

# MAT 566: Differential Topology Spring 2018

## Problem Set 4

Due on Monday, 03/26, by 5pm in Math 3-111

Problem (v): If  $X$  is a manifold and  $Y$  is a compact oriented submanifold, the image of the fundamental homology class  $\mu_Y = [Y]$  under the inclusion map into  $X$  determines a homology class in  $X$ , which will be denoted by  $[Y]_X$ . Throughout this problem, all manifolds will be assumed to be smooth, compact, and oriented; all vector bundles are smooth. If  $Y$  is a submanifold of  $X$ , let

$$\mathcal{N}_X Y \equiv TX|_Y / TY$$

be the normal bundle of  $Y$  in  $X$ .

(a) Suppose  $Y^k$  is a submanifold of  $X^n$ . Let

$$u'' \equiv u'|_X \in H^{n-k}(X; \mathbb{Z})$$

be the dual class to  $Y$ ; see the bottom of p118. Show that  $u''$  is the Poincaré dual of  $[Y]_X$ , i.e. the Poincaré duality isomorphism

$$\cap[X]: H^{n-k}(X; \mathbb{Z}) \longrightarrow H_k(X; \mathbb{Z}), \quad \alpha \longrightarrow \alpha \cap [X],$$

takes  $u''$  to  $[Y]_X$ , at least up to a sign in odd dimensions depending on conventions (see 11-C). Conclude that

$$e(\mathcal{N}_X Y) = \text{PD}_X([Y]_X)|_Y \in H^{n-k}(Y; \mathbb{Z}).$$

(b) Suppose  $V \longrightarrow X^n$  is an orientable vector bundle of rank  $k$  and  $s: X \longrightarrow V$  is a smooth section which is transverse to the zero set. Show that  $s^{-1}(0)$  is a smooth compact orientable submanifold of  $X$  and

$$[s^{-1}(0)]_X = \text{PD}_X(e(V)) \in H_{n-k}(X; \mathbb{Z}),$$

provided  $s^{-1}(0)$  is suitably oriented (describe how). If  $k = n$ ,  $s^{-1}(0)$  is a finite set and the image of  $[s^{-1}(0)]_X$  under the natural homomorphism

$$H_0(X; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is the number of points in  $s^{-1}(0)$  *counted with sign*. How is the sign of each point of  $s^{-1}(0)$  determined?

(c) Suppose  $Y_1^k, Y_2^m$  are compact oriented submanifolds of  $X^n$  that intersect transversally. Show that  $Y_1 \cap Y_2$  is a smooth compact orientable submanifold of  $X$  and

$$[Y_1 \cap Y_2]_X = \text{PD}_X(\text{PD}_X([Y_1]_X) \cup \text{PD}_X([Y_2]_X)) \in H_{k+m-n}(X; \mathbb{Z}),$$

provided  $Y_1 \cap Y_2$  is suitably oriented (describe how).