MAT 545: Complex Geometry
Fall 2008

Problem Set 5
Due on Tuesday, 11/18, at 2:20pm in Math P-131
(or by 2pm on 11/18 in Math 3-111)

Please write up **clear** *and concise solutions to problems worth 20 pts.*

Problem 1 (10 pts)

Suppose (M, J) is an almost complex manifold, g is a J-compatible Riemannian metric on M, and ∇ is the Levi-Civita connection of g (thus, J is a complex structure in the fibers of the vector bundle $TM \longrightarrow M$ which preserves g; ∇ is g-compatible and $[X, Y] = \nabla_X Y - \nabla_Y X$ for any two vector fields X, Y on M). Show that $\nabla J = 0$ if and only if (M, J, g) is Kahler (you can use either of the equivalent conditions in PS2,#1 as the integrability criterion for J).

Problem 2 (5 pts)

Let $M_n = (\mathbb{C}^n - 0)/\sim$, where $z \sim 2^k z$ for every $k \in \mathbb{Z}$.
(a) Show that M_n is a complex manifold (with the complex structure inherited from $\mathbb{C}^n$). What simple smooth manifold is M diffeomorphic to?
(b) Give at least two reasons why M_n does not admit a Kahler metric for $n \geq 2$.

Problem 3 (10 pts)

Let $M = \mathbb{R}^4/\sim$, where

$$(s, t, x, y) \sim \big(s+k, t+l, x+m, y+lx+n\big) \qquad \forall\ (s, t, x, y) \in \mathbb{R}^4,\ (k, l, m, n) \in \mathbb{Z}^4 .$$

Show that
(a) this is an equivalence relation;
(b) M is a compact symplectic manifold (with the symplectic form, i.e. closed non-degenerate 2-form, inherited from the standard symplectic form on $\mathbb{R}^4$, i.e. $ds \wedge dt + dx \wedge dy$).
(c) M does not admit an integrable complex structure compatible with this symplectic form.

Note 1: M in (c) is the first known example (due to W. Thurston'76) of a symplectic manifold that admits no Kahler structure.
Note 2: in contrast, every symplectic manifold (M, ω) admits an *almost* complex structure compatible with ω; the space of ω-compatible almost complex structures is contractible.

Problem 4 (5 pts)

Let (X, J, g) be a Kahler manifold and ω its symplectic form. Show that ω is harmonic with respect to g.

Problem 5 (10 pts)

Let M be a compact complex manifold that admits a Kahler metric.
(a) Let α be a (p, q)-form on M such that $d\alpha = 0$. Show that the following are equivalent:

(i) $\alpha = d\beta$ for some $(p+q-1)$-form β;

(ii) $\alpha = \partial\beta$ for some $(p-1, q)$-form β;

(ii) $\alpha = \bar{\partial}\beta$ for some $(p, q-1)$-form β;

(ii) $\alpha = \partial\bar{\partial}\beta$ for some $(p-1, q-1)$-form β.

(b) Let ω and ω' be symplectic forms compatible with the complex structure on M (thus ω and ω' arise from Kahler metrics on M). If $[\omega] = [\omega'] \in H^2_{deR}(M)$, show that $\omega' = \omega + \mathfrak{i}\partial\bar{\partial}f$ for some $f \in C^\infty(M; \mathbb{R})$.