Problem (v): A homogeneous cubic polynomial in four variables, $f(x_0, x_1, x_2, x_3) = \sum_{i+j+k+l=3} a_{ijkl} x_0^i x_1^j x_2^k x_3^l$, naturally corresponds to an element

$$\tilde{f} \in (\text{Sym}^3((\mathbb{C}^4)^*))^* \approx \text{Sym}^3((\mathbb{C}^4)^*) \equiv (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^*/\sim, \quad \alpha \otimes \beta \otimes \gamma \sim \beta \otimes \alpha \otimes \gamma \sim \alpha \otimes \gamma \otimes \beta.$$  

This is because each $X_i$ defines a linear map $X_i : \mathbb{C}^4 \to \mathbb{C}$, i.e. the projection onto the $i$th coordinate. Since every fiber of the tautological 2-plane bundle $\gamma_2 \to \text{Gr}_2 \mathbb{C}^4$ is a linear subspace of $\mathbb{C}^4$, by restriction $f$ induces a section

$$s_f \in \Gamma(\text{Gr}_2 \mathbb{C}^4; \text{Sym}^3(\gamma_2^*)).$$

Note that the complex rank of the bundle $\text{Sym}^3(\gamma_2^*) \to \text{Gr}_2 \mathbb{C}^4$ equals to the dimension of $\text{Gr}_2 \mathbb{C}^4$. Thus, if $s_f$ is transverse to the zero set (as is the case for a generic $f$), the set $s_f^{-1}(0)$ is finite and its signed cardinality is

$$\#|s_f^{-1}(0)| = \langle e(\text{Sym}^3(\gamma_2^*)), \text{Gr}_2 \mathbb{C}^4 \rangle \in \mathbb{Z}.$$  

In fact, $s_f$ is a holomorphic section of a holomorphic vector bundle and thus all its zeros are positive.

A cubic surface $Y$ in $\mathbb{C}P^3$ is the zero set of a homogeneous cubic polynomial in four variables, i.e.

$$Y \equiv Y_f = \{ [x_0, x_1, x_2, x_3] \in \mathbb{C}P^3 : \sum_{i+j+k+l=3} a_{ijkl} x_0^i x_1^j x_2^k x_3^l = 0 \} = (f^{-1}(0) - 0)/\mathbb{C}^*.$$  

A projective line $\ell$ in $\mathbb{C}P^3$ corresponds to a 2-plane $P$ in $\mathbb{C}^4$, i.e. an element of $\text{Gr}_2 \mathbb{C}^4$. Such a line $\ell$ lies in $Y_f$ if and only if $f|_P$ vanishes identically, i.e. $P \in s_f^{-1}(0)$. Thus, if $f$ is generic, the number of lines in $Y_f$ is finite and is given by the euler class of $\text{Sym}^3(\gamma_2^*)$.

(a) Formulate and prove a splitting principle for Chern classes.
(b) Use it to determine the number of lines that lie on a generic cubic surface in $\mathbb{C}P^3$.
(c) Determine the number of lines that lie on a generic quintic threefold in $\mathbb{C}P^4$.