

MAT 566: Characteristic Classes

Problem Set 2

Due by Tuesday, 2/25, 5pm, in class

(if you have not passed the orals yet)

Do 7-C plus any *one* of the following problems: 5-B,5-E,6-B,7-A,7-B, (ii), or (iii) below.

Problem (ii): Let X be a paracompact locally contractible topological space. Thus, $\check{H}^1(X; \mathbb{Z}_2)$ is naturally isomorphic to $H^1(X; \mathbb{Z}_2)$; see Chapter 5 in Warner. An equivalence class $[L]$ of real line bundles corresponds to some element $\check{w}_1(L) \in \check{H}^1(X; \mathbb{Z}_2)$. Show that $\check{w}_1(L)$ corresponds to $w_1(L) \in H^1(X; \mathbb{Z}_2)$ under the natural isomorphism.

Hint: there is a *very* short solution via naturality.

Problem (iii): Let $\gamma_{\mathbb{C}} \equiv \{(\ell, v) \in \mathbb{C}\mathbb{P}^1 \times \mathbb{C}^2 : v \in \ell \subset \mathbb{C}^2\}$ be the total space of the complex tautological line bundle over $\mathbb{C}\mathbb{P}^1 \approx S^2$. For each $a \in \mathbb{Z}$, define

$$V_a \equiv \gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}} 2a} / \mathbb{Z}_2 \longrightarrow \mathbb{C}\mathbb{P}^1 / \mathbb{Z}_2 \approx \mathbb{R}\mathbb{P}^2, \\ ([z_0, z_1], (v_0, v_1)^{\otimes_{\mathbb{C}} 2a}) \sim ([-\bar{z}_1, \bar{z}_0], (-\bar{v}_1, \bar{v}_0)^{\otimes_{\mathbb{C}} 2a}), \quad [z_0, z_1] \sim [-\bar{z}_1, \bar{z}_0].$$

Show that V_a is a rank 2 real vector bundle, is not orientable for every $a \in \mathbb{Z}$, and does not split as a sum of line bundles if $a \neq 0$. Furthermore, V_a and $V_{a'}$ are not isomorphic as real vector bundles if $a \neq a'$. For the last statement, you may need to use that

$$\langle e(\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}} 2a}), [\mathbb{C}\mathbb{P}^1] \rangle = \langle c_1(\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}} 2a}), [\mathbb{C}\mathbb{P}^1] \rangle = -2a,$$

as will be shown later in the course (this implies that $\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}} 2a}$ and $\gamma_{\mathbb{C}}^{\otimes_{\mathbb{C}} 2a'}$ are not isomorphic as real vector bundles if $a \neq a'$).

Remark 1: Problem 7-C is an example of the *Splitting Principle*: if a natural formula involving characteristic classes holds for split vector bundles (i.e. direct sums of line bundles), then it holds for all vector bundles.

Remark 2: Problem (ii) implies that the real line bundles over a paracompact topological space are classified by their w_1 . Problem (iii) implies that the real vector bundles of rank 2 (and higher) are generally not distinguished by their total Stiefel-Whitney classes w , because $H^2(\mathbb{R}\mathbb{P}^2; \mathbb{Z}_2)$ contains only 2 elements in total.