+\theta_x\cdot f(x)^t\big).
\end{aligned} \tag{B.13}$$

By (B.12) and (B.13), the property (B.8) is satisfied for all $\xi\in\Gamma(\mathrm{U};E)$ if and only if

$$\widetilde{J}^{\mathrm{vh}}_{(x,\underline{c})} = 2\big(\theta_x\cdot\underline{c}^t\big)\circ(-\mathfrak{j}_x) = 2\mathfrak{i}\,\theta_x\cdot\underline{c}^t \qquad \forall\ (x,\underline{c})\in\mathrm{U}\times\mathbb{C}^n.$$

In summary, the almost complex structure $J=J_{\bar\partial}$ on E has the three desired properties if and only if for every trivialization of E over an open subset U of Σ

$$\begin{gathered}
\widetilde{J}_{(x,\underline{c})}\big(w_1,w_2\big) = \big(\mathfrak{j}_x w_1, \mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)\\
\forall\ (x,\underline{c})\in\mathrm{U}\times\mathbb{C}^n,\ (w_1,w_2)\in T_x\mathrm{U}\oplus T_{\underline{c}}\mathbb{C}^n = T_{(x,\underline{c})}(\mathrm{U}\times\mathbb{C}^n),
\end{gathered} \tag{B.14}$$

where $\widetilde{J}$ is the almost complex structure on $\mathrm{U}\times\mathbb{C}^n$ induced by J via the trivialization and θ is the connection 1-form corresponding to $\bar\partial$ with respect to the frame inducing the trivialization.

(2) By (B.14), there exists at most one almost complex structure J satisfying the three properties. Conversely, (B.14) determines such an almost complex structure on E. Since

$$\begin{aligned}
\widetilde{J}^2_{(x,\underline{c})}\big(w_1,w_2\big) = \widetilde{J}_{(x,\underline{c})}\big(\mathfrak{j}w_1,\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big) &= \big(\mathfrak{j}^2w_1, \mathfrak{i}\big(\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)+2\mathfrak{i}\theta_x(\mathfrak{j}w_1)\cdot\underline{c}^t\big)\\
&= -(w_1,w_2),
\end{aligned}$$

$\widetilde{J}$ is indeed an almost complex structure on E. The almost complex structure induced by $\widetilde{J}$ on $E|_{\mathrm{U}}$ satisfies the three properties by part (a). By the uniqueness property, the almost complex structures on E induced by the different trivializations agree on the overlaps. Therefore, they define an almost complex structure $J=J_{\bar\partial}$ on the total space of E with the desired properties. □

B.3 Connections and $\bar\partial$-operators

Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle, and

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$. A $\mathbb{C}$-linear connection ∇ in $(E,\mathfrak{i})$ is $\bar\partial$-compatible if

$$\bar\partial\xi = \bar\partial_\nabla\xi\equiv\frac{1}{2}\big(\nabla\xi+\mathfrak{i}\nabla\xi\circ\mathfrak{j}\big) \qquad \forall\ \xi\in\Gamma(X;\Sigma). \tag{B.15}$$

Lemma B.3. *Suppose* $(\Sigma, \mathfrak{j})$ *is an almost complex manifold,* $\pi: (E, \mathfrak{i}) \longrightarrow \Sigma$ *is a complex vector bundle,*

$$\bar{\partial}\colon \Gamma(\Sigma; E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} E)$$

is a $\bar{\partial}$*-operator on* $(E, \mathfrak{i})$*, and* $J_{\bar{\partial}}$ *is the complex structure in the vector bundle* $TE \longrightarrow E$ *provided by Lemma B.2. A* $\mathbb{C}$*-linear connection* ∇ *in* $(E, \mathfrak{i})$ *is* $\bar{\partial}$*-compatible if and only if the splitting (A.14) determined by* ∇ *respects the complex structures.*

Proof. Since $J_{\bar{\partial}} = \pi^*\mathfrak{i}$ on $\pi^*E \subset TE$, the splitting (A.14) determined by ∇ respects the complex structures if and only if

$$J_{\bar{\partial}}|_v \circ \big\{\mathrm{d}\xi - \nabla\xi\big\}\big|_x = \big\{\mathrm{d}\xi - \nabla\xi\big\}\big|_x \circ \mathfrak{j}_x \colon T_x\Sigma \longrightarrow T_vE$$

for all $x \in \Sigma$, $v \in E_x$, and $\xi \in \Gamma(\Sigma; E)$ such that $\xi(x) = 0$; see the proof of Lemma A.1. This identity is equivalent to

$$\bar{\partial}_{J_{\bar{\partial}},\mathfrak{j}}\xi = \bar{\partial}_\nabla\xi \qquad \forall\ \xi \in \Gamma(\Sigma; E). \tag{B.16}$$

On the other hand, by the proof of Lemma B.2,

$$\bar{\partial}_{J_{\bar{\partial}},\mathfrak{j}}\xi = \bar{\partial}\xi \qquad \forall\ \xi \in \Gamma(\Sigma; E); \tag{B.17}$$

see (B.12)-(B.14). The lemma follows immediately from (B.16) and (B.17). □

B.4 Holomorphic vector bundles

Let $(\Sigma, \mathfrak{j})$ be a complex manifold. A holomorphic vector bundle $(E, \mathfrak{i})$ on $(\Sigma, \mathfrak{j})$ is a complex vector bundle with a collection of trivializations that overlap holomorphically.

A collection of holomorphically overlapping trivializations of (E, i) determines a holomorphic structure J on the total space of E and a $\bar{\partial}$-operator

$$\bar{\partial}\colon \Gamma(\Sigma; E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} E).$$

The latter is defined as follows. If $\xi_1, \ldots, \xi_n$ is a holomorphic complex frame for E over an open subset U of X, then

$$\bar{\partial} \sum_{k=1}^{k=n} f^k \xi_k = \sum_{k=1}^{k=n} \bar{\partial} f^k \otimes \xi_k \qquad \forall\ f^1, \ldots, f^k \in C^\infty(\mathrm{U}; \mathbb{C}).$$

In particular, for all $\xi \in \Gamma(X; E)$

$$\bar{\partial}_{J,\mathfrak{j}}\xi = 0 \qquad \Longleftrightarrow \qquad \bar{\partial}\xi = 0.$$

Thus, $J = J_{\bar{\partial}}$; see Lemma B.2.

Lemma B.4. *Suppose* $(\Sigma, \mathfrak{j})$ *is a Riemann surface and* $\pi\colon (E, \mathfrak{i}) \longrightarrow \Sigma$ *is a complex vector bundle. If*

$$\bar{\partial}\colon \Gamma(\Sigma; E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} E)$$

is a $\bar{\partial}$*-operator on* $(E, \mathfrak{i})$*, the almost complex structure* $J = J_{\bar{\partial}}$ *on* E *is integrable. With this complex structure,* $\pi\colon E \longrightarrow \Sigma$ *is a holomorphic vector bundle and* $\bar{\partial}$ *is the corresponding* $\bar{\partial}$*-operator.*

Proof. By (B.8), it is sufficient to show that there exists a $(J,\mathfrak{j})$-holomorphic local section through every point $v\in E$, i.e. there exist a neighborhood U of $x\equiv\pi(v)$ in Σ and $\xi\in\Gamma(\mathrm{U};E)$ such that

$$\xi(x)=v \qquad \text{and} \qquad \bar{\partial}_{J,\mathfrak{j}}\xi=0.$$

By Lemma B.2 and (B.13), this is equivalent to showing that the equation

$$\big\{\bar{\partial}+\theta\big\}f^t=0, \qquad f(x)=v, \qquad f\in C^\infty(\mathrm{U};\mathbb{C}^n), \tag{B.18}$$

has a solution for every $v\in\mathbb{C}^n$. We can assume that U is a small disk contained in S^2. Let

$$\eta\colon S^2\longrightarrow[0,1]$$

be a smooth function supported in U and such that $\eta\equiv1$ on a neighborhood of x. Then,

$$\eta\theta\in\Gamma(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{Mat}_n\mathbb{C}).$$

Choose $p>2$. The operator

$$\Theta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathbb{C}^n\big)\oplus\mathbb{C}^n, \qquad \Theta(f)=\big(\bar{\partial}_{\mathfrak{i},\mathfrak{j}}f,f(x)\big),$$

is surjective. If η has sufficiently small support, so is the operator

$$\Theta_\eta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathbb{C}^n\big)\oplus\mathbb{C}^n, \qquad \Theta_\eta(f)=\big(\{\bar{\partial}_{\mathfrak{i},\mathfrak{j}}+\eta\theta\}f,f(x)\big).$$

Then, the restriction of $\Theta_\eta^{-1}(0,v)$ to a neighborhood of x on which $\eta\equiv1$ is a solution of (B.18). By elliptic regularity, $\Theta_\eta^{-1}(0,v)\in C^\infty(S^2;\mathbb{C}^n)$. □

B.5 Deformations of almost complex submanifolds

If (X,J) is a complex manifold, holomorphic coordinate charts on (X,J) determine a holomorphic structure in the vector bundle $(TX,\mathfrak{i})\longrightarrow X$. If $(\Sigma,\mathfrak{j})\subset(X,J)$ is a complex submanifold, holomorphic coordinate charts on Σ can be extended to holomorphic coordinate charts on X. Thus, the holomorphic structure in $T\Sigma\longrightarrow\Sigma$ induced from $(\Sigma,\mathfrak{j})$ is the restriction of the holomorphic structure in $TX|_\Sigma$. It follows that

$$\bar{\partial}_X=\bar{\partial}_\Sigma\colon\Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma)\subset\Gamma\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TX|_\Sigma\big),$$

where $\bar{\partial}_X$ and $\bar{\partial}_\Sigma$ are the $\bar{\partial}$-operators in $TX|_\Sigma$ and $T\Sigma$ induced from the holomorphic structures in Σ and X. Therefore, $\bar{\partial}_X$ descends to a $\bar{\partial}$-operator on the quotient

$$\bar{\partial}\colon\Gamma(\Sigma;\mathcal{N}_X\Sigma)=\Gamma(\Sigma;TX|_\Sigma)\big/\Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_X\Sigma),$$

where

$$\mathcal{N}_X\Sigma\equiv TX|_\Sigma/T\Sigma\longrightarrow\Sigma$$

is the normal bundle of Σ in X. This vector bundle inherits a holomorphic structure from that of $TX|_\Sigma$ and Σ. The above $\bar{\partial}$-operator on $\mathcal{N}_X\Sigma$ is the $\bar{\partial}$-operator corresponding to this induced holomorphic structure on $\mathcal{N}_X\Sigma$.

Suppose (X,J) is an almost complex manifold and $(\Sigma,\mathfrak{j})\subset(X,J)$ is an almost complex submanifold. Let ∇ be a torsion-free connection in TX. Define

$$\begin{gathered}D_{J;\Sigma}\colon\Gamma(\Sigma;TX|_\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}TX|_\Sigma)\qquad\text{by}\\ D_{J;\Sigma}\xi=\frac12\big(\nabla\xi+J\circ\nabla\xi\circ\mathfrak{j}\big)-\frac12J\circ\nabla_\xi J\colon T\Sigma\longrightarrow TX|_\Sigma.\end{gathered}\tag{B.19}$$

If ∇ is the Levi-Civita connection (the connection of Lemma A.2) for a J-compatible metric on X (and Σ is a Riemann surface), then $D_{J;\Sigma}$ is the linearization of the $\bar\partial_J$-operator at the inclusion map $\iota\colon\Sigma\longrightarrow X$; see [29, Proposition 3.1.1].

In fact, $D_{J;\Sigma}$ is independent of the choice of a torsion-free connection in TX. Let

$$\widetilde\nabla=\nabla+\theta,\qquad\theta\in\Gamma\big(X;T^*X\otimes_{\mathbb R}\mathrm{Hom}_{\mathbb R}(TX,TX)\big),\tag{B.20}$$

be another torsion-free connection; see (A.9). Since $\widetilde\nabla$ and ∇ are torsion-free connections,

$$\big\{\theta(X)\big\}Y=\big\{\theta(Y)\big\}X\qquad\forall X,Y\in T_xX,\ x\in X.\tag{B.21}$$

If $x\in X$ and $X,Y\in\Gamma(X;TX)$,

$$\begin{gathered}\big\{\nabla_YJ\big\}X=\nabla_Y(JX)-J\nabla_YX\,,\quad\big\{\widetilde\nabla_YJ\big\}X=\widetilde\nabla_Y(JX)-J\widetilde\nabla_YX\qquad\Longrightarrow\\ \big\{\widetilde\nabla_YJ\big\}X-\big\{\nabla_YJ\big\}X=\big\{\theta(Y)\big\}(JX)-J\big\{\theta(Y)\big\}X=\big\{\theta(JX)\big\}Y-J\big\{\theta(X)\big\}Y\end{gathered}\tag{B.22}$$

by (B.20) and (B.21). On the other hand, by (B.20) for all $X\in T\Sigma$ and $\xi\in\Gamma(\Sigma;TX|_\Sigma)$,

$$\begin{split}\big\{\widetilde\nabla\xi+J\circ\widetilde\nabla\xi\circ\mathfrak{j}\big\}(X)-\big\{\nabla\xi+J\circ\nabla\xi\circ\mathfrak{j}\big\}(X)&=\big\{\theta(X)\big\}\xi+J\big\{\theta(\mathfrak{j}X)\big\}\xi\\&=J\big(\big\{\theta(JX)\big\}\xi-J\big\{\theta(X)\big\}\xi\big),\end{split}\tag{B.23}$$

since $\mathfrak{j}=J|_{T\Sigma}$ and $J^2=-\mathrm{Id}$. By (B.22) and (B.23), $D_{J,\Sigma}$ is independent of the choice of torsion-free connection ∇.

Since any torsion-free connection on Σ extends to a torsion-free connection on X, the above observation implies that

$$D_{J;\Sigma}\colon\Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}T\Sigma)\subset\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}TX|_\Sigma).\tag{B.24}$$

Thus, an almost complex submanifold $(\Sigma,\mathfrak{j})$ of an almost complex manifold (X,J) induces a well-defined generalized Cauchy-Riemann operator[1] on the normal bundle of Σ in X,

$$D^{\mathcal N}_{J;\Sigma}\colon\Gamma(\Sigma;\mathcal N_X\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}\mathcal N_X\Sigma),\qquad D^{\mathcal N}_{J;\Sigma}\big(\pi(\xi)\big)=\pi\big(D_{J;\Sigma}(\xi)\big)\ \ \forall\,\xi\in\Gamma(\Sigma;TX|_\Sigma),$$

where $\pi\colon TX|_\Sigma\longrightarrow\mathcal N_X\Sigma$ is the quotient projection map. The $\mathbb C$-linear part of $D^{\mathcal N}_{J;\Sigma}$ determines a $\bar\partial$-operator on the normal bundle of Σ in X:

$$\begin{gathered}\bar\partial^{\mathcal N}_{J;\Sigma}\colon\Gamma(\Sigma;\mathcal N_X\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}\mathcal N_X\Sigma),\\ \bar\partial^{\mathcal N}_{J;\Sigma}(\xi)=\frac12\big(D^{\mathcal N}_{J;\Sigma}(\xi)-JD^{\mathcal N}_{J;\Sigma}(J\xi)\big)\ \ \forall\,\xi\in\Gamma(\Sigma;\mathcal N_X\Sigma).\end{gathered}$$

[1]see Section D.3

Both operators are determined by the almost complex submanifold $(\Sigma,\mathfrak{j})$ of the almost complex manifold (X,J) only and are independent of the choice of torsion-free connection ∇ in (B.19).

Any connection ∇ in TX induces a J-linear connection in TX by

$$\nabla^J_v\xi=\nabla_v\xi-\frac{1}{2}J(\nabla_vJ)\xi \qquad \forall\, v\in TX,\ \xi\in\Gamma(X;TX). \tag{B.25}$$

If ∇ is as in (B.19),

$$\big\{D_{J;\Sigma}\xi\big\}(v)=\big\{\bar\partial_{\nabla^J}\xi\big\}(v)+A_J(v,\xi)-\frac{1}{4}\big\{(\nabla_{J\xi}J)+J(\nabla_\xi J)\big\}(v) \tag{B.26}$$

for all $\xi\in\Gamma(\Sigma;TX|_\Sigma)$ and $v\in T\Sigma$, where A_J is the Nijenhuis tensor of J; see (2.3). Since the sum of the terms in the curly brackets in (B.26) is $\mathbb{C}$-linear in ξ, while the Nijenhuis tensor is $\mathbb{C}$-antilinear, the $\mathbb{C}$-linear operator

$$\Gamma(\Sigma;TX|_\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TX|_\Sigma), \qquad \xi\longrightarrow\bar\partial_{\nabla^J}(\xi)-\frac{1}{4}\big\{(\nabla_{J\xi}J)+J(\nabla_\xi J)\big\}, \tag{B.27}$$

takes $\Gamma(\Sigma;T\Sigma)$ to $\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma)$ by (B.24). Thus, it induces a $\bar\partial$-operator on $\mathcal{N}_X\Sigma$ and this induced operator is $\bar\partial^{\mathcal{N}}_{J;\Sigma}$. If the image of the homomorphism

$$TX\longrightarrow T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TX|_\Sigma\,, \qquad \xi\longrightarrow\nabla_\xi J-J\nabla_{J\xi}J\,,$$

is contained in $T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma$, then $\bar\partial_{\nabla^J}$ preserves $T\Sigma$ and induces a $\bar\partial$-operator $\bar\partial^{\mathcal{N}}_{\nabla^J}$ on $\mathcal{N}_X\Sigma$ with $\bar\partial^{\mathcal{N}}_{\nabla^J}=\bar\partial^{\mathcal{N}}_{J;\Sigma}$. In this case,

$$D^{\mathcal{N}}_{J;\Sigma}\big(\pi(\xi)\big)=\pi\big(\bar\partial_{\nabla^J}\xi+A_J(\cdot,\xi)\big)\colon T\Sigma\longrightarrow\mathcal{N}_X\Sigma \qquad \forall\ \xi\in\Gamma(\Sigma;TX|_\Sigma).$$

This is the case in particular if J is compatible with a symplectic form ω on X and ∇ is the Levi-Civita connection for the metric $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$, as the sum in the curly brackets in (B.26) then vanishes by [29, (C.7.5)].

It is immediate that A_J takes $T\Sigma\otimes_{\mathbb{R}}T\Sigma$ to $T\Sigma$ and thus induces a bundle homomorphism

$$A^{\mathcal{N}}_J\colon T\Sigma\otimes_{\mathbb{R}}\mathcal{N}_X\Sigma\longrightarrow\mathcal{N}_X\Sigma\,.$$

If ζ is any vector field on X such that $\zeta(x)=X\in T_x\Sigma$ for some $x\in\Sigma$, then

$$\begin{aligned}\big\{D_{J;\Sigma}\xi\big\}(X)&=\frac{1}{2}\big([\zeta,\xi]+J[J\zeta,\xi]\big)\big|_x,\\ \Big\{\bar\partial_{\nabla^J}(\xi)-\frac{1}{4}\big((\nabla_{J\xi}J)+J(\nabla_\xi J)\big)\Big\}(X)&=\frac{1}{4}\big([\zeta,\xi]+J[J\zeta,\xi]-J[\zeta,J\xi]+[J\zeta,J\xi]\big)\big|_x,\end{aligned} \tag{B.28}$$

since ∇ is torsion-free.[2] These two identities immediately imply that the operators (B.19) and (B.27) preserve $T\Sigma\subset TX|_\Sigma$ and thus induce operators

$$\Gamma(\Sigma;\mathcal{N}_X\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_X\Sigma)$$

[2]Since LHS and RHS of these identities depend only ξ and $X=\zeta(x)$, and not on ζ, it is sufficient to verify them under the assumption that $\nabla\zeta|_x=0$.

as claimed above.

If g is a J-compatible metric on $TX|_\Sigma$ and $\pi^\perp\colon TX|_\Sigma \longrightarrow T\Sigma^\perp$ is the projection to the g-orthogonal complement of $T\Sigma$ in $TX|_\Sigma$, the composition $\nabla^\perp$

$$\Gamma(\Sigma; T\Sigma^\perp) \hookrightarrow \Gamma\big(\Sigma; TX|_\Sigma\big) \xrightarrow{\nabla^J} \Gamma\big(\Sigma; T^*\Sigma \otimes_{\mathbb{R}} TX|_\Sigma\big) \xrightarrow{\pi^\perp} \Gamma\big(\Sigma; T^*\Sigma \otimes_{\mathbb{R}} T\Sigma^\perp\big),$$

with ∇^J as in (B.25), is a g-compatible J-linear connection in $T\Sigma^\perp$. Via the isomorphism $\pi: T\Sigma^\perp \longrightarrow \mathcal{N}_X\Sigma$, it induces a J-linear connection $\nabla^{\mathcal{N}}$ in $\mathcal{N}_X\Sigma$ which is compatible with the metric $g^{\mathcal{N}}$ induced via this isomorphism from $g|_{T\Sigma^\perp}$. If the image of the homomorphism

$$T\Sigma^\perp \longrightarrow T^*\Sigma^{0,1} \otimes_{\mathbb{C}} TX|_\Sigma\,, \qquad \xi \longrightarrow \nabla_\xi J - J\nabla_{J\xi} J\,, \tag{B.29}$$

is contained in $T^*\Sigma^{0,1} \otimes_{\mathbb{C}} T\Sigma$, then $\bar\partial_{\nabla^{\mathcal{N}}} = \bar\partial^{\mathcal{N}}_{J;\Sigma}$ and so

$$D^{\mathcal{N}}_{J;\Sigma}\big(\pi(\xi)\big) = \pi\big(\bar\partial_{\nabla^\perp}\xi + A_J(\cdot,\xi)\big)\colon T\Sigma \longrightarrow \mathcal{N}_X\Sigma \qquad \forall\ \xi \in \Gamma(\Sigma; T\Sigma^\perp).$$

This is the case if Σ is a divisor in X, i.e. $\mathrm{rk}_{\mathbb{C}}\mathcal{N} = 1$, since $(\nabla_\zeta J)\xi$ is g-orthogonal to ξ and $J\xi$ for all $\xi, \zeta \in T_xX$ and $x \in X$ by [29, (C.7.1)]. This is also the case if J is compatible with a symplectic form ω on X and $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, as the homomorphism (B.29) is then trivial by [29, (C.7.5)].

C Riemannian geometry estimates

This section is based on [7, Chapter 1] and [10, Section 3] and culminates in a Poincare lemma for closed curves in Proposition C.6 and an expansion for the $\bar\partial$-operator in Proposition C.13. If $u\colon \Sigma \longrightarrow X$ is a smooth map between smooth manifolds and $E \longrightarrow X$ is a smooth vector bundle, let

$$\Gamma(u; E) = \Gamma(\Sigma; u^*E), \qquad \Gamma^1(u; E) = \Gamma(\Sigma; T^*\Sigma \otimes_{\mathbb{R}} u^*E).$$

We denote the subspace of compactly supported sections in $\Gamma(u; E)$ by $\Gamma_c(u; E)$.

An exponential-like map on a smooth manifold X is a smooth map $\exp: TX \longrightarrow X$ such that $\exp|_X = \mathrm{id}_X$ and

$$\mathrm{d}_x \exp = \big(\mathrm{id}_{T_xX}\ \mathrm{id}_{T_xX}\big)\colon T_x(TX) = T_xX \oplus T_xX \longrightarrow T_xX \qquad \forall\ x \in X,$$

where the second equality is the canonical splitting of $T_x(TX)$ into the horizontal and vertical tangent space along the zero section. Any connection ∇ in TX gives rise to a smooth map $\exp^\nabla$: $W \longrightarrow X$ from some neighborhood W of the zero section X in TX; see [7, Section 1.3]. If $\eta\colon TX \longrightarrow \mathbb{R}$ is a smooth function which equals 1 on a neighborhood of X in TX and 0 outside of W, then

$$\exp\colon TX \longrightarrow X, \qquad v \longrightarrow \exp^\nabla\big(\eta(v)v\big),$$

is an exponential-like map. If X is compact, then W can be taken to be all of TX and $\exp = \exp^\nabla$.

If $(X, g, \exp)$ is a Riemannian manifold with an exponential-like map and $x \in X$, let $r_{\exp}(x) \in \mathbb{R}^+$ be the supremum of the numbers $r \in \mathbb{R}$ such that the restriction

$$\exp\colon \big\{v \in T_xX\colon |v| < r\big\} \longrightarrow X$$

is a diffeomorphism onto an open subset of X. Set

$$r^g_{\exp}(x) = \inf\left\{d_g(x,\exp(v))\colon\, v\in T_xX,\ |v|=r_{\exp}(x)\right\} \in \mathbb{R}^+,$$

where d_g is the metric on X induced by g. If $K\subset X$, let

$$r^g_{\exp}(K) = \inf_{x\in K} r^g_{\exp}(x);$$

this number is positive if $\bar{K}\subset X$ is compact.

C.1 Parallel transport

Let $(E,\langle,\rangle,\nabla)\longrightarrow X$ be a vector bundle, real or complex, with an inner-product $\langle,\rangle$ and a metric-compatible connection ∇. If $\alpha\colon (a,b)\longrightarrow X$ is a piecewise smooth curve, denote by

$$\Pi_\alpha\colon E_{\alpha(a)} \longrightarrow E_{\alpha(b)}$$

the parallel-transport map along α with respect to the connection ∇. If $\exp: TX \longrightarrow X$ is an exponential-like map, $x\in X$, and $v\in T_xX$, let

$$\Pi_v\colon E_x \longrightarrow E_{\exp(v)}$$

be the parallel transport along the curve

$$\gamma_v\colon [0,1] \longrightarrow X, \qquad \gamma_v(t) = \exp(tv).$$

If $u\colon [a,b]\times[c,d]\longrightarrow X$ is a smooth map, let

$$\Pi_{\partial u} : E_{u(a,c)} \longrightarrow E_{u(a,c)}$$

be the parallel transport along u restricted to the boundary of the rectangle traversed in the positive direction. If $u\colon \Sigma\longrightarrow X$ is any smooth map, ∇ induces a connection

$$\nabla^u\colon \Gamma(u;E) \longrightarrow \Gamma^1(u;E)$$

in the vector bundle $u^*E\longrightarrow\Sigma$. If α is a smooth curve as above and $\zeta\in\Gamma(\alpha;E)$, let

$$\frac{D}{\mathrm{d}t}\zeta = \nabla^\alpha_{\partial_t}\zeta \in \Gamma(\alpha;E),$$

where ∂_t is the standard unit vector field on $\mathbb{R}$.

Lemma C.1. *If (X,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, for every compact subset $K\subset X$ there exists $C_K\in\mathbb{R}^+$ such that for every smooth map $u\colon [a,b]\times[c,d]\longrightarrow X$ with $\operatorname{Im} u\subset K$*

$$|\Pi_{\partial u} - \mathbb{I}| \le C_K \int_c^d\int_a^b |u_s||u_t|\mathrm{d}s\mathrm{d}t,$$

where the norm of $(\Pi_{\partial u}-\mathbb{I})\in\operatorname{End}(E_{u(a,c)})$ is computed with respect to the inner-product in $E_{u(a,c)}$.

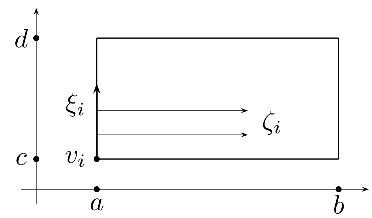

Figure 13: Extending a basis $\{v_i\}$ for $E_{u(a,c)}$ to a frame $\{\zeta_i\}$ over $[a,b]\times[c,d]$

Proof. (1) Choose an orthonormal frame $\{v_i\}$ for $E_{u(a,c)}$. Extend each v_i to

$$\xi_i \in \Gamma\big(u|_{a\times[c,d]};E\big)$$

by parallel-transporting along the curve $t\longrightarrow u(a,t)$ and then to $\zeta_i\in\Gamma(u;E)$ by parallel-transporting $\xi_i(a,t)$ along the curve $s\longrightarrow u(s,t)$; see Figure 13. By construction,

$$\frac{D}{\mathrm{d}s}\zeta_i = 0 \in \Gamma(u;E).$$

Let A be the matrix-valued function on $[a,b]\times[c,d]$ such that

$$\frac{D}{\mathrm{d}t}\zeta_i\Big|_{(s,t)} = \sum_{l=1}^{l=k} A_{il}(s,t)\zeta_l(s,t), \tag{C.1}$$

where k is the rank of E. Note that $A_{ij}(a,t)=0$ and

$$\big\langle \mathcal{R}_\nabla(u_s,u_t)\zeta_i,\zeta_j\big\rangle = \left\langle \frac{D}{\mathrm{d}s}\frac{D}{\mathrm{d}t}\zeta_i - \frac{D}{\mathrm{d}t}\frac{D}{\mathrm{d}s}\zeta_i,\zeta_j\right\rangle = \sum_{l=1}^{l=k}\left\langle\Big(\frac{\partial}{\partial s}A_{il}\Big)\zeta_l,\zeta_j\right\rangle = \frac{\partial}{\partial s}A_{ij}\,, \tag{C.2}$$

where $\mathcal{R}_\nabla$ is the curvature tensor of the connection of ∇. Since K is compact and the image of u is contained in K, it follows that

$$|A_{ij}(b,t)| \le C_K\int_a^b |u_s|_{(s,t)}|u_t|_{(s,t)}\mathrm{d}s. \tag{C.3}$$

(2) The parallel transport of ζ_i along the curves

$$\tau\longrightarrow u(\tau,c), \quad \tau\longrightarrow u(\tau,d), \quad \tau\longrightarrow u(a,\tau)$$

is ζ_i itself. Thus, it remains to estimate the parallel transport of each ζ_i along the curve $\tau\longrightarrow u(b,\tau)$. Let h_{ij} be the SO_k-valued function (U_k-valued function if E is complex) on $[c,d]$ such that

$$h(c)=\mathbb{I}, \qquad \sum_{j=1}^{j=k}\frac{D}{\mathrm{d}t}(h_{ij}\zeta_j)\Big|_{(b,t)} = 0 \ \ \forall\, i,t.$$

The second equation is equivalent to

$$\sum_{j=1}^{j=k}h'_{ij}(t)\zeta_j(b,t) + \sum_{j=1}^{j=k}\sum_{l=1}^{l=k}h_{ij}(t)A_{jl}(b,t)\zeta_l(b,t) = 0 \qquad\Longleftrightarrow\qquad h' = -hA(b,\cdot). \tag{C.4}$$

Since (the real part of) the trace of (A_{ij}) is zero by (C.2), equation (C.4) has a unique solution in SO_k (or U_k) such that $h(c)=\mathbb{I}$. Furthermore, by (C.3)

$$\big|h(d)-\mathbb{I}\big| \le \int_c^d |h'(t)|\mathrm{d}t \le \int_c^d |h||A|\mathrm{d}t \le k^2\int_c^d\!\!\int_a^b C_K|u_s||u_t|\mathrm{d}s\mathrm{d}t. \tag{C.5}$$

Since $\Pi_{\partial\alpha}v_i=\sum_{j=1}^{j=k}h_{ij}(d)v_j$ by the above, the claim follows from equation (C.5). □

Corollary C.2. *If (X,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, for every compact subset $K\subset X$ there exists $C_K\in\mathbb{R}^+$ such that for every smooth closed curve $\alpha\colon [a,b]\longrightarrow X$ with $\mathrm{Im}\,\alpha\subset K$*

$$\big|\Pi_\alpha-\mathbb{I}\big| \le C_K\min\big(\|\mathrm{d}\alpha\|_1,(b-a)\|\mathrm{d}\alpha\|_2^2\big).$$

Proof. Let $\exp\colon TX\longrightarrow X$ be an exponential-like map. Since the group SO_k (or U_k if E is complex) is compact and

$$\|\mathrm{d}\alpha\|_1^2 \le (b-a)\|\mathrm{d}\alpha\|_2^2$$

by Hölder's inequality, it is enough to assume that

$$\|\mathrm{d}\alpha\|_1 \le \min(r^g_{\exp}(K)/2,1).$$

Thus, there exists

$$\widetilde{\alpha}\in C^\infty\big([a,b];T_{\alpha(a)}X\big) \qquad \text{s.t.} \qquad \alpha(t)=\exp(\widetilde{\alpha}(t)), \quad |\widetilde{\alpha}(t)|_{\alpha(a)}<r_{\exp}(\alpha(a)).$$

Define

$$u\colon [0,1]\times[a,b]\longrightarrow K\subset X \qquad \text{by} \quad u(s,t)=\exp\big(s\widetilde{\alpha}(t)\big).$$

Using

$$|\widetilde{\alpha}(t)| \le C_K d_g\big(\alpha(a),\alpha(t)\big) \le C_K\|\mathrm{d}\alpha\|_1\,,$$
$$|\widetilde{\alpha}'(t)| = \big|\{\mathrm{d}_{\widetilde{\alpha}(t)}\exp\}^{-1}(\alpha'(t))\big| \le C_K|\mathrm{d}_t\alpha|\,,$$

we find that

$$u_s(s,t)=\big\{\mathrm{d}_{s\widetilde{\alpha}(t)}\exp\big\}\big(\widetilde{\alpha}(t)\big) \qquad \Longrightarrow \qquad |u_s|_{(s,t)} \le C'_K\|\mathrm{d}\alpha\|_1\,; \tag{C.6}$$
$$u_t(s,t)=s\big\{\mathrm{d}_{s\widetilde{\alpha}(t)}\exp\big\}\big(\widetilde{\alpha}'(t)\big) \qquad \Longrightarrow \qquad |u_t|_{(s,t)} \le C'_K|\mathrm{d}_t\alpha|. \tag{C.7}$$

Thus, by Lemma C.1,

$$\big|\Pi_\alpha-\mathbb{I}\big| = \big|\Pi_{\partial u}-\mathbb{I}\big| \le C_K\int_0^1\!\!\int_a^b |u_s||u_t|\mathrm{d}s\mathrm{d}t \le C'_K\|\mathrm{d}\alpha\|_1^2 \le C'_K(b-a)\|\mathrm{d}\alpha\|_2^2.$$

Since $\|\mathrm{d}\alpha\|_1\le r^g_{\exp}(K)$, it follows that $|\Pi_\alpha-\mathbb{I}|\le C_K\|\mathrm{d}\alpha\|_1$. □

Corollary C.3. *If $(X,g,\exp)$ is a Riemannian manifold with an exponential-like map and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, for every compact subset $K\subset X$ there exists $C_K\in C^\infty(\mathbb{R};\mathbb{R})$ such that for all $x\in K$ and smooth maps $\widetilde{\alpha}\colon(-\epsilon,\epsilon)\longrightarrow T_xX$ and $\xi\colon(-\epsilon,\epsilon)\longrightarrow E_x$*

$$\left|\frac{D}{\mathrm{d}t}\Big(\Pi_{\widetilde{\alpha}(t)}\xi(t)\Big)\Big|_{t=0}-\Pi_{\widetilde{\alpha}(0)}\xi'(0)\right| \le C_K\big(|\widetilde{\alpha}(0)|\big)|\widetilde{\alpha}(0)||\widetilde{\alpha}'(0)||\xi(0)|. \tag{C.8}$$

Proof. Define

$$u\colon [0,1]\times\big[0,\epsilon/2\big] \longrightarrow K\subset X \qquad \text{by} \quad u(s,t)=\exp\big(s\widetilde{\alpha}(t)\big).$$

Let $\{v_i\}$ be an orthonormal basis for E_x. Extend each v_i to

$$\zeta_i\in\Gamma\big(u|_{[0,1]\times t};E\big)$$

by parallel-transporting along the curves $s\longrightarrow u(s,t)$. If

$$\xi(t)=\sum_{i=1}^{i=k} f_i(t)v_i\,,$$

where k is the rank of E, then

$$\begin{aligned}
\Pi_{\widetilde{\alpha}(t)}\xi(t) &= \sum_{i=1}^{i=k} f_i(t)\zeta_i(1,t) \qquad \Longrightarrow\\
\frac{D}{\mathrm{d}t}\Big(\Pi_{\widetilde{\alpha}(t)}\xi(t)\Big)\Big|_{t=0} &= \sum_{i=1}^{i=k} f_i'(0)\zeta_i(1,0)+\sum_{i=1}^{i=k} f_i(0)\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0}\\
&= \Pi_{\widetilde{\alpha}(0)}\xi'(0)+\sum_{i=1}^{i=k} f_i(0)\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0}.
\end{aligned} \tag{C.9}$$

On the other hand, by (C.1), (C.3), and the first identities in (C.6) and (C.7),

$$\begin{aligned}
\Big|\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0} &= \sum_{j=1}^{j=k}\big|A_{ij}(1,0)\big| \le kC_K'\big(|\widetilde{\alpha}(0)|\big)\int_0^1 |u_s|_{(s,0)}|u_t|_{(s,0)}\mathrm{d}s\\
&\le C_K\big(|\widetilde{\alpha}(0)|\big)|\widetilde{\alpha}(0)||\widetilde{\alpha}'(0)|.
\end{aligned} \tag{C.10}$$

The claim follows from (C.9) and (C.10). □

Remark C.4. Note that (C.3) is applied above with K replaced by the compact set

$$\exp\big(\{v\in T_xX\colon\; x\in K,\ |v|\le|\widetilde{\alpha}(0)|\}\big).$$

Thus, the constants $C_K'(|\widetilde{\alpha}(0)|)$ and $C_K(|\widetilde{\alpha}(0)|)$ may depend on $|\widetilde{\alpha}(0)|$. If X is compact, then the first constant does not depend on $|\widetilde{\alpha}(0)|$, since (C.3) can then be applied with $K=X$. The second constant is then also independent of K and $|\widetilde{\alpha}(0)|$ if $\exp=\exp^{\nabla}$ for some connection ∇ in TX. So, in this case, the function C_K in (C.8) can be taken to be a constant independent of K.

C.2 Poincare lemmas

Lemma C.5 (Poincare Inequality). *If $\zeta\colon S^1\longrightarrow\mathbb{R}^k$ is a smooth function such that $\int_0^{2\pi}\zeta(\theta)\mathrm{d}\theta=0$, then*

$$\int_0^{2\pi}|\zeta(\theta)|^2\mathrm{d}\theta \le \int_0^{2\pi}|\zeta'(\theta)|^2\mathrm{d}\theta.$$

Proof. Write

$$\zeta(\theta)=\sum_{n>-\infty}^{n<\infty}\zeta_n e^{\mathrm{i}n\theta}\,;$$

see [42, Section 6.16]. Since ζ integrates to 0, $\zeta_0=0$. Thus,

$$\int_0^{2\pi}|\zeta(\theta)|^2\mathrm{d}\theta = 2\pi\sum_{n>-\infty}^{n<\infty}|\zeta_n|^2 \le 2\pi\sum_{n>-\infty}^{n<\infty}|n\zeta_n|^2 = \int_0^{2\pi}|\zeta'(\theta)|^2\mathrm{d}\theta,$$

as claimed. □

Proposition C.6. *If (X,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, for every compact subset $K\subset X$ there exists $C_K\in\mathbb{R}^+$ with the following property. If $\alpha\in C^\infty(S^1;X)$ is such that $\mathrm{Im}\,\alpha\subset K$ and $\xi,\zeta\in\Gamma(\alpha;E)$, then*

$$\big|\langle\!\langle\nabla_\theta\xi,\zeta\rangle\!\rangle\big| \le \|\nabla_\theta\xi\|_2\|\nabla_\theta\zeta\|_2 + C_K\min\big(\|\mathrm{d}\alpha\|_1,\|\mathrm{d}\alpha\|_2^2\big)\|\xi\|_{2,1}\|\zeta\|_2\,,$$

where $\nabla_\theta\equiv\nabla^\alpha_{\partial_\theta}$ is the covariant derivative with respect to the oriented unit field on S^1 and all the norms are computed with respect to the standard metric on S^1.

Proof. Identify $E_{\alpha(0)}$ with $\mathbb{R}^k$ (or $\mathbb{C}^k$), preserving the metric. Denote by $so(E_{\alpha(0)})\approx so_k$ (or $u(E_{\alpha(0)})\approx u_k$) the Lie algebra of the Lie group $\mathrm{SO}(E_{\alpha(0)})\approx\mathrm{SO}_k$ (or of $\mathrm{U}(E_{\alpha(0)})\approx\mathrm{U}_k$). For each $\chi\in so(E_{\alpha(0)})$ (or $\chi\in u(E_{\alpha(0)})$), let $\mathrm{e}^\chi\in\mathrm{SO}(E_{\alpha(0)})$ (or $\mathrm{e}^\chi\in\mathrm{U}(E_{\alpha(0)})$) be the exponential of χ. Let

$$\Pi_\theta\colon E_{\alpha(0)}\longrightarrow E_{\alpha(\theta)}$$

be the parallel transport along the curve $t\longrightarrow\alpha(t)$ with $t\in[0,\theta]$. By Corollary C.2, there exists $\chi\in so(E_{\alpha(0)})$ (or $\chi\in u(E_{\alpha(0)})$) such that

$$\Pi_{2\pi}=\mathrm{e}^\chi \qquad\text{and}\qquad |\chi|\le C_K\min\big(\|\mathrm{d}\alpha\|_1,\|\mathrm{d}\alpha\|_2^2\big)\,. \tag{C.11}$$

By the first statement in (C.11),

$$\Psi\colon S^1\times E_{\alpha(0)}\longrightarrow\alpha^*E\,,\qquad(\theta,v)\longrightarrow\mathrm{e}^{-\theta\chi/2\pi}\Pi_\theta(v),$$

is a smooth isometry. Let $\Phi_2=\pi_2\circ\Psi^{-1}\colon\alpha^*E\longrightarrow E_{\alpha(0)}$ and

$$\bar\zeta=\frac{1}{2\pi}\int_0^{2\pi}\{\Phi_2\zeta\}(\theta)\mathrm{d}\theta\in E_{\alpha(0)}.$$

By Hölder's inequality and Lemma C.5,

$$\begin{aligned}\big|\langle\!\langle\nabla_\theta\xi,\zeta-\Psi\bar\zeta\rangle\!\rangle\big| &\le \|\nabla_\theta\xi\|_2\|\zeta-\Psi\bar\zeta\|_2\\ &= \|\nabla_\theta\xi\|_2\|\Phi_2\zeta-\bar\zeta\|_2\le\|\nabla_\theta\xi\|_2\|\mathrm{d}(\Phi_2\zeta)\|_2.\end{aligned} \tag{C.12}$$

By the product rule,

$$\begin{aligned}\|\mathrm{d}(\Phi_2\zeta)\|_2 &\le \big\|\mathrm{d}(\Pi^{-1}\zeta)\big\|_2+|\chi/2\pi|\big\|\Pi^{-1}\zeta\big\|_2=\|\nabla_\theta\zeta\|_2+|\chi/2\pi|\|\zeta\|_2\\ &\le \|\nabla_\theta\zeta\|_2+C_K\min\big(\|\mathrm{d}\alpha\|_1,\|\mathrm{d}\alpha\|_2^2\big)\|\zeta\|_2.\end{aligned} \tag{C.13}$$

On the other hand, by integration by parts, we obtain

$$\langle\!\langle\nabla_\theta\xi,\zeta-\Psi\bar\zeta\rangle\!\rangle=\langle\!\langle\nabla_\theta\xi,\zeta\rangle\!\rangle+\langle\!\langle\xi,\nabla_\theta(\Psi\bar\zeta)\rangle\!\rangle. \tag{C.14}$$

Since $\Psi\bar\zeta$ is the parallel transport of $\mathrm{e}^{\theta\chi/2\pi}\bar\zeta$,

$$\begin{aligned}\big|\langle\!\langle\xi,\nabla_\theta(\Psi\bar\zeta)\rangle\!\rangle\big| &\le \|\xi\|_2\|\nabla_\theta(\Psi\bar\zeta)\|_2=\|\xi\|_2|\chi/2\pi|\big\|\Psi\bar\zeta\big\|_2\\ &\le C_K\min\big(\|\mathrm{d}\alpha\|_1,\|\mathrm{d}\alpha\|_2^2\big)\|\xi\|_2\|\zeta\|_2.\end{aligned} \tag{C.15}$$

The claim follows from equations (C.12)-(C.15). □

Let $B_{R,r}\subset\mathbb{R}^2$ denote the open annulus with radii $r<R$ centered at the origin.

Corollary C.7 (of Lemma C.5)**.** *There exists $C\in C^\infty(\mathbb{R};\mathbb{R})$ such that for all $R\in\mathbb{R}^+$*

$$r\in(0,R],\quad \zeta\in C^\infty\big(B_{R,r};\mathbb{R}^k\big),\quad \int_{B_{R,r}}\zeta=0 \qquad\Longrightarrow\qquad \|\zeta\|_1\le C(R/r)R^2\|\mathrm{d}\zeta\|_2.$$

Proof. It is sufficient to assume that $k=1$. Define

$$\xi\colon S^1\longrightarrow\mathbb{R}\qquad\text{by}\qquad \xi(\theta)=\int_r^R\zeta(\rho,\theta)\rho\mathrm{d}\rho.$$

By Hölder's inequality and Lemma C.5,

$$\begin{aligned}\bigg(\int_0^{2\pi}\bigg|\int_r^R\zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|\mathrm{d}\theta\bigg)^2&\le 2\pi\int_0^{2\pi}\big|\xi(\theta)\big|^2\mathrm{d}\theta\le 2\pi\int_0^{2\pi}\big|\xi'(\theta)\big|^2\mathrm{d}\theta\\&\le 2\pi\int_0^{2\pi}\bigg(\int_r^R\big|\mathrm{d}_{(\rho,\theta)}\zeta\big|\rho^2\mathrm{d}\rho\bigg)^2\mathrm{d}\theta\\&\le\frac{\pi R^4}{2}\int_0^{2\pi}\int_r^R\big|\mathrm{d}_{(\rho,\theta)}\zeta\big|^2\rho\mathrm{d}\rho\mathrm{d}\theta=\frac{\pi R^4}{2}\|\mathrm{d}\zeta\|_2^2\,.\end{aligned}\tag{C.16}$$

If the function $\rho\longrightarrow\zeta(\rho,\theta)$ does not change sign on (r,R), then

$$\int_r^R\big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho=\bigg|\int_r^R\zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|.$$

On the other hand, if this function vanishes somewhere on (r,R), then

$$\big|\zeta(\rho,\theta)\big|\le\int_r^R\big|\mathrm{d}_{(t,\theta)}\zeta\big|\mathrm{d}t\ \ \forall\,\rho\quad\Longrightarrow\quad\int_r^R\big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho\le\frac{R^2}{2}\int_r^R\big|\mathrm{d}_{(t,\theta)}\zeta\big|\mathrm{d}t\,.$$

Combining these two cases and using (C.16) and Hölder's inequality, we obtain

$$\begin{aligned}\int_0^{2\pi}\!\!\int_r^R\big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho\mathrm{d}\theta&\le\int_0^{2\pi}\bigg|\int_r^R\zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|\mathrm{d}\theta+\frac{R^2}{2}\int_0^{2\pi}\!\!\int_r^R\big|\mathrm{d}_{(\rho,\theta)}\zeta\big|\mathrm{d}\rho\mathrm{d}\theta\\&\le\frac{\sqrt{\pi}R^2}{\sqrt{2}}\|\mathrm{d}\zeta\|_2+\frac{R^2}{2}\|\mathrm{d}\zeta\|_2\bigg(\int_0^{2\pi}\!\!\int_r^R\rho^{-1}\mathrm{d}\rho\mathrm{d}\theta\bigg)^{1/2}\\&=\sqrt{\frac{\pi}{2}}\Big(1+\sqrt{\ln(R/r)}\Big)R^2\|\mathrm{d}\zeta\|_2\,,\end{aligned}\tag{C.17}$$

as claimed. □

Remark C.8. By Corollary D.7 below, C can in fact be chosen to be a constant function. Corollary C.7 suffices for gluing J-holomorphic maps in symplectic topology, but Corollary D.7 leads to a sharper version of Proposition D.14; see Remark D.13.

C.3 Exponential-like maps and differentiation

Let $(X,g,\exp,\nabla)$ be a smooth Riemannian manifold with an exponential-like map exp and connection ∇ in TX, which is g-compatible, but not necessarily torsion-free. Let

$$T_\nabla\big(\xi(x),\zeta(x)\big)\equiv\big(\nabla_\xi\zeta-\nabla_\zeta\xi-[\xi,\zeta]\big)\big|_x\qquad\forall\,x\in X,\ \xi,\zeta\in\Gamma(X;TX),$$

be the torsion tensor of ∇. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow X$ is a smooth curve and $\xi\in\Gamma(\alpha;TX)$, put

$$\Phi_{\alpha(0)}\Big(\alpha'(0);\xi(0),\frac{D}{\mathrm{d}s}\xi\Big|_{s=0}\Big)=\Pi_{\xi(0)}^{-1}\bigg(\frac{\mathrm{d}}{\mathrm{d}s}\exp\big(\xi(s)\big)\Big|_{s=0}\bigg)=\Pi_{\xi(0)}^{-1}\big(\{\mathrm{d}_{\xi(0)}\exp\}(\xi'(0))\big),$$

where $\xi'(0)\in T_{\xi(0)}(TX)$ is the tangent vector to the curve $\xi\colon(-\epsilon,\epsilon)\longrightarrow TX$ at $s=0$.

Lemma C.9. *If $(X,g,\exp,\nabla)$ is a smooth Riemannian manifold with an exponential-like map and a g-compatible connection, there exists $C\in C^\infty(TX;\mathbb{R})$ such that*

$$\Big|\Phi_x(v;w_0,w_1)-\big(v+w_1-T_\nabla(v,w_0)\big)\Big|\le C(w_0)\big(|v||w_0|^2+|w_0||w_1|\big)$$

for all $x\in X$ and $v,w_0,w_1\in T_xX$.

Proof. Let $\alpha\colon(-\epsilon,\epsilon)\longrightarrow X$ be a smooth curve and $\xi\in\Gamma(\alpha;TX)$ such that

$$\alpha(0)=x,\quad\alpha'(0)=v,\quad\xi(0)=w_0,\quad\frac{D}{\mathrm{d}s}\xi(s)\Big|_{s=0}=w_1.$$

Put

$$\begin{aligned}F_{v,w_0,w_1}(t)&=\frac{\mathrm{d}}{\mathrm{d}s}\exp\big(t\xi(s)\big)\Big|_{s=0}=\{\mathrm{d}_{tw_0}\exp\}\big(\mathrm{d}_{w_0}m_t(\xi'(0))\big),\\ H_{v,w_0,w_1}(t)&=\Pi_{tw_0}\big(v+tw_1-tT_\nabla(v,w_0)\big),\end{aligned}$$

where $m_t\colon TX\longrightarrow TX$ is the scalar multiplication by t. Then,

$$F_{v,w_0,w_1}(0)=\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s)\Big|_{s=0}=v=H_{v,w_0,w_1}(0),$$
$$\frac{D}{\mathrm{d}t}F_{v,w_0,w_1}(t)\Big|_{t=0}=\frac{D}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\exp\big(t\xi(s)\big)\Big|_{t=0}\Big|_{s=0}-T_\nabla\big(v,w_0\big)=w_1-T_\nabla(v,w_0)=\frac{D}{\mathrm{d}t}H_{v,w_0,w_1}(t)\Big|_{t=0};$$

see Corollary C.3. Since

$$F_{\cdot,w_0,\cdot}(t)-H_{\cdot,w_0,\cdot}(t)\in\mathrm{Hom}(T_xX\oplus T_xX,T_{\exp(tw_0)}X),$$

combining the last two equations, we obtain

$$\big|F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)\big|\le C(w_0,t)t^2\big(|v|+|w_1|\big)\quad\forall\ v,w_0,w_1\in T_xX,\ x\in X,\ t\in\mathbb{R},$$

where C is a smooth function on $TX\times\mathbb{R}$. Since

$$F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)=F_{v,tw_0,tw_1}(1)-H_{v,tw_0,tw_1}(1),$$

we conclude that there exists $C\in C^\infty(TX)$ such that

$$\big|F_{v,w_0,w_1}(1)-H_{v,w_0,w_1}(1)\big|\le C(w_0)\big(|w_0|^2|v|+|w_0||w_1|\big)\quad\forall\ v,w_0,w_1\in T_xX,\ x\in X,\tag{C.18}$$

as claimed. $\square$

For any $v,w_0,w_1\in T_xX$, let $\widetilde{\Phi}_x(v;w_0,w_1)=\Phi_x(v;w_0,w_1)-\big(v+w_1-T_\nabla(v,w_0)\big)$.

Corollary C.10. *If $(X,g,\exp,\nabla)$ is a smooth Riemannian manifold with an exponential-like map and a g-compatible connection, there exists $C\in C^\infty(TX\times_X TX;\mathbb{R})$ such that*

$$\begin{split}\Big|\widetilde{\Phi}_x(v;w_0,w_1)-\widetilde{\Phi}_x(v;w_0',w_1')\Big| \\ \le C(w_0,w_0')\Big(\big((|w_0|+|w_0'|)|v|+|w_1|+|w_1'|\big)|w_0-w_0'|+\big(|w_0|+|w_0'|\big)|w_1-w_1'|\Big)\end{split}$$

for all $x\in X$ and $v,w_0,w_1,w_0',w_1'\in T_xX$.

Proof. By the proof of Lemma C.9,

$$\widetilde{\Phi}(v;w_0,w_1)=\widetilde{\Phi}_1(w_0;v)+\widetilde{\Phi}_2(w_0;w_1)$$

for some smooth bundle sections $\widetilde{\Phi}_1,\widetilde{\Phi}_2\colon TX\longrightarrow\pi_{TX}^*\mathrm{Hom}(TX,TX)$ such that

$$\big|\widetilde{\Phi}_1(w_0;\cdot)\big|\le C_1(w_0)|w_0|^2, \qquad \big|\widetilde{\Phi}_2(w_0;\cdot)\big|\le C_2(w_0)|w_0| \qquad \forall\, w_0\in TX.$$

Thus,

$$\begin{aligned}&\big|\widetilde{\Phi}_1(w_0;\cdot)-\widetilde{\Phi}_1(w_0';\cdot)\big|\le C_1'(w_0,w_0')\big(|w_0|+|w_0'|\big)|w_0-w_0'|\\ &\big|\widetilde{\Phi}_2(w_0;\cdot)-\widetilde{\Phi}_2(w_0';\cdot)\big|\le C_2'(w_0,w_0')|w_0-w_0'|\end{aligned} \qquad \forall\, w_0,w_0'\in T_xX.$$

From the linearity of $\widetilde{\Phi}_1(w_0;\cdot)$ and $\widetilde{\Phi}_2(w_0;\cdot)$ in the second input, we conclude that

$$\begin{aligned}\big|\widetilde{\Phi}_1(w_0;v)-\widetilde{\Phi}_1(w_0';v)\big|&\le C_1'(w_0,w_0')\big(|w_0|+|w_0'|\big)|w_0-w_0'||v|,\\ \Big|\widetilde{\Phi}_2(w_0;w_1)-\widetilde{\Phi}_2(w_0;w_1')\Big|&\le C_2'(w_0,w_0')|w_0-w_0'||w_1|+C_2(w_0')|w_0'||w_1-w_1'|.\end{aligned}$$

This establishes the claim. □

C.4 Expansion of the $\bar\partial$-operator

Let (X,J) and $(\Sigma,\mathfrak{j})$ be almost-complex manifolds. If $u\colon\Sigma\longrightarrow X$ is a smooth map, let

$$\begin{gathered}\Gamma(u)=\Gamma(\Sigma;u^*TX), \qquad \Gamma^{0,1}_{J,\mathfrak{j}}(u)=\Gamma\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TX\big),\\ \bar\partial_{J,\mathfrak{j}}u=\frac12\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma^{0,1}_{J,\mathfrak{j}}(u),\end{gathered}$$

as in (B.5). If ∇ is a connection in TX, define

$$D^\nabla_{J,\mathfrak{j};u}\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u) \qquad \text{by} \qquad D^\nabla_{J,\mathfrak{j};u}\xi=\frac12\big(\nabla^u\xi+J\nabla^u_{\mathfrak{j}}\xi\big)-\frac12\big(T_\nabla(\mathrm{d}u,\xi)+JT_\nabla(\mathrm{d}u\circ\mathfrak{j},\xi)\big).$$

If in addition $\exp\colon TX\longrightarrow X$ is an exponential-like map and $\nabla J=0$, define

$$\begin{gathered}\exp_u\colon\Gamma(u)\longrightarrow C^\infty(\Sigma;X), \quad \bar\partial_u,N^\nabla_{\exp}\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u) \qquad \text{by}\\ \big\{\exp_u(\xi)\big\}(z)=\exp\big(\xi(z)\big)\ \ \forall\,z\in\Sigma, \quad \big\{\bar\partial_u\xi\big\}_z(v)=\Pi^{-1}_{\xi(z)}\big(\big\{\bar\partial_{J,\mathfrak{j}}(\exp_u(\xi))\big\}_z(v)\big)\ \ \forall\,z\in\Sigma,\ v\in T_z\Sigma,\\ \bar\partial_u\xi=\bar\partial_{J,\mathfrak{j}}u+D^\nabla_{J,\mathfrak{j};u}\xi+N^\nabla_{\exp}(\xi).\end{gathered}$$

Lemma C.11. *If $(X,J,g,\exp,\nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a g-compatible connection in (TX,J), there exists $C\in C^\infty(TX\times_X TX;\mathbb{R})$ with the following property. If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $u:\Sigma\longrightarrow X$ is a smooth map, and $\xi,\xi'\in\Gamma(u)$, then*

$$\begin{aligned}\Big|\big\{N_{\exp}^{\nabla}(\xi)\big\}_z(v)-\big\{N_{\exp}^{\nabla}(\xi')\big\}_z(v)\Big| \le C\big(\xi(z),\xi'(z)\big)\Big(\big(|\xi(z)|+|\xi'(z)|\big)\big(|\nabla_v(\xi-\xi')|+|\nabla_{\mathfrak{j}v}(\xi-\xi')|\big)\\ +\big((|\mathrm{d}_zu(v)|+|\mathrm{d}_zu(\mathfrak{j}v)|)(|\xi(z)|+|\xi'(z)|)+(|\nabla_v\xi|+|\nabla_{\mathfrak{j}v}\xi|+|\nabla_v\xi'|+|\nabla_{\mathfrak{j}v}\xi|)\big)\big|\xi(z)-\xi'(z)\big|\Big)\end{aligned}$$

for all $z\in\Sigma$, $v\in T_z\Sigma$. Furthermore, $N_{\exp}^{\nabla}(0)=0$.

Proof. Since the connection ∇ commutes with J, so does the parallel transport Π. Thus, with notation as in Section C.3,

$$\big\{N_{\exp}^{\nabla}(\xi)\big\}_z(v)=\frac{1}{2}\Big(\widetilde{\Phi}\big(\mathrm{d}_zu(v);\xi(z),\nabla_v\xi\big)+J\big(u(z)\big)\widetilde{\Phi}\big(\mathrm{d}_zu(\mathfrak{j}v);\xi(z),\nabla_{\mathfrak{j}v}\xi\big)\Big).$$

The claim now follows from Corollary C.10. □

Definition C.12. Let X be a smooth manifold and $(E,\langle,\rangle,\nabla)$ a normed vector bundle with connection over X. If $C_0\in\mathbb{R}^+$, $(\Sigma,\mathfrak{j})$ is an almost complex manifold, and $u:\Sigma\longrightarrow X$ is a smooth map, norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ on $\Gamma(u;E)$ and $\Gamma^1(u;E)$, respectively, are C_0-admissible if for all $\xi\in\Gamma(u;E)$, $\eta\in\Gamma^1(u;E)$, and every continuous function $f:\Sigma\longrightarrow\mathbb{R}$,

$$\|f\eta\|_p\le\|f\|_{C^0}\|\eta\|_p,\quad \|\eta\circ\mathfrak{j}\|_p=\|\eta\|_p,\quad \|\nabla^u\xi\|_p\le\|\xi\|_{p,1},\quad \|\xi\|_{C^0}\le C_0\|\xi\|_{p,1}.$$

Proposition C.13. *If $(X,J,g,\exp,\nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a g-compatible connection in (TX,J), for every compact subset $K\subset X$ there exists $C_K\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $u:\Sigma\longrightarrow K$ is a smooth map, and $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ are C_0-admissible norms on $\Gamma(u;TX)$ and $\Gamma^1(u;TX)$, respectively, then*

$$\big\|N_{\exp}^{\nabla}(\xi)-N_{\exp}^{\nabla}(\xi')\big\|_p\le C_K\big(C_0+\|\mathrm{d}u\|_p+\|\xi\|_{p,1}+\|\xi'\|_{p,1}\big)\big(\|\xi\|_{p,1}+\|\xi'\|_{p,1}\big)\|\xi-\xi'\|_{p,1}$$

for all $\xi,\xi'\in\Gamma(u)$. Furthermore, $N_{\exp}^{\nabla}(0)=0$. If the g-ball $B_{g;\delta}(u(z))$ of radius δ around $f(z)$ for some $z\in\Sigma$ is isomorphic to an open subset of $\mathbb{C}^n$ and $|\xi(z)|<\delta$, then $\{N_{\exp}^{\nabla}\xi\}_z=0$.

Proof. The first two statements follow from Lemma C.11 and Definition C.12. The last claim is clear from the definition of $N_{\exp}^{\nabla}$. □

Remark C.14. As the notation suggests, one possibility for the norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ is the usual Sobolev L^p_1 and L^p-norms with respect to some Riemannian metric on Σ, where $p>\dim_{\mathbb{R}}\Sigma$. Another natural possibility in the $\dim_{\mathbb{R}}\Sigma=2$ case is the modified Sobolev norms introduced in [26, Section 3]; these are particularly suited for gluing pseudo-holomorphic curves. By Proposition D.10 below, in the $\dim_{\mathbb{R}}\Sigma=2$ case the constant C_0 itself is a function of $\|\mathrm{d}u\|_p$ only for either of these two choices of norms.

Remark C.15. By Proposition C.13, the operator $D^{\nabla}_{J,\mathfrak{j};u}$ defined above is a linearization of the $\bar\partial$-operator on the space of smooth maps to X at u. If ∇' is any connection in TX, the connection

$$\nabla:\Gamma(X;TX)\longrightarrow\Gamma(X;T^*X\otimes_{\mathbb{R}}TX),\quad \nabla_v\xi=\frac{1}{2}\Big(\nabla'_v\xi-J\nabla'_v(J\xi)\Big)\quad\forall\,v\in TX,\ \xi\in\Gamma(X;TX),$$

is J-compatible. If in addition ∇' and J are compatible with a Riemannian metric g on X, then so is ∇. If ∇' is also the Levi-Civita connection of the metric g (i.e. $T_{\nabla'}=0$),

$$T_\nabla(v,w)=\frac{1}{2}\big(J(\nabla'_w J)v-J(\nabla'_v J)w\big)\qquad \forall\, v,w\in T_xX,\ x\in X.$$

If the 2-form $\omega(\cdot,\cdot)\equiv g(J\cdot,\cdot)$ is closed as well, then

$$\nabla'_{Jv}J=-J\nabla'_v J\qquad \forall\, v\in TX$$

by [29, (C.7.5)] and thus

$$T_\nabla(v,w)=-\frac{1}{4}\big(J(\nabla'_v J)w-J(\nabla'_w J)v-(\nabla'_{Jv}J)w+(\nabla'_{Jw}J)v\big)=-A_J(v,w)\quad \forall\, v,w\in T_xX,\ x\in X,$$

where A_J is the Nijenhuis tensor of J as in (2.3). The operator $D^\nabla_{J,\mathfrak{j};u}$ then becomes

$$D^\nabla_{J,\mathfrak{j};u}\colon \Gamma(u)\longrightarrow \Gamma^{0,1}_{J,\mathfrak{j}}(u),\qquad D^\nabla_{J,\mathfrak{j};u}\xi=\bar\partial_{\nabla^u}\xi+A_J(\partial_{J,\mathfrak{j}}u,\xi),\tag{C.19}$$

where

$$\begin{aligned}\bar\partial_{\nabla^u}\xi&=\frac{1}{2}\big(\nabla^u\xi+J\nabla^u_{\mathfrak{j}}\xi\big)\in\Gamma^{0,1}_{J,\mathfrak{j}}(u),\\ \partial_{J,\mathfrak{j}}u&=\frac{1}{2}\big(\mathrm{d}u-J\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma\big(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}u^*TX\big).\end{aligned}$$

This agrees with [29, (3.1.5)], since the Nijenhuis tensor of J is defined to be $-4A_J$ in [29, p18].

D Sobolev and elliptic inequalities

This appendix refines, in the $n=2$ case, the proofs of Sobolev Embedding Theorems given in [32] to obtain a C^0-estimate in Proposition D.10 and elliptic estimates for the $\bar\partial$-operator in Propositions D.14 and D.16. If $R,r\in\mathbb{R}$, let

$$B_R=\big\{x\in\mathbb{R}^2\colon |x|<R\big\},\qquad B_{R,r}=B_R-\bar B_r\,,\qquad \widetilde{B}_{R,r}=B_R-B_r\,.$$

D.1 Euclidean case

If ξ is an $\mathbb{R}^k$-valued function defined on a subset B of $\mathbb{R}^2$, let $\mathrm{supp}_{\mathbb{R}^2}(\xi)$ be the closure of $\mathrm{supp}(\xi)\subset B$ in $\mathbb{R}^2$. If U is an open subset of $\mathbb{R}^2$, $\xi\in C^\infty(U;\mathbb{R}^k)$, and $p\geq 1$, let

$$\|\xi\|_p\equiv\bigg(\int_U|\xi|^p\bigg)^{1/p},\qquad \|\xi\|_{p,1}\equiv\|\xi\|_p+\|\mathrm{d}\xi\|_p\,,$$

be the usual Sobolev norms of ξ.

Lemma D.1. *For every bounded convex domain $\mathcal{D}\subset\mathbb{R}^2$, $\xi\in C^\infty(\mathcal{D};\mathbb{R}^k)$, and $x\in\mathcal{D}$,*

$$\big|\xi_{\mathcal{D}}-\xi(x)\big|\leq\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y,$$

where $2r_0$ is the diameter of $\mathcal{D}$, $|\mathcal{D}|$ is the area of $\mathcal{D}$, and

$$\xi_{\mathcal{D}}=\frac{1}{|\mathcal{D}|}\bigg(\int_{\mathcal{D}}\xi(y)\mathrm{d}y\bigg)$$

is the average value of ξ on $\mathcal{D}$.

Proof. For any $y\in\mathcal{D}$,

$$\xi(y)-\xi(x)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\xi\big(x+t(y-x)\big)\mathrm{d}t=\int_0^1\mathrm{d}_{x+t(y-x)}\xi(y-x)\mathrm{d}t.$$

Putting $g(z)=|\mathrm{d}_z\xi|$ if $z\in\mathcal{D}$ and $g(z)=0$ otherwise, we obtain

$$\big|\xi_{\mathcal{D}}-\xi(x)\big|\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}|\xi(y)-\xi(x)|\mathrm{d}y\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}\int_0^\infty g\big(x+t(y-x)\big)|y-x|\mathrm{d}t\mathrm{d}y.$$

Rewriting the last integral in polar coordinates (r,θ) centered at x, we obtain

$$\begin{aligned}\big|\xi_{\mathcal{D}}-\xi(x)\big|&\le\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(tr,\theta)r^2\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\\&=\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(t,\theta)r\mathrm{d}t\mathrm{d}r\mathrm{d}\theta=\frac{2r_0^2}{|\mathcal{D}|}\int_0^{2\pi}\int_0^\infty g(t,\theta)\mathrm{d}t\mathrm{d}\theta\\&=\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y.\end{aligned}$$

This establishes the claim. □

Corollary D.2. *For every $p>2$, there exists $C_p>0$ such that*

$$r\in\big[0,R/2\big],\quad\xi\in C^\infty(B_{R,r};\mathbb{R}^k)\qquad\Longrightarrow\qquad\big|\xi(x)-\xi(y)\big|\le C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\quad\forall\,x,y\in B_{R,r}\,.$$

Proof. For any $x\in B_{R,r}$, put

$$\mathcal{D}_x=\big\{y\in B_{R,r}:\langle x,|x|y-rx\rangle>0\big\}.$$

If $x\neq0$, $\mathcal{D}_x$ is the part of the annulus on the same side of the line $\langle x,y-rx/|x|\rangle=0$ as x; see Figure 14. In particular,

$$\mathrm{diam}(\mathcal{D}_x)\le2R\,,\qquad|\mathcal{D}_x|\ge\Big(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\Big)R^2.$$

Thus, by Lemma D.1 and Hölder's inequality,

$$\begin{aligned}\big|\xi(x)-\xi_{\mathcal{D}_x}\big|&\le12\int_{y\in\mathcal{D}_x}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y\\&\le12\bigg(\int_{y\in B_{2R}(x)}|y-x|^{-\frac{p}{p-1}}\bigg)^{\frac{p-1}{p}}\|\mathrm{d}\xi\|_p\le C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p,\end{aligned}\tag{D.1}$$

since $\frac{p}{p-1}<2$. Let

$$x_\pm=\big(\pm(R-r)/2,0\big),\qquad y_\pm=\big(0,\pm(R-r)/2\big).$$

Since each of the convex regions $\mathcal{D}_{x_\pm}$ intersects $\mathcal{D}_{y_+}$ and $\mathcal{D}_{y_-}$ and $\mathcal{D}_x$ intersects at least one (in fact precisely two if $r\neq0$) of these four convex regions for every $x\in B_{R,r}$,

$$\big|\xi(x)-\xi(y)\big|\le8C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\quad\forall\,x,y\in B_{R,r}$$

by (D.1) and triangle inequality. □

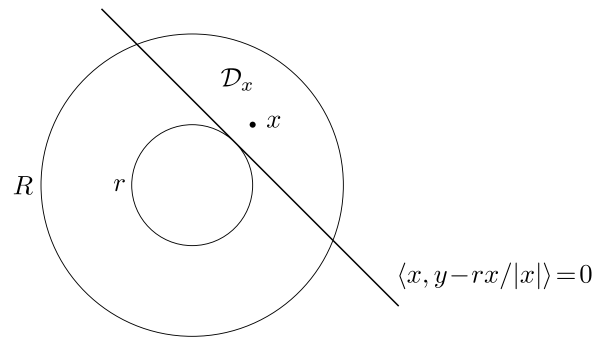

Figure 14: A convex region $\mathcal{D}_x$ of the annulus $\mathcal{D}_{R,r}$ containing x

Corollary D.3 (Sobolev Embedding Theorem). *For every $p>2$, there exists $C_p \in C^\infty(\mathbb{R}^+;\mathbb{R})$ such that*

$$r \in [0, R/2], \quad \xi \in C^\infty(B_{R,r};\mathbb{R}^k) \quad \Longrightarrow \quad \|\xi\|_{C^0} \le C_p(R)\|\xi\|_{p,1}.$$

Proof. By Corollary D.2 and Hölder's inequality, for every $x \in B_{R,r}$

$$\begin{aligned} |\xi(x)| &\le \big|\xi_{B_{R,r}}\big| + C_p R^{\frac{p-2}{p}} \|\mathrm{d}\xi\|_p \le \frac{1}{|B_{R,r}|}\|\xi\|_1 + C_p R^{\frac{p-2}{p}} \|\mathrm{d}\xi\|_p \\ &\le |B_{R,r}|^{-\frac{1}{p}} \|\xi\|_p + C_p R^{\frac{p-2}{p}} \|\mathrm{d}\xi\|_p \le (1+C_p) R^{-\frac{2}{p}} \big(\|\xi\|_p + R\|\mathrm{d}\xi\|_p\big). \end{aligned} \tag{D.2}$$

This implies the claim. □

By Example 3.15, the bound of Corollary D.3 does not hold for $p=2$.

Lemma D.4. *For all $R>0$ and $r \in [0, R)$,*

$$\zeta \in C^\infty(B_{R,r};\mathbb{R}^k), \quad \mathrm{supp}_{\mathbb{R}^2}(\zeta) \subset \widetilde{B}_{R,r} \quad \Longrightarrow \quad \|\zeta\|_2 \le \|\mathrm{d}\zeta\|_1.$$

Proof. Such a function ζ can be viewed as a function on the complement of the ball B_r in $\mathbb{R}^2$. Since ζ vanishes at infinity, for any $(x,y) \in B_{R,r}$

$$\zeta(x,y) = \begin{cases} \int_{-\infty}^x \zeta_s(s,y)\mathrm{d}s, & \text{if } x \le 0; \\ -\int_x^\infty \zeta_s(s,y)\mathrm{d}s, & \text{if } x \ge 0; \end{cases} \qquad \zeta(x,y) = \begin{cases} \int_{-\infty}^y \zeta_t(x,t)\mathrm{d}t, & \text{if } y \le 0; \\ -\int_y^\infty \zeta_t(x,t)\mathrm{d}t, & \text{if } y \ge 0. \end{cases}$$

Taking the absolute value in these equations, we obtain

$$\big|\zeta(x,y)\big| \le \int_{-\infty}^\infty \big|\mathrm{d}_{(s,y)}\zeta\big|\mathrm{d}s \quad \text{and} \quad \big|\zeta(x,y)\big| \le \int_{-\infty}^\infty \big|\mathrm{d}_{(x,t)}\zeta\big|\mathrm{d}t, \tag{D.3}$$

where we formally set ζ and $\mathrm{d}\zeta$ to be zero on the smaller disk. Multiplying the two inequalities in (D.3) and integrating with respect to x and y, we conclude

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \big|\zeta(x,y)\big|^2 \mathrm{d}x\mathrm{d}y \le \Big(\int_{-\infty}^\infty \int_{-\infty}^\infty \big|\mathrm{d}_{(x,y)}\zeta\big|\mathrm{d}x\mathrm{d}y\Big)^2,$$

as claimed. □

Corollary D.5. *For all $p,q\geq 1$ with $1-2/p\geq -2/q$, there exists $C_{p,q}\in\mathbb{R}^+$ such that*

$$r\in[0,R),\quad \xi\in C^\infty(B_{R,r};\mathbb{R}^k),\quad \mathrm{supp}_{\mathbb{R}^2}(\xi)\subset\widetilde{B}_{R,r}\quad\Longrightarrow\quad \|\xi\|_q\leq C_{p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\|\mathrm{d}\xi\|_p.$$

Proof. We can assume that $k=1$. For $\epsilon>0$, let $\zeta_\epsilon=(\xi^2+\epsilon)^{\frac{q}{4}}-\epsilon^{\frac{q}{4}}$. By Lemma D.4 and Hölder's inequality,

$$\begin{aligned}\|\xi\|_q^q&\leq\big\|\zeta_\epsilon+\epsilon^{\frac{q}{4}}\big\|_2^2\leq 2\|\mathrm{d}\zeta_\epsilon\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2=2\Big\|\frac{q}{2}(\xi^2+\epsilon)^{\frac{q}{4}-1}\xi\mathrm{d}\xi\Big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\\&\leq q^2\big\|(\xi^2+\epsilon)^{\frac{q}{4}-\frac{1}{2}}\mathrm{d}\xi\big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\leq q^2\|\mathrm{d}\xi\|_p^2\big\|(\xi^2+\epsilon)^{\frac{q-2}{4}}\big\|_{\frac{p}{p-1}}^2+2\epsilon^{\frac{q}{2}}\pi R^2.\end{aligned}\tag{D.4}$$

Note that

$$1-\frac{2}{p}=-\frac{2}{q}\quad\Longrightarrow\quad\frac{q-2}{4}\frac{p}{p-1}=\frac{q-2}{4}\frac{2q}{q-2}=\frac{q}{2}.$$

Thus, letting ϵ go to zero in (D.4), we obtain

$$\|\xi\|_q^q\leq q^2\|\mathrm{d}\xi\|_p^2\|\xi\|_q^{q-2}\quad\Longrightarrow\quad\|\xi\|_q\leq q\|\mathrm{d}\xi\|_p.$$

The case $1-\frac{2}{p}>-\frac{2}{q}$ follows by Hölder's inequality. □

Remark D.6. By Hölder's inequality, the constant $C_{p,q}$ can be taken to be

$$C_{p,q}=\max(2,q)\pi^{\frac{1}{2}\left(1-\frac{2}{p}+\frac{2}{q}\right)}.$$

Corollary D.7 (of Lemmas D.1, D.4)**.** *There exists $C>0$ such that for all $R\in\mathbb{R}^+$*

$$r\in[0,R],\quad\zeta\in C^\infty\big(B_{R,r};\mathbb{R}^k\big),\quad\int_{B_{R,r}}\zeta=0\qquad\Longrightarrow\qquad\|\zeta\|_1\leq CR^2\|\mathrm{d}\zeta\|_2.$$

Proof. (1) If $\zeta\in C^\infty(B_{R,r};\mathbb{R}^k)$ integrates to 0 over its domain, then so does the function

$$\widetilde{\zeta}\in C^\infty\big(B_{1,r/R};\mathbb{R}^k\big),\qquad\widetilde{\zeta}(z)=\zeta(Rz).$$

Furthermore, $\|\widetilde{\zeta}\|_1=\|\zeta\|_1/R^2$ and $\|\mathrm{d}\widetilde{\zeta}\|_2=\|\mathrm{d}\zeta\|_2$. Thus, it is sufficient to prove the claim for $R=1$.

(2) If $r=0$, for some open half-disk $\mathcal{D}\subset B_{1,0}$

$$\int_{\mathcal{D}}\zeta=0,\qquad\big\|\zeta|_{\mathcal{D}}\big\|_1\geq\frac{1}{2}\|\zeta\|_1\,.\tag{D.5}$$

By the first condition, Lemma D.1, and Hölder's inequality

$$\big\|\zeta|_{\mathcal{D}}\big\|_1\leq\frac{4}{\pi}\int_{\mathcal{D}}\int_{\mathcal{D}}|\mathrm{d}_y\zeta||y-x|^{-1}\mathrm{d}y\mathrm{d}x\leq 16\int_{\mathcal{D}}|\mathrm{d}_y\zeta|\mathrm{d}y\leq 8\sqrt{2\pi}\|\mathrm{d}\zeta\|_2\,.$$

Along with the second assumption in (D.5), this implies the claim for $r=0$ with $C=16\sqrt{2\pi}$.

(3) Let $\beta\colon\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\beta(t)=\begin{cases}1, & \text{if } t\leq 1/2;\\ 0, & \text{if } t\geq 1.\end{cases}$$

It remains to prove the claim for all $r>0$ and $R=1$. By (C.17), we can assume that

$$r \le \frac{1}{48\sqrt{3\pi}\|\beta'\|_{C^0}} < \frac{1}{96\sqrt{3\pi}}\,. \tag{D.6}$$

We first consider the case

$$\big\|\zeta|_{B_{2r,r}}\big\|_1 \ge \frac{1}{25}\|\zeta\|_1. \tag{D.7}$$

Using polar coordinates, define $\widetilde{\zeta}\in C^\infty(B_{1,r};\mathbb{R}^k)$ by

$$\widetilde{\zeta}(\rho,\theta) = \beta(\rho)\zeta(\rho,\theta).$$

By Hölder's inequality and Lemma D.4,

$$\big\|\zeta|_{B_{2r,r}}\big\|_1 \le \sqrt{3\pi}r\|\widetilde{\zeta}\|_2 \le \sqrt{3\pi}r\|\mathrm{d}\widetilde{\zeta}\|_1 \le \sqrt{3\pi}r\big(\|\mathrm{d}\zeta\|_1 + \|\beta'\|_{C^0}\big\|\zeta|_{B_{1,1/2}}\big\|_1\big).$$

Along with the assumptions (D.6) and (D.7), this implies the bound with

$$C = 25\frac{\sqrt{3\pi}r}{1-24\sqrt{3\pi}\|\beta\|_{C^0}r} \le \frac{25}{48}\,.$$

Finally, suppose

$$\big\|\zeta|_{B_{2r,r}}\big\|_1 \le \frac{1}{25}\|\zeta\|_1. \tag{D.8}$$

Split the annulus $B_{1,r}$ into 3 wedges of equal area; split each wedge into a large convex outer portion and a small inner portion by drawing the line segment tangent to the circle of radius r and with the end points on the sides of the wedges $2r$ from the center as in Figure 15. By (D.8),

$$A \equiv \big\|\zeta|_{\mathcal{D}_+}\big\|_1 \ge \frac{8}{25}\|\zeta\|_1 \tag{D.9}$$

for the outer piece $\mathcal{D}_+$ of some wedge $\mathcal{D}$. If

$$\bigg|\int_{\mathcal{D}_+}\zeta\bigg| \le \frac{3}{10}A\,,$$

then by Lemma D.1, (D.6), and Hölder's inequality,

$$\begin{aligned}
A &\le \frac{3}{10}A + \frac{2\left(\frac{\sqrt{3}}{2}\right)^2}{\frac{\pi}{3}\left(1-\left(\frac{1}{96\sqrt{3\pi}}\right)^2\right)}\int_{\mathcal{D}_+}\int_{\mathcal{D}_+}|\mathrm{d}_y\zeta||y-x|^{-1}\mathrm{d}y\mathrm{d}x\\
&\le \frac{3}{10}A + \frac{9}{2\pi}\cdot\frac{7\sqrt{2}}{9}\cdot 2\pi\sqrt{3}\int_{\mathcal{D}}|\mathrm{d}_y\zeta|\mathrm{d}y \le \frac{3}{10}A + 7\sqrt{2\pi}\|\mathrm{d}\zeta\|_2\,.
\end{aligned}$$

Along with the assumption (D.9), this implies the bound with $C=125\sqrt{2\pi}/4$. If

$$\bigg|\int_{\mathcal{D}_+}\zeta\bigg| \ge \frac{3}{10}A\,,$$

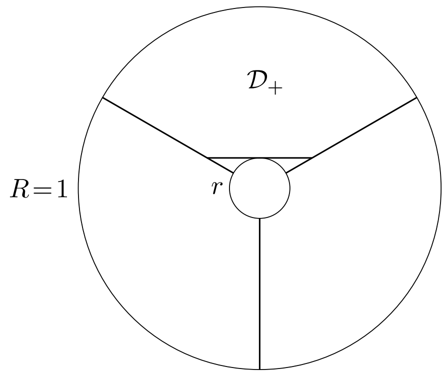

Figure 15: A large convex region $\mathcal{D}_+$ of an annulus $\mathcal{D}$

then by (D.8), (D.9), and (C.16),

$$\begin{aligned} A \le \big\|\xi|_{\mathcal{D}}\big\|_1 \le \big\|\zeta|_{\mathcal{D}}\big\|_1 - \bigg|\int_{\mathcal{D}}\zeta\bigg| + \int_0^{2\pi}\bigg|\int_r^1 \zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|\mathrm{d}\theta \\ \le \Big(A+\frac{1}{8}A\Big) - \Big(\frac{3}{10}A-\frac{1}{8}A\Big) + \sqrt{\frac{\pi}{2}}\|\mathrm{d}\zeta\|_2 = \frac{19}{20}A + \sqrt{\frac{\pi}{2}}\|\mathrm{d}\zeta\|_2\,. \end{aligned}$$

Along with the assumption (D.9), this implies the bound with $C=125\sqrt{2\pi}/4$. Since β can be chosen so that $\|\beta'\|_{C^0}<3$ (actually arbitrarily close to 2), comparing with (C.17) for $R/r=144\sqrt{3\pi}$ we conclude that the claim holds with $C=125\sqrt{2\pi}/4$ for all r. $\square$

D.2 Bundle sections along smooth maps

Let (X,g) be a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ a normed vector bundle with connection over X. If $u\in C^\infty(\widetilde{B}_{R,r};X)$, $\xi\in\Gamma(u;E)$, and $p\ge1$, let

$$\|\xi\|_p \equiv \bigg(\int_{\widetilde{B}_{R,r}}|\xi|^p\bigg)^{1/p}, \qquad \|\xi\|_{p,1}\equiv\|\xi\|_p+\|\nabla^u\xi\|_p\,.$$

Lemma D.8. *If (X,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, and $p,q\ge1$ are such that $1-2/p\ge-2/q$, for every compact subset $K\subset X$ there exists $C_{K;p,q}\in\mathbb{R}^+$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R)$, $u\in C^\infty(\widetilde{B}_{R,r};X)$ is such that* $\operatorname{Im}u\subset K$, *and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_q \le C_{K;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\big(\|\nabla^u\xi\|_p+\|\xi\otimes\mathrm{d}u\|_p\big).$$

Proof. Let $\exp\colon TX\longrightarrow X$ be an exponential-like map and $\{U_i\colon i\in[N]\}$ a finite open cover of K such that the g-diameter of each set U_i is at most $r^g_{\exp}(K)/2$. Let $\{W_i\colon i\in[N]\}$ be an open cover of K such that $\overline{W}_i\subset U_i$. Choose smooth functions $\eta_i\colon X\longrightarrow[0,1]$ such that $\eta_i=1$ on W_i and $\eta_i=0$ outside of U_i. For each $i\in[N]$, pick $x_i\in W_i$. For each $z\in u^{-1}(U_i)\subset\widetilde{B}_{R,r}$, define $\widetilde{u}_i(z)\in T_{x_i}X$ and $\xi_i(z)\in E_{x_i}$ by

$$\exp_{x_i}\widetilde{u}_i(z)=u(z),\quad |\widetilde{u}_i(z)|<r_{\exp}(x_i); \qquad \Pi_{\widetilde{u}_i(z)}\xi_i(z)=\xi(z).$$

For any $z\in B_{R,r}$, put $\widetilde{\xi}_i(z)=\eta_i(u(z))\xi_i(z)$. Since $\widetilde{\xi}_i\in C_c^\infty(\widetilde{B}_{R,r};E_{x_i})$, by Corollary D.5 there exists $C_{i;p,q}>0$ such that

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_q = \big\|\widetilde{\xi}_i|_{u^{-1}(W_i)}\big\|_q \le \|\widetilde{\xi}_i\|_q \le C_{i;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\|\mathrm{d}\widetilde{\xi}_i\|_p\,. \tag{D.10}$$

Since $\mathrm{d}\widetilde{\xi}_i=(\mathrm{d}\eta_i\circ\mathrm{d}u)\xi_i+(\eta\circ u)\mathrm{d}\xi_i$ on $u^{-1}(U_i)$ and vanishes outside of $u^{-1}(U_i)$,

$$\|\mathrm{d}\widetilde{\xi}_i\|_p\le\big\|\mathrm{d}\xi_i|_{u^{-1}(U_i)}\big\|_p+C_i\|\xi_i\otimes\mathrm{d}u\|_p. \tag{D.11}$$

On the other hand, by Corollary C.3, if $u(z)\in U_i$

$$\Big|\nabla^u\xi|_z-\Pi_{\widetilde{u}_i(z)}\circ\mathrm{d}_z\xi_i\Big|\le C_K|\mathrm{d}_zu||\xi(z)|. \tag{D.12}$$

Combining equations (D.10)-(D.12), we obtain

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_q\le\widetilde{C}_{i;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\big(\|\xi\|_{p,1}+\|\xi\otimes\mathrm{d}u\|_p\big).$$

The claim follows by summing the last inequality over all i. □

Lemma D.9. *If (X,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, and $p>2$, for every compact subset $K\subset X$ there exists $C_{K;p}\in C^\infty(\mathbb{R}^+;\mathbb{R})$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R/2]$, $u\in C^\infty(B_{R,r};X)$ is such that $\mathrm{Im}\,u\subset K$, and $\xi\in\Gamma(u;E)$, then*

$$\|\xi\|_{C^0}\le C_{K;p}(R)\big(\|\xi\|_{p,1}+\|\xi\otimes\mathrm{d}u\|_p\big).$$

Proof. We continue with the setup in the proof of Lemma D.8. By Corollary D.3,

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_{C^0}\le\|\widetilde{\xi}_i\|_{C^0}\le C_{i;p}(R)\|\widetilde{\xi}_i\|_{p,1}\le C_{i;p}(R)\big(\big\|\xi|_{u^{-1}(U_i)}\big\|_p+\|\mathrm{d}\widetilde{\xi}_i\|_p\big).$$

As above, we obtain

$$\|\mathrm{d}\widetilde{\xi}_i\|_p\le C_i\big(\|\nabla^u\xi\|_p+\|\xi\otimes\mathrm{d}u\|_p\big),$$

and the claim follows. □

Proposition D.10. *If (X,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over X, and $p>2$, for every compact subset $K\subset X$ there exists $C_{K;p}\in C^\infty(\mathbb{R}^+\times\mathbb{R};\mathbb{R})$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R/2]$, $u\in C^\infty(B_{R,r};X)$ is such that $\mathrm{Im}\,u\subset K$, and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_{C^0}\le C_{K;p}\big(R,\|\mathrm{d}u\|_p\big)\|\xi\|_{p,1}.$$

The same statement holds if $B_{R,r}$ is replaced by a fixed compact Riemann surface (Σ,g_Σ).

Proof. By Lemma D.9 applied with $\widetilde{p}=(p+2)/2$ and Hölder's inequality,

$$\|\xi\|_{C^0}\le C_{K;\widetilde{p}}(R)\big(\|\xi\|_{\widetilde{p},1}+\|\xi\otimes\mathrm{d}u\|_{\widetilde{p}}\big)\le\widetilde{C}_{K;\widetilde{p}}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_1}\big), \tag{D.13}$$

where $q_1=p(p+2)/(p-2)$. If $q_1\le p$, then the proof is complete. Otherwise, apply Lemma D.8 with $p_1=2q_1/(q_1+2)$ and Hölder's inequality:

$$\|\xi\|_{q_1}\le C_{K;p_1,q_1}(R)\big(\|\xi\|_{p_1,1}+\|\xi\otimes\mathrm{d}u\|_{p_1}\big)\le C_{K;1}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_2}\big), \tag{D.14}$$

where $q_2=pp_1/(p-p_1)$. If $q_2\le p$, then the claim follows from equations (D.13) and (D.14). Otherwise, we can continue and construct sequences $\{p_i\},\{q_i\},\{C_{K;i}\}$ such that

$$p_i=\frac{2q_i}{q_i+2},\quad q_{i+1}=\frac{pp_i}{p-p_i}; \tag{D.15}$$

$$\|\xi\|_{q_i}\le C_{K;i}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_{i+1}}\big). \tag{D.16}$$

The recursion (D.15) implies that

$$q_{i+1}=\frac{2p}{2p+(p-2)q_i}q_i \quad\Longrightarrow\quad \text{if } q_i>0, \text{ then } \ 0<q_{i+1}<q_i.$$

Thus, if $q_i>2$ for all i, then the sequence $\{q_i\}$ must have a limit $q\geq 2$ with

$$q=\frac{2p}{2p+(p-2)q}q \quad\Longrightarrow\quad (p-2)q=0 \quad\Longrightarrow\quad q=0,$$

since $p>2$ by assumption. Thus, $q_N\leq p$ for N sufficiently large and the first claim follows from (D.13) and the equations (D.16) with i running from 1 to N, where N is the smallest integer such that $q_{N+1}\leq p$. The second claim follows immediately from the first. □

D.3 Elliptic estimates

If $A_1=B_{R_1,r_1}$ and $A_2=\bar{B}_{R_2,r_2}$ are two annuli in $\mathbb{R}^2$, we write $A_2\Subset_\delta A_1$ if $R_1-R_2>\delta$ and $r_2-r_1\geq\delta$.

Lemma D.11. *For any $\delta>0$, $p\geq 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$ and $\xi\in C^\infty(A_1;\mathbb{C}^k)$,*

$$\big\|\xi|_{A_2}\big\|_{p,1}\leq C_{\delta,p}(A_1)\big(\|\bar\partial\xi\|_p+\|\mathrm{d}\xi\|_2+\|\xi\|_1\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. We can assume that A_2 is the maximal annulus such that $A_2\Subset_\delta A_1$. Let $\eta\colon A_1\longrightarrow[0,1]$ be a compactly supported smooth function such that $\eta|_{A_2}=1$. By the fundamental elliptic inequality for the $\bar\partial$-operator on S^2 [29, Lemma C.2.1],

$$\begin{aligned}\big\|\xi|_{A_2}\big\|_{p,1}\leq\|\eta\xi\|_{p,1}&\leq C_p(A_1)\big(\|\bar\partial(\eta\xi)\|_p+\|\eta\xi\|_p\big)\\&\leq C_p(A_1)\big(\|\bar\partial\xi\|_p+\|(\mathrm{d}\eta)\xi\|_p+\|\eta\xi\|_p\big).\end{aligned}\tag{D.17}$$

By Corollary D.5 with $(p,q)=(2,p)$ and $(p,q)=(1,2)$ and Hölder's inequality,

$$\begin{aligned}\|\eta\xi\|_p&\leq C_p(A_1)\|\mathrm{d}(\eta\xi)\|_2\leq C_p(A_1)\big(\|\mathrm{d}\xi\|_2+\|(\mathrm{d}\eta)\xi\|_2\big)\\&\leq\widetilde{C}_p(A_1)\big(\|\mathrm{d}\xi\|_2+\|\mathrm{d}((\mathrm{d}\eta)\xi)\|_1\big)\leq\widetilde{C}_{p,\delta}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\mathrm{d}\xi\|_1+\|\xi\|_1\big)\\&\leq C_{\delta,p}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\xi\|_1\big).\end{aligned}\tag{D.18}$$

Similarly,

$$\|(\mathrm{d}\eta)\xi\|_p\leq C_{\delta,p}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\xi\|_1\big).\tag{D.19}$$

The claim follows by plugging (D.18) and (D.19) into (D.17). □

Corollary D.12. *For any $\delta>0$, $p\geq 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$, and $\xi\in C^\infty(A_1;\mathbb{C}^n)$,*

$$\|\mathrm{d}\xi|_{A_2}\|_p\leq C_{\delta,p}(A_1)\big(\|\bar\partial\xi\|_p+\|\mathrm{d}\xi\|_2\big).$$

Proof. With $|A_1|$ denoting the area of A_1, let

$$\bar\xi=\frac{1}{|A_1|}\int_{A_1}\xi$$

be the average value of ξ. By Lemma D.11,

$$\begin{aligned}\|\mathrm{d}\xi|_{A_2}\|_p = \|\mathrm{d}(\xi-\bar{\xi})|_{A_2}\|_p &\le C_{\delta,p}(A_1)\big(\|\bar{\partial}(\xi-\bar{\xi})\|_p + \|\mathrm{d}(\xi-\bar{\xi})\|_2 + \|\xi-\bar{\xi}\|_1\big)\\ &= C_{\delta,p}(A_1)\big(\|\bar{\partial}\xi\|_p + \|\mathrm{d}\xi\|_2 + \|\xi-\bar{\xi}\|_1\big).\end{aligned} \tag{D.20}$$

The claim follows by applying Corollary D.7 with $\zeta=\xi-\bar{\xi}$. □

Remark D.13. The case $r_1>0$ (which is the case needed for gluing pseudo-holomorphic maps in symplectic topology) follows from Corollary C.7; Corollary D.7 can be used to obtain a sharper statement in this case (that $C_{\delta,p}(A_1)$ does not depend on r_1). The $r_1=0$ case requires only the first two steps in the proof of Corollary D.7.

A smooth generalized CR-operator in a smooth complex vector bundle (E,∇) with connection over an almost complex manifold (X,J) is an operator of the form

$$D = \bar{\partial}_\nabla + A\colon \Gamma(X;E) \longrightarrow \Gamma(X;T^*X^{0,1}\otimes_{\mathbb{C}}E),$$

where

$$\bar{\partial}_\nabla\xi = \frac{1}{2}\big(\nabla\xi + \mathfrak{i}\nabla_J\xi\big) \quad \forall\,\xi\in\Gamma(X;TX), \qquad A\in\Gamma\big(X;\mathrm{Hom}(E;T^*X^{0,1}\otimes_{\mathbb{C}}E)\big).$$

If in addition $u\colon\Sigma\longrightarrow X$ is a smooth map from an almost complex manifold $(\Sigma,\mathfrak{j})$, the pull-back CR-operator is given by

$$D_u = \bar{\partial}_{\nabla^u} + A\circ\partial u\colon \Gamma(u;E)\longrightarrow\Gamma^{0,1}(u;E).$$

Proposition D.14. *If (X,g) is a Riemannian manifold with an almost complex structure J, $(E,\langle,\rangle,\nabla)$ is a normed complex vector bundle with connection over X and a smooth generalized CR-operator D, and $p\ge1$, then for every compact subset $K\subset X$, $\delta>0$, and open annulus $A_1\subset\mathbb{R}^2$, there exists $C_{K;\delta,p}(A_1)\in\mathbb{R}^+$ with the following property. If $u\in C^\infty(A_1;X)$ is such that $\mathrm{Im}\,u\subset K$, $\xi\in\Gamma(u;E)$, and $A_2\Subset_\delta A_1$ is an annulus, then*

$$\big\|\nabla^u\xi|_{A_2}\big\|_p \le C_{K;\delta,p}(A_1)\big(\|D_u\xi\|_p + \|\nabla^u\xi\|_2 + \|\xi\otimes\mathrm{d}u\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. We continue with the setup in the proof of Lemma D.8. By Corollary D.12,

$$\begin{aligned}\big\|\mathrm{d}\widetilde{\xi}_i|_{A_2}\big\|_p &\le C_{i;\delta,p}(A_1)\big(\|\bar{\partial}\widetilde{\xi}_i\|_p + \|\mathrm{d}\widetilde{\xi}_i\|_2\big)\\ &\le C'_{i;\delta,p}(A_1)\big(\big\|\bar{\partial}\xi_i|_{u^{-1}(U_i)}\big\|_p + \big\|\mathrm{d}\xi_i|_{u^{-1}(U_i)}\big\|_2 + \|\xi\otimes\mathrm{d}u\|_p\big).\end{aligned} \tag{D.21}$$

Since ∇ commutes with the complex structure in E and $\widetilde{\xi}_i=\xi_i$ on $u^{-1}(W_i)$, it follows from (D.12) and (D.21) that

$$\begin{aligned}\big\|\nabla^u\xi|_{A_2\cap u^{-1}(W_i)}\big\|_p &\le \big\|\mathrm{d}\widetilde{\xi}_i|_{A_2}\big\|_p + C_K\|\xi\otimes\mathrm{d}u\|_p\\ &\le \widetilde{C}_{i;\delta,p}(A_1)\big(\|\bar{\partial}_{\nabla^u}\xi\|_p + \|\nabla^u\xi\|_2 + \|\xi\otimes\mathrm{d}u\|_p\big)\\ &\le \widetilde{C}'_{i;\delta,p}(A_1)\big(\|D_u\xi\|_p + \|\nabla^u\xi\|_2 + \|\xi\otimes\mathrm{d}u\|_p\big).\end{aligned} \tag{D.22}$$

The claim is obtained by summing the last equation over all i. □

Lemma D.15. *If (X,g) is a Riemannian manifold with an almost complex structure J, $(E,\langle,\rangle,\nabla)$ is a normed complex vector bundle with connection over X and a smooth generalized CR-operator D, and $p>2$, then for every compact subset $K\subset X$ and open ball $B\subset\mathbb{R}^2$, there exists $C_{K;B,p}\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $u\in C^\infty(B;X)$ is such that $\mathrm{Im}\,u\subset K$ and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_{p,1}\le C_{K;B,p}(\|\mathrm{d}u\|_p)\big(\|D_u\xi\|_p+\|\xi\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. By an argument nearly identical to the proof of Proposition D.14,

$$\|\xi\|_{p',1}\le C_{K;p'}(B)\big(\|D_u\xi\|_{p'}+\|\xi\|_{p'}+\|\xi\otimes\mathrm{d}u\|_{p'}\big)$$

for any $p'\ge 1$. On the other hand, by Proposition D.10,

$$\|\xi\|_{C^0}\le C_{K;B,\widetilde{p}}(\|\mathrm{d}u\|_{\widetilde{p}})\|\xi\|_{\widetilde{p},1},$$

where $\widetilde{p}=(p+2)/2$. Proceeding as in the proof of Proposition D.10, we then obtain

$$\begin{aligned}
\|\xi\|_{p,1}&\le C_{K;B,p}(\|\mathrm{d}u\|_{\widetilde{p}})\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{\widetilde{p},1}\big),\\
\|\xi\|_{\widetilde{p},1}&\le C_{K;\widetilde{p}}(B)\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{q_1}\big),\\
\|\xi\|_{q_i}&\le C_{K;p_i,q_i}(B)\big(\|\xi\|_{p_i,1}+\|\xi\otimes\mathrm{d}u\|_{p_i}\big)\\
&\le C_{K;B,i}(\|\mathrm{d}u\|_p)\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{q_{i+1}}\big);
\end{aligned}$$

we stop the recursion at the same value of $i=N$ as in the proof of Proposition D.10. □

Proposition D.16. *If (X,g) is a Riemannian manifold with an almost complex structure J, $(E,\langle,\rangle,\nabla)$ is a normed complex vector bundle with connection over X and a smooth generalized CR-operator D, and $p>2$, then for every compact subset $K\subset X$ and compact Riemann surface (Σ,g_Σ), there exists $C_{K;\Sigma,p}\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $u\in C^\infty(\Sigma;X)$ is such that $\mathrm{Im}\,u\subset K$ and $\xi\in\Gamma(u;E)$, then*

$$\|\xi\|_{p,1}\le C_{K;\Sigma,p}\big(\|\mathrm{d}u\|_p)(\|D_u\xi\|_p+\|\xi\|_p\big).$$

Proof. This statement is immediate from Lemma D.15. □

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