MAT 644: Complex Curves and Surfaces
Notes for 04/20/20

Last time: $S=$ $\mathbb{C}$-surface, $M=$ $\mathbb{C}$-mfld
(1) $f\colon S \dashrightarrow M$ is rational map if $f\colon S-F\longrightarrow M$ is holomorphic for some finite $F\subset S$
(2) rational $f\colon S\dashrightarrow S'$ is birational if $\exists g\colon S'\dashrightarrow S$ rational s.t. $g\circ f=\mathrm{id}_S$ and $f\circ g=\mathrm{id}_{S'}$
outside of divisors on S, S'

Prp: $S,S'=$ projective surface. Then $f\colon S\dashrightarrow S'$ is birational
iff $\exists$ blowups $\pi\colon\widetilde{S}\longrightarrow S$ and $\pi'\colon\widetilde{S}\longrightarrow S'$ s.t. $\pi'=f\circ\pi$ wherever RHS is defined

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ S \arrow[rr, dashed, "f"] & & S' \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2=\mathrm{Bl}_q(\mathbb{P}^1\times\mathbb{P}^1) \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^1\times\mathbb{P}^1 \end{tikzcd}$$

$\boxed{\textbf{HW4 \#4}}$

$$[Z_0,Z_1,Z_2] \xdashrightarrow[p_1=[0,1,0],\quad p_2=[0,0,1],\quad q=([0,1],[0,1])]{} ([Z_0,Z_1],[Z_0,Z_2])$$

S,S' birational $\Longrightarrow$ (1) $\pi_1(S)\approx\pi_1(S')$: π_1 does not change under blowups
(2) $h^{p,q}(S)=h^{p,q}(S')$ if $(p,q)\neq(1,1)$: these $h^{p,q}$ do not change under blowups
(3) $P_n(S)=P_n(S')$: $H^0(K_S^{\otimes n})\xrightarrow{\{\mathrm{d}\pi\}^*}H^0(K_{\widetilde{S}}^{\otimes n})$ isomor. if $\pi\colon\widetilde{S}\longrightarrow S$ is blowup
$P_n(S)\equiv\dim H^0(K_S^{\otimes n})$ n-th pluri-genus, $P_1(S)=h^{2,0}(S)$ if S is cmpt Kähler

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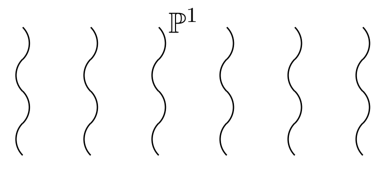
Today: Rational Surfaces
Dfn: Projective surface S is rational if S is birational to $\mathbb{P}^2$
$\Longrightarrow$ $S=$ cmpt connected, $\pi_1(S)=0$, $P_n(S)\equiv\dim H^0(K_S^{\otimes n})=0$ $\forall\, n\in\mathbb{Z}^+$
Next Monday (?): converse to (weaker version of) this, Castelnuovo-Enriques Thm

Examples: $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ minimal (no exceptional curves); blowups of these
Today: more minimal rational surfaces

Dfn: $\mathbb{C}$-mfld S is ruled if $\exists$ holomor. $\pi\colon S\longrightarrow C$ s.t. $\pi^{-1}(p)\approx\mathbb{P}^1$ $\forall\, p\in C$
(can assume $\dim_{\mathbb{C}}S=2\Longrightarrow\dim_{\mathbb{C}}C=1$)
Example 0: $S=C\times\mathbb{P}^1$

$$\begin{tikzcd} S \arrow[d, "\pi"'] \\ C \end{tikzcd}$$



Lemma 1: If $\pi\colon S\longrightarrow C$ is holomor. s.t. $\pi^{-1}(p)\approx\mathbb{P}^1$ $\forall\,p\in C$ and $\dim_{\mathbb{C}}S=2$ (or π is submersion), then π is a $\mathbb{P}^1$-bundle over C ($\pi\colon S\longrightarrow C$ locally trivial).

Need to show: $\forall p\in C$, $\exists$ neighborhood $U_p\subset C$ of p and holomorphic trivialization Φ_p of $S|_{U_p}$:

$$\begin{tikzcd} S|_{U_p} \arrow[rr, dashed, "\Phi_p", "\approx"'] \arrow[dr, "\pi"'] & & U_p\times\mathbb{P}^1 \arrow[dl, "\pi_1"] \\ & U_p & \end{tikzcd}$$

No problem in smooth category if π is submersion:

horizontal lifts of paths $\Phi_p=\exp\colon\mathcal{N}_S S_p\approx S_p\times T_pC\longrightarrow S$

$\mathcal{N}_S S_p\equiv$ normal bundle of S_p in S

Claim: if $\dim_{\mathbb{C}}S=2$, then π is submersion

Proof of "Need to show" in holomor. category.

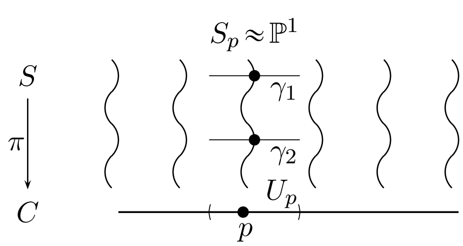
Choose positive l.b. $L\longrightarrow S$ s.t. $H^1(S;L(-S_p))=0$ (can do by Thm B, G&H p159)
(if $\dim_{\mathbb{C}}S>2$, want $H^1(S;\mathcal{I}_{S_p}(L))=0$ with $\mathcal{I}_{S_p}\equiv\{f\in\mathcal{O}_S\colon f|_{S_p}=0\}$)
L positive $\Longrightarrow$ $L|_{S_p}=\mathcal{O}_{\mathbb{P}^1}(d)$ with $d\in\mathbb{Z}^+$ ($S_p\approx\mathbb{P}^1$)
π submersion $\Longrightarrow$ $[S_q]=[S_p]\in H_2(S;\mathbb{Z})$ $\forall\,q\in C$
$\Longrightarrow$ $L|_{S_q}=\mathcal{O}_{\mathbb{P}^1}(d)$ $\forall\,q\in C$ b/c $d=\langle c_1(L),S_q\rangle$

$H^1(S;L(-S_p))=0\Longrightarrow H^0(S;L)\longrightarrow H^0(S_p;L|_{S_p})\approx\mathbb{C}^{d+1}$ onto:

$$0\longrightarrow\mathcal{O}_S\big(L(-S_p)\big)\longrightarrow\mathcal{O}_S(L)\longrightarrow\mathcal{O}_S(L_p)\big|_{S_p}\longrightarrow 0$$

$\therefore$ Can choose $\sigma_0,\ldots,\sigma_d\in H^0(S;L)$ s.t. $\sigma_0|_{S_p},\ldots,\sigma_d|_{S_p}$ is basis for $H^0(S_p;L|_{S_p})$
$\Longrightarrow$ $\sigma_0|_{S_q},\ldots,\sigma_d|_{S_q}$ is basis for $H^0(S_q;L|_{S_q})$ $\forall\,q\in C$ near p
$\Longrightarrow$ $\iota_q\colon S_q\hookrightarrow\mathbb{P}^d$, $x\longrightarrow[\sigma_0(x),\ldots,\sigma_d(x)]$, is embedding as nondegen. degree d curve
$\Longrightarrow$ can choose $\widetilde{p}_1,\ldots,\widetilde{p}_{d-1}\in S_p$ spanning $\mathbb{P}^{d-2}\subset\mathbb{P}^d$ ($\widetilde{p}_i\neq\widetilde{p}_j$)

Choose lifts $\gamma_i\colon(U_p,p)\longrightarrow(S,\widetilde{p}_i)$ $\Longrightarrow$ (i) $\gamma_i\pitchfork S_p$
(ii) $\iota_q(\gamma_1(q)),\ldots,\iota_q(\gamma_{d-1}(q))\in\mathbb{P}^d$ span some $H_q^2\equiv\mathbb{P}^{d-2}\subset\mathbb{P}^d$ $\forall\,q\in C$ near p



Pick $\mathbb{P}^1\subset\mathbb{P}^d-H_p^2\Longrightarrow\mathbb{P}^1\cap H_q^2=\varnothing$ $\forall\,q\in C$ near p
$\Longrightarrow$ get projection $\pi_q\colon\mathbb{P}^d-H_q^2\longrightarrow\mathbb{P}^1$, $x\longrightarrow(\overline{H_q^2x})\cap\mathbb{P}^1$
if $x,x'\in\iota_q(S_q)-H_q^2$ with $\pi(x)=\pi(x')$,
then H_q^2,x,x' lie in hypersurface $H\subset\mathbb{P}^n$ with $|\iota_q(S_q)\cap H|>d\Longrightarrow\iota_q(S_q)\subset H$ X

Define $\Phi_p\colon S|_{U_p}\longrightarrow U_p\times\mathbb{P}^1$ by $\Phi_p|_{S_q}\equiv\pi_q\circ\iota_q\colon S_q-\{\gamma_i(q)\}\xrightarrow{1:1}\{q\}\times\mathbb{P}^1$
extends over $\{\gamma_i(q)\}$ b/c S_q is a curve; injective by above
Φ_p is holomorphic b/c π_q,ι_q depend holomor. on q $\square$

Proof of Claim. Let $p\in C$.

$[S_p]=k[S_q]\in H_2(X;\mathbb{Q})$ for some $k\in\mathbb{Z}^+$ and generic $q\in C \implies \deg\mathcal{N}_S S_p=0$
$-2=K_S\cdot S_q=(K_S\cdot S_p)/k=\big(\langle K_{S_p},S_p\rangle-\deg\mathcal{N}_S S_p\big)/k=-2/k \implies k=1$
$k=\min\{l\in\mathbb{Z}^+ : D^l_x\pi\neq 0 \text{ for some } x\in S_p\} \implies (\mathcal{N}_S S_p)^{\otimes k}\longrightarrow S_p\times\mathbb{C}$ isom. $\square$

Lemma 2: if $S\longrightarrow C$ is a $\mathbb{P}^{k-1}$-bundle and C is a smooth $\mathbb{C}$-curve,
then $\exists\ E\longrightarrow C$ holomor. v.b. of rank $k+1$ s.t. $S\approx\mathbb{P}E$ as $\mathbb{P}^{k-1}$-bundles over C.

Proof. $E\longleftrightarrow\big\{\widetilde{g}_{\alpha\beta}: U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}_k\mathbb{C}\big\}\in\check{H}^1(C;\mathrm{GL}_k\mathbb{C})$
$S\longleftrightarrow\big\{g_{\alpha\beta}: U_\alpha\cap U_\beta\longrightarrow \mathrm{PGL}_k\mathbb{C}\big\}\in\check{H}^1(C;\mathrm{PGL}_k\mathbb{C})$
$\therefore$ *enough to show* $\check{H}^1(C;\mathrm{GL}_k\mathbb{C})\longrightarrow\check{H}^1(C;\mathrm{PGL}_k\mathbb{C})$ onto

Note: $\check{H}^1(C;G)\equiv\check{H}^1$ for sheaf of holomorphic functions with values in group G
$\mathrm{GL}_k\mathbb{C},\mathrm{PGL}_k\mathbb{C}$ not abelian $\Longrightarrow \check{H}^1(C;\cdot)$ not a group, $\check{H}^2(C;\cdot)$ not defined

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Short exact sequence

$$\{1\}\longrightarrow\mathbb{C}^*\longrightarrow\mathrm{GL}_k\mathbb{C}\longrightarrow\mathrm{PGL}_k\mathbb{C}\longrightarrow\{1\}$$

of groups gives exact sequence in $\check{H}^*$:

$$\check{H}^1(C;\mathrm{GL}_k\mathbb{C})\longrightarrow\check{H}^1(C;\mathrm{PGL}_k\mathbb{C})\longrightarrow\check{H}^2(C;\mathbb{C}^*)\equiv\check{H}^2(C;\mathcal{O}^*)$$

Claim: $\check{H}^2(C;\mathcal{O}^*)=0 \qquad\Longrightarrow$ Lemma 2

Proof of Claim. Short exact sequence

$$\{0\}\longrightarrow\mathbb{Z}\longrightarrow\mathbb{C}\xrightarrow{\exp(2\pi\mathrm{i}\cdot)}\mathbb{C}^*\longrightarrow\{1\}$$

of abelian groups gives exact sequence in $\check{H}^*$:

$$\check{H}^2(C;\mathbb{C})\equiv\underbrace{\check{H}^2(C;\mathcal{O})}_{H^{0,2}(C)=\{0\}}\longrightarrow\check{H}^2(C;\mathbb{C}^*)\equiv\check{H}^2(C;\mathcal{O}^*)\longrightarrow\underbrace{\check{H}^3(C;\mathbb{Z})}_{H^3(C;\mathbb{Z})=\{0\}}$$

More on $\check{H}^1$ with values in non-abelian groups: MAT 545 HW2 #6
math/1905.11316 Appendix A

Crl: if $S\longrightarrow\mathbb{P}^1$ is a projective ruled surface,
then $\exists\ k\in\mathbb{Z}^{\geqslant 0}$ s.t. $S\approx\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))\equiv\mathbb{F}_k$ (k-th Hirzebruch surface) as $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.

Proof. Lemmas 1,2 $\Longrightarrow S\approx E$ for some $E\longrightarrow\mathbb{P}^1$ holomor. v.b. of rank 2
Grothendieck's Thm (MAT 545 HW6 #6) $\Longrightarrow E\approx\underbrace{\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(b)}_{\approx(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(b-a))\otimes\mathcal{O}_{\mathbb{P}^1}(a)}$ for some $a,b\in\mathbb{Z}$, $a\leqslant b$
$\Longrightarrow S\approx\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))$ with $k\equiv b-a\in\mathbb{Z}^{\geqslant 0}$ $\square$

Examples: $\mathbb{F}_0\approx\mathbb{P}^1\times\mathbb{P}^1$ (trivial)
$\mathbb{F}_1\approx\mathrm{Bl}_p\mathbb{P}^2$ (later)

Properties of Hirzebruch Surfaces $\pi\colon \mathbb{F}_k \equiv \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)) \longrightarrow \mathbb{P}^1$

(1) $\pi_1(\mathbb{F}_k)=\{0\}$: homotopy exact sequence of the fiber bundle $\mathbb{P}^1 \longrightarrow \mathbb{F}_k \longrightarrow \mathbb{P}^1$ and $\pi_1(\mathbb{P}^1)=\{0\}$

(2) $H^*(\mathbb{F}_k)\approx H^*(\mathbb{P}^1)\otimes H^*(\mathbb{P}^1)$ as graded $\mathbb{Z}$-modules (*not* algebras, unless $k=0$)

$$\Longrightarrow h^{p,q}(\mathbb{F}_k) = \begin{cases} 1, & \text{if } (p,q)=(0,0),(2,2); \\ 2, & \text{if } (p,q)=(1,1); \\ 0, & \text{otherwise.} \end{cases}$$

Reason for $\approx$. Let $E=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)$,

$\tilde{\gamma}\equiv\big\{(\ell,v)\in\mathbb{P}E\times E\colon v\in\ell\big\}\longrightarrow\mathbb{P}E$ tautological line bundle,

$u\equiv c_1(\tilde{\gamma}^*)\in H^2(\mathbb{P}E;\mathbb{Z}) \Longrightarrow \{1, u|_{\mathbb{P}E_p}\}$ is basis for $H^*(\mathbb{P}E_p;\mathbb{Z})$ for all $p\in C$

Take $\theta\colon \underbrace{\mathbb{Z}\{1,u\}}_{\mathbb{Z}-\text{Span of } 1,u} \longrightarrow H^*(\mathbb{P}E;\mathbb{Z})$ obvious map

$\mathbb{Z}\{1,u\} \xrightarrow{\theta} H^*(\mathbb{P}E;\mathbb{Z}) \longrightarrow H^*(\mathbb{P}E_p;\mathbb{Z})$ isom. $\forall\, p\in C$

$\therefore\ \theta$ is a cohomology extension of the fiber for fiber bundle $\mathbb{P}^1\longrightarrow\mathbb{P}E\xrightarrow{\pi}\mathbb{P}^1$

Spanier Thm 5.7.9 (MAT 566?) $\Longrightarrow H^*(\mathbb{P}^1;\mathbb{Z})\otimes\mathbb{Z}\{1,u\}\longrightarrow H^*(\mathbb{P}E_p;\mathbb{Z})$, $\alpha\otimes\beta\longrightarrow(\pi^*\alpha)\cup\theta(\beta)$,
is isom. of graded $\mathbb{Z}$-modules

Note: $\tilde{\gamma}\subset\pi^*E\longrightarrow\mathbb{P}E \Longrightarrow \tilde{\gamma}^*\otimes\pi^*E$ contains trivial $\mathbb{C}$-subbundle
$\Longrightarrow u^2+(\pi^*c_1(E))u+\pi^*c_2(E)=e(\tilde{\gamma}^*\otimes\pi^*E)=0$

$$\therefore \qquad H^*(\mathbb{P}^1;\mathbb{Z})[u]/\big(u^2+c_1(E)u+c_2(E)\big) \longrightarrow H^*(\mathbb{P}E;\mathbb{Z}), \qquad \alpha\otimes\beta \longrightarrow (\pi^*\alpha)\cup\theta(\beta),$$

is isomorphism of graded $\mathbb{Z}$-algebras (the product is preserved)